Finite Difference Scheme for One Mixed Problem With Integral Condition

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Abstract: A mixed problem with integral restrictions and with Dirichlet conditions on a part of the boundary for Poisson equation is considered. A unique solvability of the corresponding difference scheme is studied.It is proved that the difference scheme converges in the discrete $W_2^1(\omega,\rho)$ norm with the rate $O(h^2)$, when the solution of the problem belongs to the space $W_2^3(\Omega)$.

Key–Words: nonlocal boundary value problem; difference scheme; weighted space

Boundary value problems for differential equations with nonlocal condition occur in many applications. Problems with integral conditions were considered by various authors. In the present paper, a nonlocal boundary problem with integral condition is considered. The corresponding difference scheme is constructed. Under assumption that a solution to the original problem belongs to Sobolev spaces, the estimate of convergence rate

$$\|y-u\|_{W_2^1(\omega,\rho)} \le ch^2\|u\|_{W_2^3(\Omega)} \qquad (1)$$

is obtained, where weight function $\rho(x_1) = 1 - x_1$. The error estimate is derived using certain well-known techniques (see, e.g., [1,2,3,4]) that employ the Bramble-Hilbert lemma.

Let $\Omega = \{(x_1,x_2): \quad 0 < x_k < 1, k = 1,2\}$ be a unit square with boundary Γ, and $\Gamma_1 = \{(0,x_2): 0 < x_2 < 1\}, \quad \Gamma_0 = \Gamma\backslash\Gamma_1$.

Consider the nonlocal boundary value problem

$$-\Delta u = f(x),\ x\in\Omega, \quad u(x)=0,$$
$$x\in\Gamma_0, \quad l(u)=0,\ 0<x_2<1, \qquad (2)$$

where

$$l(u) := \int_\xi^1 u(x)\,dx_1, \quad \xi\in(0;1).$$

Consider the following grid domains in $\bar\Omega$: $\bar\omega_\alpha = \{x_\alpha = i_\alpha h:\ i_\alpha = 0,1,\cdots,n, h=1/n\}, \omega_\alpha = \bar\omega_\alpha \cap (0,1),\ \omega_\alpha^+ = \bar\omega_\alpha \cap (0,1], \alpha=1,2, \omega=\omega_1\times\omega_2, \bar\omega = \bar\omega_1\times\bar\omega_2, \gamma = \bar\omega\cap\bar\Omega, \gamma_0 = \Gamma_0\cap\bar\omega$.

For grid functions and difference ratios, we use the notation

$$v_{x_i} = (v^{(+1_i)} - v)/h, \quad v_{\bar x_i} = (v - v^{(-1_i)})/h,$$

where $v^{(\pm 1_1)}(x) = v(x_1 \pm h, x_2)$, $v^{(\pm 1_2)}(x) = v(x_1, x_2 \pm h)$. For simplicity let us assume, that the index written below grid functions corresponds to the first coordinate: $y_i = y(ih, x_2)$.

Let

$$\xi = (m+\theta)h, \quad 0\le\theta<1,$$

where m is nonnegative integer $0\le m<n$.

Define the averaging operators

$$T_1 u = \frac{1}{h^2}\int_{x_1-h}^{x_1+h}(h-|x_1-t|)u((t,x_2)\,dt,$$

$$T_2 u = \frac{1}{h^2}\int_{x_2-h}^{x_2+h}(h-|x_2-t|)u((x_1,t)\,dt.$$

A set of grid functions given on $\bar\omega$ and satisfying the condition

$$y=0,\ x\in\gamma_0, \quad l_h(y) := \frac{h(1-\theta)}{2}((1-\theta)y_m + (1$$

$$+\theta)y_{m+1}) + h\sum_{k=m+2}^{n-1} y_k + \frac{h}{2}y_{m+1} + \frac{h}{2}y_n = 0, x_2\in\omega_2$$

will be denoted by H. On the set H let us introduce the inner product and the norm

$$(y,v)_{\tilde{\omega}} = \sum_{\tilde{\omega}} h^2 yv, \quad \|y\|_{\tilde{\omega}} = (y,y)^{1/2}_{\tilde{\omega}}, \quad \tilde{\omega} \subseteq \bar{\omega}.$$

Let, moreover,

$$\|y\|^2_{\rho} = \sum_{\omega} h^2 \rho y^2, \ \|\nabla y\|^2 = \|y_{\bar{x}_1}\|^2_{(1)} + \|y_{\bar{x}_2}\|^2_{(2)},$$

$$\|y_{\bar{x}_1}\|^2_{(1)} = (\rho y_{\bar{x}_1}, y_{\bar{x}_1})_{\omega_1^+ \times \omega_2},$$

$$\|y_{\bar{x}_2}\|^2_{(2)} = (\rho y_{\bar{x}_2}, y_{\bar{x}_2})_{\omega_1 \times \omega_2^+}.$$

We approximate problem (2) by the difference scheme

$$\Delta_h y := y_{\bar{x}_1 x_1} + y_{\bar{x}_2 x_2}$$
$$= -\varphi(x), \quad x \in \omega, y \in H, \qquad (3),$$

where $\varphi = T_1 T_2 f$. Let

$$G_h y_i = y_i, \quad i \le m, \quad G_h y_i = \widetilde{\rho}_i y_i + \frac{1}{n-m-\theta} P y_i,$$
$$i \ge (m+1),$$

where

$$\widetilde{\rho}_i = \frac{2(n-i)+1}{2(n-m-\theta)}, \quad Py_i = -\sum_{k=i}^{n} y_k + \frac{1}{2} y_n,$$

Lemma 1. *The estimate*

$$(y, G_h y)_{\omega} \ge c\|y\|^2_{\rho}$$

is true for grid functions $y(x)$, satisfying the conditions $l_h(y) = 0, \ y(0,x_2) = 0, \ x_2 \in \omega_2$.

proof. Having noted that $y_i = Py_{i+1} - Py_i$ and $Py_i = \frac{1}{2}(Py_i + Py_{i+1}) - \frac{1}{2} y_i$, we receive

$$\sum_{i=m+1}^{n-1} h y_i P y_i = -\frac{h}{2} \sum_{i=m+1}^{n-1} y_i^2 + \frac{h}{2}$$

$$\times \sum_{i=m+1}^{n-1} (Py_{i+1} + Py_i)(Py_{i+1} - Py_i) =$$

$$-\frac{h}{2} \sum_{i=m+1}^{n-1} y_i^2 + \frac{h}{2}((Py_n)^2 - (Py_{m+1})^2).$$

Taking into account the nonlocal condition we have

$$Py_{m+1} = -\sum_{k=m+1}^{n} y_k + \frac{1}{2} y_n = \frac{1}{2}((1-\theta)^2 y_m - \theta^2 y_{m+1}),$$

since

$$((1-\theta)^2 y_m - \theta^2 y_{m+1})^2 \le (1-\theta)^4 y_m^2 + \theta^4 y_{m+1}^2$$

$$+(1-\theta)^2\theta^2 y_m^2 + (1-\theta)^2\theta^2 y_{m+1}^2 =$$

$$((1-\theta)^2+\theta^2)((1-\theta)^2 y_m^2 + \theta^2 y_{m+1}^2) \le \frac{1}{2}((1-\theta)^2 y_m^2$$
$$+\theta^2 y_{m+1}^2).$$

Therefore

$$(Py_{m+1})^2 \le \frac{1}{8}((1-\theta)^2 y_m^2 + \theta^2 y_{m+1}^2).$$

Besides, due to $Py_n = -0.5 y_n$, we have

$$\sum_{i=m+1}^{n-1} h y_i P y_i \ge -\frac{h}{2} \sum_{i=m+1}^{n-1} y_i^2 + \frac{h}{8} y_n^2 - \frac{h}{16}((1-\theta)^2 y_m^2$$
$$+\theta^2 y_{m+1}^2).$$

This estimate, taking into account the form of operator G_h, proves the lemma.

Lemma 2. *For any $y \in H$ the inequality*

$$-(\Delta_h y, G_h y)_{\omega} \ge c\|\nabla y\|^2$$

holds.

Proof. It is not difficult to verify that

$$G_h y_i - G_h y_{i-1} = h y_{\bar{x}_1,i}, i \le m,$$

$$G_h y_{m+1} - G_h y_m = \frac{2(n-m) - 1 - \theta^2}{2(n-m-\theta)} h y_{\bar{x}_1,m+1},$$

$$G_h y_i - G_h y_{i-1} = h \widetilde{\rho}_i y_{\bar{x}_1,i}, \quad i \ge m+2.$$

Therefore, using summation by parts, we get

$$-\sum_{i=1}^{n-1} h y_{\bar{x}_1 x_1,i} G_h y_i = \sum_{i=1}^{n} h y^2_{\bar{x}_1,i} (\widetilde{\rho}_i$$

$$-\frac{\theta^2}{2(n-m-\theta)} \delta_{i,m+1})$$

where $\widetilde{\rho}_i = 1$ if $i \le m$ and $\delta_{\cdot,\cdot}$ is the Kronecker delta. Therefore

$$-(y_{\bar{x}_1 x_1}, G_h y)_{\omega} \ge c \sum_{\omega_1^+ \times \omega_2} h^2 \rho (y_{\bar{x}_1})^2. \qquad (4)$$

Besides, applying Lemma 1, we have

$$-(y_{\bar{x}_2 x_2}, G_h y)_{\omega} = (y_{\bar{x}_2}, G_h y_{\bar{x}_2})_{\omega_1 \times \omega_2^+}$$

$$\ge c \sum_{\omega_1 \times \omega_2^+} h^2 \rho (y_{\bar{x}_2})^2,$$

which with estimate (4) proves the Lemma 2.

Thus, if $\varphi(x) = 0, \ x \in \omega$, then $y(x) = 0, \ x \in \bar{\omega}$ and, consequently, the solution of difference scheme (3) exists and it is unique.

To investigate the convergence and accuracy of scheme (3), we consider the error of the method $z = y - u$, where y is a solution to problem (3) and $u(x)$ is a solution to problem (2). Substituting $y = u + z$ into (3), we obtain the problem

$$\Delta_h z = \psi,\ x \in \omega,\ z = 0,\ x \in \gamma_0,$$
$$l_h(z) = \chi(x_2),\ x_2 \in \omega_2, \qquad (5)$$

where

$$\psi = \eta^{(1)}_{x_1} + \eta^{(2)}_{x_2}, \quad \chi = l(u) - l_h(u),$$

$$\eta^{(\alpha)} = (u - T_{3-\alpha}u)_{\bar{x}_\alpha}, \quad \alpha = 1, 2.$$

It is evident, that $\chi = 0$ for $u(x) = x_1$. Consequently, $l_h(x_1) = l(x_1) = (1 - \xi^2)/2$ and the substitution

$$z(x) = \tilde{z}(x) + \frac{2x_1}{1-\xi^2}\chi(x_2) \qquad (6)$$

turns problem (5) (in which the nonlocal condition is not homogeneous) into the problem with homogeneous conditions

$$\Delta_h \tilde{z} = \tilde{\psi},\ x \in \omega,\ \tilde{z} = 0,\ x \in \gamma_0,$$
$$l_h(\tilde{z}) = 0,\ x_2 \in \omega_2, \qquad (7)$$

where

$$\tilde{\psi} = \psi - \frac{2x_1}{1-\xi^2}\chi_{\bar{x}_2 x_2}.$$

Lemma 3.*If the grid function y defined on $\bar{\omega}$ satisfies the conditions $l_h(y) = 0,\ y(0, x_2) = 0,\ x_2 \in \omega_2$, then*

$$|(\eta_{x_\alpha}, G_h y)|_\omega \le c\|\eta\|_\omega \|y_{\bar{x}_\alpha}\|_{(\alpha)},\ \alpha = 1,\ 2,$$

where $\eta(x)$ is an arbitrary grid function.

The proof of this lemma for case $\alpha = 1$ is not difficult. When $\alpha = 2$, we use inequalities received by taking into account condition $l_h(y) = 0$:

$$|Py_{m+1}| \le \frac{1}{2}(|y_m| + |y_{m+1}|),$$

$$|Py_{m+2}| \le \frac{1}{2}|y_m| + |y_{m+1}|,$$

$$|Py_{i+1}| \le |Py_i| + |y_i|,\ i \ge m+2,$$

whence

$$\sum_{i=m+1}^{n-1}(Py_i)^2 \le (n-m-1)\sum_{i=m}^{n-2}(n-i)(y_i)^2.$$

Thus

$$\sum_{i=m+1}^{n-1}\frac{h}{(n-m-\theta)^2}(Py_i)^2 \le$$

$$\frac{h(n-m-1)}{(n-m-\theta)^2}\sum_{i=m}^{n-2}(n-i)(y_i)^2 \le$$

$$\frac{h}{n-m-\theta}\sum_{i=m}^{n-2}(n-i)(y_i)^2 = \frac{h}{1-\xi}\sum_{i=m}^{n-2}\rho_i(y_i)^2.$$

Applying Lemma 2 to the solution of problem (7) we come to

$$\|\nabla\tilde{z}\|^2 \le c(\tilde{\psi}, G_h\tilde{z})_\omega.$$

Using Lemma 3 gives

$$\|\nabla\tilde{z}\| \le c(\|\eta^{(1)}\|_{\omega_1^+\times\omega_2} + \|\eta^{(2)}\|_{\omega_1\times\omega_2^+} + \|\chi_{\bar{x}_2}\|_{\omega_2^+}). \qquad (8)$$

For the error of the method, according to (6), we can write

$$\|\nabla z\| \le \|\nabla\tilde{z}\| + c(\|\chi\|_{\omega_2} + \|\chi_{\bar{x}_2}\|_{\omega_2^+}),$$

which together with (8) gives

$$\|\nabla z\| \le c(\|\eta^{(1)}\|_{\omega_1^+\times\omega_2} + \|\eta^{(2)}\|_{\omega_1\times\omega_2^+} + \|\chi\|_{\omega_2} + \|\chi_{\bar{x}_2}\|_{\omega_2^+}). \qquad (9)$$

In order to estimate the convergence rate of finite-difference scheme (3), it is enough to estimate the norm of error functionals on right-hand side of (9). As a result, we arrive at the following

Theorem 4. *The finite difference scheme (3) converges, and for its convergence rate the estimate (1) holds.*

References:

[1] A.A. Samarskii, R.D. Lazarov and V.L. Makarov, *Difference schemes for differential equations with generalized solutions*, Visshaya Shkola, Moscow, 1987 (in Russian).

[2] R.D. Lazarov, V.L. Makarov, and W. Weinelt, On the convergence of difference schemes for the approximation of solutions $u \in W_2^m$ $(m > 0.5)$ of elliptic equations with mixed derivatives, *Numer. Math.*, 44, 1984, pp. 223-232.

[3] G. Berikelashvili, On a nonlocal boundary-value problem for two-dimensional elliptic equation, *Comput. Methods Appl. Math.* 3, 2003, No. 1, 35–44.

[4] G. Berikelashvili, Construction and analysis of difference schemes for some elliptic problems, and consistence estimates of rate of convergence, *Mem. Differential Equations Math. Phys.*, 38, 2006, pp. 1–131.