Convergence of Linearized Difference Schemes for Two-Dimensional Saint-Venant Equations (Shallow Water)

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Abstract: - The convergence of linearized difference scheme in Eulerian variables with non-linear regularizator to the smooth solutions for linear analog of two-dimensional Saint-Venant equations are considered for Cauchy problem with periodic (in spatial variables) solutions. The proof of convergence of difference scheme is performed by energetic method. In the class of sufficiently smooth solutions of the difference scheme is proved the convergence of solution of considered difference scheme in mesh norm $L^2$ with speed $O(h^2)$.

Key-Words: Difference scheme, Differential equation, Mathematical model, Convergence of the scheme.

1 Introduction
At the present-day the solution of two-dimensional hydrodynamic problems is actual. Also the problems related to catastrophes, for example, two-dimensional problem of neighbour dams collapse caused by earthquake, is of special interest. We call dams neighbour if they are located in the canyons outgoing to the same plain. The corresponding two-dimensional mathematical problem is formulated as following: at the initial moment of time the dams situated at left down and right upper corners of the effective domain instantly collapse and two strong water flows moving toward each other generate. The effective domain for the considered problem is the rectangular

$$D = \{0 \leq x \leq a, 0 \leq y \leq a\}.$$

2 Problem Formulation
The corresponding two-dimensional mathematical model can be written in divergent form for conservative variables - components of flow impulses $j_1(x, y, t)$ and $j_2(x, y, t)$ by axis x and y, and flow depth $H(x, y, t)$. In vector form they are as following [1]:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} F_1(Q) + \frac{\partial}{\partial y} F_2(Q) = W(Q),$$

$$J_1 = Hu, \ J_2 = Hv, \ Q = (J_1, J_2, H)^T,$$

$$F_1(Q) = \left(J_1 u + g H^2/2, \ J_1 v, \ J_1 \right)^T,$$

$$F_2(Q) = (J_2 u, \ J_2 v + g H^2/2, \ J_2)^T,$$

$$W(Q) = \begin{pmatrix} f_1(x, y, t, H, u, v) \\ f_2(x, y, t, H, u, v) \\ f_3(x, y, t, H, u, v) \end{pmatrix},$$

$$H(x, y, 0) = H_0(x, y),$$

$u(x, y, 0) = u_0(x, y) \quad v(x, y, 0) = v_0(x, y),$

where $x$ and $y$ are Eulerian coordinates, $t$- time, $g$ - acceleration of gravity, $v(x, y, t)$ and $u(x, y, t)$ - projections of velocity vector by x and y coordinates accordingly. $a_1, a_2, a_3 = const$, $v_0(x, y)$ and $H_0(x, y)$, $u_0(x, y)$ -- are relatively distribution functions of flow depth and velocities for the initial time moment.,

$$f_1(x, y, t, H, u, v), \ f_2(x, y, t, H, u, v), \ f_3(x, y, t, H, u, v)$$

are functions, expressing sources or sinks of mass or impulse.

The essential difficulties, arising during the investigation of partial differential equations, are due to non-linearity of considerable system. Therefore the investigation of solution behavior and moreover, the obtaining one in inexplicit form, in general case, is a difficult problem. Therefore we consider a linear approximation of Saint-Vennane (shallow water) two-dimensional equations (*) and the convergence investigation are made for linearized difference scheme with non-linear regularizator.

In domain

$$\{0 \leq x \leq a, 0 \leq y \leq a\}$$
\( \Omega = \{(x, y, t), -\infty < x < +\infty, -\infty < y < +\infty, 0 \leq t \leq T\} \)

we consider the Cauchy problem for non-homogeneous system of Saint-Vennane two-dimensional equations with Eulerian variables taking into account the sinks and sources:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial H}{\partial x} + v \frac{\partial u}{\partial y} = f_1(x, y, t, H, u, v)
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + u \frac{\partial v}{\partial y} + g \frac{\partial H}{\partial y} = f_2(x, y, t, H, u, v, u, v)
\]

where \( f_i, i = 1, 3 \) -- functions, expressing mass and impulse sources and sinks, \( H_0(x, y), v_0(x, y), v_0(x, y) \) -- sufficiently smooth periodic functions with period \( L \), with respect to \( x \), and with period \( L \) with respect to \( y \).

We assume that

(A1) Functions \( H(x, y, t), u(x, y, t), v(x, y, t) \) belongs to \( C^{1,2,3} \times \{(0,\infty) \times (0,\infty) \times [0, T] \} \) class and there exists such constant \( \mu > 0 \), that \( H(x, t) \geq \mu \), when \((x, y, t) \in \Omega \). We assume as well that the conditions guaranteeing the existence and uniqueness of periodic (with respect to \( x \) and \( y \)) solution of problem (1)-(4) are fulfilled.

(A2) Functions \( f_i(x, y, t, H, u, v) \) and \( f_3(x, y, t, H, u, v) \) are satisfy Lipshitz conditions with respect to variables \( H(x, y, t), u(x, y, t), v(x, y, t) \), with constant \( k > 0 \).

3 Problem Solution

For difference approximation of problem (1)-(3), let us consider a linear analogue of difference scheme with non-linear regularizer [2], a case a) --

\[
\frac{u}{c} \geq 1, \frac{v}{c} \geq 1:\n\]

\[
H_{ht} + H_0 u_0 + v_0 H_0 + H_0 v_0 + v_0 H_0 =
\]

\[
\varphi_1(\tau, h)H_0 u_{h,z} + u_0 H_{h,y} + \varphi_2(\tau, h)H_0 v_{h,y} + v_0 H_{h,y} + f_{3h}
\]

in case, by using \(-1 \leq u/c \leq 1, -1 \leq v/c \leq 1 \), we get:

\[
u_{ht} + u_0 v_0 + v_0 H_0 + H_0 v_0 + v_0 H_0 =
\]

\[
1.5\varphi_1(\tau, h)[c_0 u_{h,z} + 0.5(c_0 u_0 / H_0)H_{h,y}] +
\]

\[
+ \varphi_2(\tau, h)c_0 v_{h,y} + f_{1h},
\]

\[
v_{ht} + u_0 v_0 + v_0 H_0 + H_0 v_0 + v_0 H_0 =
\]

\[
1.5\varphi_1(\tau, h)c_0 v_{h,y} v_{h,y} + f_{2h}.
\]

where \( c = \sqrt{gH} \) -- propagation velocity, \( c_0 = \sqrt{gH_0} \), functions \( H_h = H_h(ih, jh, k\tau), u_h = u_h(ih, jh, k\tau), v_h = v_h(ih, jh, k\tau), f_{3h} = f_{3h}(ih, jh, k\tau), i = 1, 3 \) are the approximate functions of \( H, u, v, f_i, i = 1, 3 \), correspondently and are defined in the domain \( D = \omega_h \times \omega_r : \)

\[
\omega_r = \left\{(x, y) : x = ih, y = jh, j = 0, N_2, \right\},
\]

\[
\omega_r = \left\{n \tau : N \tau = T, n = 0, N \right\}.
\]

The initial conditions of difference scheme (5)-(10) have the following form:

\[
h_h(ih, jh, 0) = H_0(ih, jh),
\]

\[
u_h(ih, jh, 0) = v_0(ih, jh), v_0(ih, jh, 0) = v_0(ih, jh)\]

\[
i = 0, N_1, j = 0, N_2.
\]

Let us consider error function

\[
\tilde{H} = H - H, \tilde{u} = u_h - u, \tilde{v} = v_h - v\]

We denote

\[
\tilde{f}_i = f_i(ih, jh, H + \tilde{H}, u + \tilde{u}, v + \tilde{v}) - f_i(ih, jh, H, u, v)
\]

and determine from equality (**) \( H_h, u_h, v_h \) and insert into the difference equations (5)-(7). Then for the case -- \( u/c \geq 1, v/c \geq 1 \), (the investigations all the rest cases are made similarly and we obtain the same result), get:
\[ \tilde{u}_t + u_0 \tilde{u}_0 + g \tilde{H}_0 + v_0 \tilde{u}_0 = \varphi_1(\tau, h) (u_0 \tilde{u}_x + h \tilde{H}_x) + \varphi_2(\tau, h) v_0 \tilde{u}_y + \tilde{f}_1 - \psi_1, \]
\[ + \varphi_2(\tau, h) v_0 \tilde{u}_y + \tilde{f}_1 - \psi_1 \]
\[ \tilde{v}_t + u_0 \tilde{v}_0 + v_0 \tilde{v}_0 + g \tilde{H}_0 = \varphi_1(\tau, h) (u_0 \tilde{v}_x + h \tilde{H}_x) + \varphi_2(\tau, h) v_0 \tilde{v}_y + \tilde{f}_2 - \psi_2, \]
\[ \tilde{H}_t + h_0 \tilde{u}_0 + u_0 \tilde{H}_0 + h_0 \tilde{v}_0 = \varphi_1(\tau, h) (H_0 \tilde{u}_x + u_0 \tilde{H}_x) + \varphi_2(\tau, h) (H_0 \tilde{v}_y + v_0 \tilde{H}_y) + \tilde{f}_3 - \psi_3, \]
\[ \tilde{H}_h(ih, jh, 0) = 0, \quad \tilde{u}_h(ih, jh, 0) = 0, \quad \tilde{v}_h(ih, jh, 0) = 0, \quad i = 0, N_1, \quad j = 0, N_2, \]
\[ \psi_1 = u_t + u_0 u_y + g H_x + v_0 u_y - f_1 - \varphi_1(\tau, h) (u_0 \tilde{u}_x + h \tilde{H}_x) - \varphi_2(\tau, h) v_0 \tilde{u}_y \]
\[ \psi_2 = v_t + u_0 v_y + v_0 v_y + g H_y - f_2 - \varphi_1(\tau, h) (u_0 \tilde{v}_x + h \tilde{H}_x) \]
\[ \psi_3 = \tilde{H}_t + H_0 \tilde{u}_0 + u_0 H_x + v_0 H_y - f_3 - \varphi_1(\tau, h) (H_0 \tilde{u}_x + u_0 \tilde{H}_x) - \varphi_2(\tau, h) (H_0 \tilde{v}_y + v_0 \tilde{H}_y) \]

The following lemma is valid for the problem (5)-(7)-(11):

Lemma 1. Let conditions (A1) and (A2) be valid. Let also, that \( \tau = h^{2+2x}, \quad (x > 0), \)
\[ \varphi_1(\tau, h) = O(h^2), \quad \varphi_2(\tau, h) = O(h^2), \]
and for the solution of problem (5)-(7)-(11) at n-th layer \( 0 \leq n \leq N - 1, \) the following estimation
\[ \max \{ \| \tilde{H} \|, \| \tilde{u} \|, \| \tilde{v} \| \} \leq h \]
is fulfilled, then there exists \( h_i > 0, \) such that when \( h \leq h_i, \) the following statements are valid:
1) On \( (n+1) \)-th layer the difference scheme (5)-(7)-(11) has unique solution.
2) solution on \( (n+1) \)-th layer satisfies inequalities
\[ \max \{ \| \tilde{H} \|, \| \tilde{u} \|, \| \tilde{v} \| \} \leq h \]
where \( M \) is independent from \( h. \)

We consider the problem (5)-(7),(11). We note in advance, that if we multiply in scalars the equation \( y_t + Ay = \phi \) by \( \alpha y^{(0.5)} \), i.e. talking into account equality \( y^{(0.5)} = y + 0.5 y_y \), we obtain
\[ \alpha(\alpha, \| y^{(1/2)} \|) = (\psi, \alpha y) - (Ay, \alpha y) + 0.5 \tau(\alpha, (\alpha - Ay)^2). \]

Let multiply equation (12) in scalars by \( g^{-1} H_0 \tilde{u}^{(0.5)} \), equation (13) -- by \( g^{-1} H_0 \tilde{v}^{(0.5)} \), and equation (14) -- by \( \tilde{H}^{(0.5)} \). After addition of obtained equations we have:
\[ 0.5 \tilde{H}^2 = \sum_{i=1}^{18} g_i + (f_1 - \psi_1, \tilde{H}) + (f_2 - \psi_2, g^{-1} H_0 \tilde{u}) + (f_3 - \psi_3, g^{-1} H_0 \tilde{v}) \]

where \( g_1 = -0.5(g^{-1} H_0, \tilde{u}^2) \),
\( g_2 = -0.5(g^{-1} H_0, \tilde{v}^2) \),
\( g_3 = (\varphi_1(\tau, h) H_0 \tilde{u}_x + u_0 \tilde{H}_x, \tilde{H}) \),
\( g_4 = (\varphi_2(\tau, h) H_0 \tilde{v}_y + v_0 \tilde{H}_x, \tilde{H}) \),
\( g_5 = - (H_0 \tilde{u}_x, \tilde{H}) \),
\( g_6 = -(u_0 \tilde{H}_x, \tilde{H}) \),
\( g_7 = -(H_0 \tilde{v}_y, \tilde{H}) \),
\( g_8 = -(v_0 \tilde{H}_y, \tilde{H}) \),
\( g_9 = 0.5 \| \varphi_1(\tau, h) (H_0 \tilde{u}_x + u_0 \tilde{H}_x) + \tilde{H} \| + - \varphi_1(\tau, h) (H_0 \tilde{v}_y + v_0 \tilde{H}_y) - \tilde{f}_3 - \psi_3 - H_0 \tilde{u}_x - u_0 \tilde{H}_x - H_0 \tilde{v}_y - v_0 \tilde{H}_y \| \)
\( g_{10} = (\varphi_1(\tau, h) u_0 \tilde{u}_x + g \tilde{H}_x, g^{-1} H_0 \tilde{u}) \),
\( g_{11} = (\varphi_2(\tau, h) v_0 \tilde{v}_y, g^{-1} H_0 \tilde{u}) \),
\( g_{12} = -(u_0 \tilde{v}_y, g^{-1} H_0 \tilde{u}) \),
\( g_{13} = -(g \tilde{H}_x, g^{-1} H_0 \tilde{u}) \),
\( g_{14} = -(v_0 \tilde{H}_y, g^{-1} H_0 \tilde{u}) \).
where

\[ G_{15} = 0.5\tau \left( g^{-1}H_0 \cdot \varphi_1(\tau, h)u_0\tilde{u}_{x\tau} + g\tilde{H}_{x\tau} \right) + \varphi_2(\tau, h)v_0\tilde{v}_{y\tau} - \tilde{f}_1 - \psi_1 - u_0\tilde{u}_x - g\tilde{H}_x - v_0\tilde{u}_y \right) \]

\[ G_{16} = \left( \varphi_1(\tau, h)u_0\tilde{u}_{x\tau}, g^{-1}H_0\tilde{v} \right), \]

\[ G_{17} = \left( \varphi_2(\tau, h)v_0\tilde{v}_{y\tau} + g\tilde{H}_{y\tau}, g^{-1}H_0\tilde{v} \right). \]

To estimate the terms in the equality (19), let us use lemma 2.4 from [3], the methodology from [2], [3], [4], and also

\[ \varphi_1(\tau, h) = O(h^2), \quad \varphi_2(\tau, h) = O(h^2). \]

Then we obtain the basic energetic inequality:

\[ \frac{\hat{Q} - Q}{\tau} \leq M \left( \hat{Q} + Q \right) + M \|\psi\|^2, Q(0) = 0, \quad (20) \]

where

\[ Q = g^{-1}(H_0, \tilde{u}^2) + g^{-1}(H_0, \tilde{v}^2) + \|\tilde{H}\|^2, \]

\[ \|\psi\|^2 = \max \{\|\psi_1\|^2, \|\psi_2\|^2, \|\psi_3\|^2\} \equiv O(h^4). \]

Using Lemma 2.3 from [5] one solves this explicit energetic inequality. Using the lemma 1 and lemma 2.3 (from [5]) the following convergence theorem is proved:

**Theorem.** Let conditions (A1) (A2) are valid for solution of problem (1)-(4) and \( \varphi(\tau, h) = O(h^2) \).

\[ \tau = h^{2+2\varepsilon} \quad (\varepsilon = \text{const} > 0). \]

Then exists \( h_0 > 0 \) such that when \( h \leq h_0 \) the following assertions are true:

1) solution of difference problem (5)-(11) on interval \([0, T]\) exist and unique;

2) solution of problem (5)-(11) converges to the solution of problem (1)-(4) in mesh norm \( L_2 \) with speed \( O(h^2) \).

Thus, in the class of sufficiently smooth solutions of Cauchy problem for linear analogue two-dimensional Saint-Vennane equations in Eulerian variables convergence of linearized scheme solution in mesh norm \( L_2 \) with speed \( O(h^2) \) is proved.

**References:**


