Finite Difference Schemes for a moving orthotropic web

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Abstract: - Recently, the examination and study of the so called moving webs has received considerable attention among applied mathematicians and engineers due to a great number of applications such as paper handling, textile manufacturing, and magnetic tape recording. The moving web in these applications, which is generally orthotropic, may experience speeds more than critical speed. We define critical speed the axial speed where the system vibration has a vanishing eigenvalue and is subject to a buckling instability. At a supercritical speed, the web could have many types of instabilities and subsequently sever out of plane vibrations. In this paper, numerical solutions of the governing partial equations are proposed and a discussion about several examples is also included.

Keywords: - Finite Difference Schemes, Stability, Partial Differential Equations, Free vibrations, Axially moving, Orthotropic web, Supercritical speed,
2 Numerical Schemes

In this paper, we start from (1) using the new variable

\[ q = \frac{\partial w}{\partial t} \]

and the replacements

\[ \bar{D}_{11} = \frac{D_{11}}{\rho h}, \quad \bar{D}_{3} = \frac{2D_{3}}{\rho h}, \quad \bar{D}_{22} = \frac{D_{22}}{\rho h} \]

\[ \bar{N}_x = \frac{N_x}{\rho h}, \quad \bar{N}_y = \frac{N_y}{\rho h}, \quad \bar{N}_v = \frac{N_v}{\rho h} \]

Then we have the System of PDEs

\[ \frac{\partial w}{\partial t} = q \quad (2) \]

\[ \frac{\partial q}{\partial t} = -\bar{D}_{11} \frac{\partial^4 w}{\partial x^4} - \bar{D}_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} - \bar{D}_{22} \frac{\partial^4 w}{\partial y^4} + \bar{N}_x \frac{\partial^2 w}{\partial x^2} + \bar{N}_y \frac{\partial^2 w}{\partial y^2} + 2 \bar{N}_v \frac{\partial^2 w}{\partial x \partial y} - (2v \frac{\partial q}{\partial x} + v^2 \frac{\partial^2 w}{\partial x^2}) \quad (3) \]

with initial conditions

\[ w(x, y, 0) = w_0(x, y) = \text{given} \]
\[ q(x, y, 0) = q_0(x, y) = \text{given} \]

Moreover, introducing the numerical differentiations

\[ \frac{\partial^2 w}{\partial x^2} \approx \frac{w_{j+1,k} - 2w_{j,k} + w_{j-1,k}}{h_i^2} \]

\[ \frac{\partial^2 w}{\partial y^2} \approx \frac{w_{j,k+1} - 2w_{j,k} + w_{j,k-1}}{h_j^2} \]

\[ \frac{\partial^4 w}{\partial x^4} \approx \frac{w_{j+2,k} - 4w_{j+1,k} + 6w_{j,k} - 4w_{j-1,k} + w_{j-2,k}}{h_i^4} \]

\[ \frac{\partial^4 w}{\partial y^4} \approx \frac{w_{j,k+2} - 4w_{j,k+1} + 6w_{j,k} - 4w_{j,k-1} + w_{j,k-2}}{h_j^4} \]

For the first-order derivatives, we use central differences

\[ \frac{\partial q}{\partial x} \approx \frac{q_{j+1,k} - q_{j-1,k}}{2h_i} \]

\[ \frac{\partial^2 w}{\partial x \partial y} \approx \frac{w_{j+1,k+1} - w_{j-1,k+1} - w_{j+1,k-1} + w_{j-1,k-1}}{2h_i^2 h_j^2} \]
Then, as 1st Numerical Scheme (Explicit) we have:

\[
\frac{w_{j,k}^{i+1} - w_{j,k}^i}{h} = q_{j,k}^i
\]

\[
\frac{q_{j,k}^{i+1} - q_{j,k}^i}{h} =
\]

\[
-D_1 w_{j+2,k}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
-D_3 w_{j+1,k}^{i+1} + w_{j+1,k-1} + w_{j+1,k-1} + w_{j-1,k-1} + w_{j-1,k-1}
\]

\[
+2\bar{D}_3 w_{j,k+1}^{i+1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1}
\]

\[
-D_{22} w_{j,k+2}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
+N_x w_{j+1,k+1}^{i+1} - 2w_{j+1,k}^{i+1} + w_{j+1,k-1}^{i+1}
\]

\[
+N_y w_{j,k+1}^{i+1} - 2w_{j,k}^{i+1} + w_{j,k-1}^{i+1}
\]

\[
+N_{xy} w_{j+1,k+1}^{i+1} - w_{j+1,k-1}^{i+1} - w_{j-1,k-1}^{i+1} + w_{j-1,k-1}^{i+1}
\]

\[
-(2u q_{j+1,k}^{i+1} - q_{j-1,k}^{i+1} + v^2 w_{j+1,k}^{i+1} - 2w_{j,k}^{i+1} + w_{j-1,k}^{i+1})
\]

Then, as 2nd Numerical Scheme (Implicit) we have:

\[
\frac{w_{j,k}^{i+1} - w_{j,k}^i}{h} = q_{j,k}^i
\]

\[
\frac{q_{j,k}^{i+1} - q_{j,k}^i}{h} =
\]

\[
-D_1 w_{j+2,k}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
+2\bar{D}_3 w_{j,k+1}^{i+1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1}
\]

\[
-D_{22} w_{j,k+2}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
+N_x w_{j+1,k+1}^{i+1} - 2w_{j+1,k}^{i+1} + w_{j+1,k-1}^{i+1}
\]

\[
+N_y w_{j,k+1}^{i+1} - 2w_{j,k}^{i+1} + w_{j,k-1}^{i+1}
\]

\[
+N_{xy} w_{j+1,k+1}^{i+1} - w_{j+1,k-1}^{i+1} - w_{j-1,k-1}^{i+1} + w_{j-1,k-1}^{i+1}
\]

\[
-(2u q_{j+1,k}^{i+1} - q_{j-1,k}^{i+1} + v^2 w_{j+1,k}^{i+1} - 2w_{j,k}^{i+1} + w_{j-1,k}^{i+1})
\]

\[
-\bar{D}_3 4w_{j+1,k}^{i+1} + w_{j+1,k-1} + w_{j+1,k-1} + w_{j-1,k-1} + w_{j-1,k-1}
\]

\[
+\bar{D}_3 w_{j,k+1}^{i+1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1}
\]

\[
+\bar{D}_{22} w_{j,k+2}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
+N_x w_{j+1,k+1}^{i+1} - 2w_{j+1,k}^{i+1} + w_{j+1,k-1}^{i+1}
\]

\[
+N_y w_{j,k+1}^{i+1} - 2w_{j,k}^{i+1} + w_{j,k-1}^{i+1}
\]

\[
+N_{xy} w_{j+1,k+1}^{i+1} - w_{j+1,k-1}^{i+1} - w_{j-1,k-1}^{i+1} + w_{j-1,k-1}^{i+1}
\]

\[
-(2u q_{j+1,k}^{i+1} - q_{j-1,k}^{i+1} + v^2 w_{j+1,k}^{i+1} - 2w_{j,k}^{i+1} + w_{j-1,k}^{i+1})
\]

and as 3rd Numerical Scheme (Implicit) the Crank-Nicholson scheme

\[
\frac{w_{j,k}^{i+1} - w_{j,k}^i}{h} = \frac{1}{2} (q_{j,k}^{i+1} + q_{j,k}^i)
\]

\[
q_{j,k}^{i+1} - q_{j,k}^i =
\]

\[
-\bar{D}_3 4w_{j+1,k}^{i+1} + w_{j+1,k-1} + w_{j+1,k-1} + w_{j-1,k-1} + w_{j-1,k-1}
\]

\[
+\bar{D}_3 w_{j,k+1}^{i+1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1}
\]

\[
+\bar{D}_{22} w_{j,k+2}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
+N_x w_{j+1,k+1}^{i+1} - 2w_{j+1,k}^{i+1} + w_{j+1,k-1}^{i+1}
\]

\[
+N_y w_{j,k+1}^{i+1} - 2w_{j,k}^{i+1} + w_{j,k-1}^{i+1}
\]

\[
+N_{xy} w_{j+1,k+1}^{i+1} - w_{j+1,k-1}^{i+1} - w_{j-1,k-1}^{i+1} + w_{j-1,k-1}^{i+1}
\]

\[
-(2u q_{j+1,k}^{i+1} - q_{j-1,k}^{i+1} + v^2 w_{j+1,k}^{i+1} - 2w_{j,k}^{i+1} + w_{j-1,k}^{i+1})
\]

Then, as 2nd Numerical Scheme (Implicit) we have:

\[
\frac{w_{j,k}^{i+1} - w_{j,k}^i}{h} = q_{j,k}^i
\]

\[
\frac{q_{j,k}^{i+1} - q_{j,k}^i}{h} =
\]

\[
-D_1 w_{j+2,k}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
+2\bar{D}_3 w_{j,k+1}^{i+1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1} + w_{j,k-1}
\]

\[
-D_{22} w_{j,k+2}^{i+1} - 4w_{j+1,k}^{i+1} + 6w_{j,k}^{i+1} - 4w_{j-1,k}^{i+1} + w_{j-2,k}^{i+1}
\]

\[
+N_x w_{j+1,k+1}^{i+1} - 2w_{j+1,k}^{i+1} + w_{j+1,k-1}^{i+1}
\]

\[
+N_y w_{j,k+1}^{i+1} - 2w_{j,k}^{i+1} + w_{j,k-1}^{i+1}
\]

\[
+N_{xy} w_{j+1,k+1}^{i+1} - w_{j+1,k-1}^{i+1} - w_{j-1,k-1}^{i+1} + w_{j-1,k-1}^{i+1}
\]

\[
-(2u q_{j+1,k}^{i+1} - q_{j-1,k}^{i+1} + v^2 w_{j+1,k}^{i+1} - 2w_{j,k}^{i+1} + w_{j-1,k}^{i+1})
\]
\[
\begin{align*}
\frac{1}{2} \Delta_t w'_{j+1,k} - 2w'_{j,k} + w'_{j-1,k} = h^2 \\
\frac{1}{2} \Delta_x w'_{j,k+1} - 2w'_{j,k} + w'_{j,k-1} = h^2 \\
\frac{1}{2} \Delta_{xy} w'_{j+1,k+1} - w'_{j+1,k-1} - w'_{j-1,k+1} + w'_{j-1,k-1} = 2h^2 \\
-(v \frac{q'_{j+1,k} - q'_{j-1,k}}{2h} + \frac{1}{2} v^2 w'_{j+1,k} - 2w'_{j,k} + w'_{j-1,k}) \\
\frac{1}{2} \Delta_{11} w'_{j+1,k} - 4w'_{j,k+1} + 6w'_{j,k} - 4w'_{j-1,k} + w'_{j-1,k-1} = h^2 h_5^2 \\
-\frac{1}{2} \Delta_{3} w'_{j,k+1} + w'_{j,k} + w'_{j+1,k} + w'_{j-1,k} = h^2 h_5^2 \\
\frac{1}{2} \Delta_{22} w'_{j,k+2} - 4w'_{j,k+1} + 6w'_{j,k} - 4w'_{j,k-1} + w'_{j,k-2} = h^2 \\
\frac{1}{2} \Delta_{xy} w'_{j+1,k+1} - 2w'_{j,k} + w'_{j,k-1} = h^2 \\
\frac{1}{2} \Delta_{x} w'_{j+1,k} - 2w'_{j,k} + w'_{j,k-1} = h^2 \\
\frac{1}{2} \Delta_{xy} w'_{j+1,k+1} - w'_{j+1,k-1} - w'_{j-1,k+1} + w'_{j-1,k-1} = 2h^2 h_5^2 \\
-(v \frac{q'_{j+1,k} - q'_{j-1,k}}{2h} + \frac{1}{2} v^2 w'_{j+1,k} - 2w'_{j,k} + w'_{j-1,k})
\end{align*}
\]

and
\[
\begin{align*}
c_2 e^{\text{th}} - c_2 &= \frac{4 \cos(k_h h_1) - 4(4 \sin^2 \frac{k_h h_1}{2} + 1)}{h^4} \\
-\tilde{D}_{11} c_1 &= \frac{4 + 4 \cos(k_h h_1) \cos(k_h h_2)}{h_1^4 h_2^2} \\
-\tilde{D}_3 c_1 &= 2 \cos(k_h h_1) + 2 \cos(k_h h_2) \\
+2 \tilde{D}_3 c_1 &= 4 \cos(k_h h_2) - 4(4 \sin^2 \frac{k_h h_2}{2} + 1) \\
-\tilde{D}_{22} c_1 &= \frac{4 \sin(k_h h_1) \sin(k_h h_2)}{2h_1^2 h_2^2} \\
+N_x c_1 &= \frac{-4 \sin^2 \frac{k_h h_1}{2}}{h_1^2} \\
+N_y c_1 &= \frac{-4 \sin^2 \frac{k_h h_2}{2}}{h_2^2} \\
+N_{xy} c_1 &= \frac{2i \sin(k_h h_1) + v^2 c_1}{2h} \frac{-4 \sin^2 \frac{k_h h_1}{2}}{h_1^2}
\end{align*}
\]

Furthermore the first equation gives:
\[
\frac{h}{e^{\text{th}} - 1} = \frac{c_1}{c_2}
\]

Introducing the notation \( H = \frac{h}{e^{\text{th}} - 1} = c_1/c_2 \)

and replacing it in the second one

\[
H^{-1} = \frac{4 \cos(k_h h_1) + 4(4 \sin^2 \frac{k_h h_1}{2} - 1)}{\tilde{D}_{11} H} \\
-\tilde{D}_3 H &= \frac{4 + 4 \cos(k_h h_1) \cos(k_h h_2)}{h_1^4 h_2^2} \\
+2 \tilde{D}_3 H &= 2 \cos(k_h h_1) + 2 \cos(k_h h_2) \\
-\tilde{D}_{22} H &= 4 \cos(k_h h_2) - 4(4 \sin^2 \frac{k_h h_2}{2} + 1)
\]

3 Von Neumann stability analysis

Introducing
\[
\begin{align*}
w &= c_1 e^{\text{th}} e^{ik_x x} e^{ik_y y} \\
q &= c_2 e^{\text{th}} e^{ik_x x} e^{ik_y y}
\end{align*}
\]

the typical Von Neumann stability analysis for the 1st Numerical Scheme (Explicit) gives

\[
\frac{c_2 e^{\text{th}} - c_2}{h} = c_2
\]
From the last equation, we find $H$ and we demand

$$|e^{ab}| = \left| \frac{h}{H} + 1 \right| < 1$$

It says that for a given $h_1, h_2$, the allowed value of $h$ must be appropriate to satisfy $|e^{ab}| = \left| \frac{h}{H} + 1 \right| < 1$

Similar Von Neumann Analysis can take place for the second and third numerical scheme

## 4 Conclusion

In this paper, we proposed three numerical finite differences' schemes for the equation of a moving orthotropic web. Also, the standard Von Neumann Stability analysis is provided for the first of them while the Von Neumann Stability analysis for the second and third is carried out in a similar way. Work is in progress by the team of the author for the numerical solution of several problems in science and engineering.

**References:**


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