FINITELY PURELY (PSEUDO)ATOMIC SET MULTIFUNCTIONS

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Abstract: In this paper, we obtain results concerning finitely purely (pseudo)atomic set multifunctions, with special consideration on their measurability and integrability.

Key–words: (finitely) purely (pseudo)atomic, (pseudo)atom, set multifunction, measurable, Gould integral, $L^p$ space.

1 Introduction

In the last years, many authors (e.g. Dobrakov [4], Drewnowski [5], Jiang and Suzuki [16], Li [17], Pap [19], Sugeno [24], Suzuki [25], Zadeh [26]) investigated the non-additive field of measure theory due to its applications in mathematical economics, statistics, theory of games etc. Fuzzy measures have applications in biology, physics, medicine, theory of probabilities, human decision making, economic mathematics.

It is well-known that notions and theorems of non-additive measure theory (such as: continuity, regularity, extensions, decompositions, measures, integrals, (pseudo)atoms, non(pseudo)atomicity, purely atomicity) are very popular in fuzzy measures theory. Purely atomic measures where studied in literature in different variants (e.g. [1,2], [4], [16], [18]). For instance, Chișescu [1,2] established interesting connections with different classical problems concerning $L^p$ spaces.

In [3], [8-10] and [18] we introduced and studied notions as (pseudo)atom, (non)(pseudo)atomicity, purely atomicity in the set valued case. In this paper, we continue our study, obtaining results concerning measurability and Gould integrability for finitely purely (pseudo)atomic set multifunctions. The Gould integral [14] was extended to the set-valued case (see [21,22,23], [6,7], [11,12]) and to the non-additive case (see [13]).

2 Basic notions and terminology

$X$ will be a real normed space, $P_0(X)$ the family of all nonvoid subsets of $X$, $P_f(X)$ the family of all nonvoid, closed subsets of $X$, $P_{bf}(X)$ the family of all nonvoid, closed, bounded subsets of $X$, $P_{bfic}(X)$ the family of all nonvoid, closed, bounded, convex subsets of $X$ and $h$ the Hausdorff pseudometric on $P_0(X)$, which becomes a metric on $P_{bf}(X)$.

It is known that $h(M,N) = \max\{e(M,N), e(N,M)\}$, where $e(M,N) = \sup_{x \in M} d(x, N)$, for every $M, N \in P_0(X)$ is the excess of $M$ over $N$ and $d(x, N)$ is the distance from $x$ to $N$ with respect to the distance induced by the norm of $X$.

We denote $|M| = h(M, \{0\})$, for every $M \in P_0(X)$, where 0 is the origin of $X$.

On $P_0(X)$ we consider the Minkowski addition

\[ (M_1, M_2) \mapsto M_1 + M_2 = \{x + y : x \in M_1, y \in M_2\} \]
µ is said to be of finite variation on C if \( \overline{\nu}(A) < +\infty \), for every \( A \in C \).

**Remark 2.4.** I) \( \overline{\nu} \) is monotone and super-additive on \( \mathcal{P}(T) \). Also (see [6,7]), if \( \mu : C \to \mathcal{P}_f(X) \) is a multi(sub)measure, then \( \overline{\nu} \) is finitely additive on \( C \) and \( |\mu| \) is a submeasure in Drewnowski’s sense [5] on \( C \).

II) ([22]) If \( \mu : C \to \mathcal{P}_f(X) \) is of finite variation, then \( \mu \) is \( \mathcal{P}_b(\mathbb{R}) \)-valued.

**Definition 2.5.** We say that \( \mu : C \to \mathcal{P}_f(\mathbb{R}) \) is induced by a set function \( m : C \to \mathbb{R}_+ \), with \( m(\emptyset) = 0 \), if \( \mu(A) = [0, m(A)] \), for every \( A \in C \).

**Definition 2.6.** Let \( \mu, \nu : C \to \mathcal{P}_0(X) \). We say that \( \mu \) is absolutely \( \nu \)-continuous (denoted by \( \mu << \nu \)) if \( \nu(A) = \{0\} \implies \mu(A) = \{0\}, A \in C \).

**Remark 2.7.** Let \( \mu, \nu : C \to \mathcal{P}_0(X) \) be null-monotone. The following statements are equivalent:

i) \( \mu << \nu \);

ii) \( |\mu| << |\nu| \);

iii) \( \overline{\mu} << \overline{\nu} \).

## 3. Finitely purely (pseudo)atomic set multifunctions

Suppose \( \mu : C \to \mathcal{P}_0(X) \) is a set multifunction, with \( \mu(\emptyset) = \{0\} \).

**Definition 3.1.** [3, 8-10] I) A set \( A \in C \) is said to be an atom (pseudo-atom, respectively) of \( \mu \) if \( \mu(A) \supseteq \{0\} \) and for every \( B \in C \), with \( B \subseteq A \), we have \( \mu(B) = \{0\} \) or \( \mu(A \setminus B) = \{0\} \) (i.e., \( A_n \supset A_{n+1} \), for every \( n \in \mathbb{N}^* \) and \( \bigcap_{n=1}^{\infty} A_n = \emptyset \)).

II) \( \mu \) is

i) finitely purely (pseudo)atomic if there is a finite disjoint family \( (A_i)_{i=1,n} \in C \) of (pseudo)atoms of \( \mu \) so that \( T = \bigcup_{i=1}^{n} A_i \).

ii) purely (pseudo)atomic if there is at most a countable number of (pseudo)atoms \( (A_i)_{i=1,n} \in C \) of \( \mu \) so that \( \mu(T \setminus \bigcup_{n=1}^{\infty} A_n) = \{0\} \) (here \( C \) is a \( \sigma \)-algebra).

iii) non(pseudo)atomic if it has no (pseudo)atoms.

**Remark 3.2.** I) If \( \mu : C \to \mathcal{P}_0(X) \) is monotone and \( A \in C \) is an atom of \( \mu \), then \( \sigma(\mu(A)) = \{|\mu(A)\}| \).

II) If \( \mu \) is monotone, then \( \mu \) is non(pseudo)atomic if and only if for every \( A \in C \), with \( \mu(A) \supseteq \{0\} \),
there exists \( B \subseteq C \) such that \( B \subseteq A, \mu(B) \not\subseteq \{0\} \) and \( \mu(A \setminus B) \not\supseteq \{0\} \) (\( \mu(A) \not\supseteq \mu(B) \)), respectively.

**Remark 3.3.** I) Let \( \mu : C \to \mathcal{P}_0(X) \) be monotone. The following statements are equivalent:

i) \( \mu \) is (finitely) purely atomic;

ii) \(|\mu|\) is (finitely) purely atomic;

iii) \( \overline{\mu} \) is (finitely) purely atomic.

II) [3] If \( \mu \) is null-additive, then any atom of \( \mu \) is, particularly, a pseudo-atom (consequently, any null-additive, (finitely) purely atomic set multifunction is (finitely) purely pseudo-atomic). The converse is not valid, as shown below:

Let \( T = \{a, b, c\}, C = \mathcal{P}(T) \) and \( \mu : C \to \mathcal{P}_f(\mathbb{R}) \)

defined for every \( A \subseteq C \) by \( \mu(A) = \begin{cases} [0, 1], A \neq \emptyset \\ \{0\}, A = \emptyset. \end{cases} \)

Then \( \mu \) is null-additive and \( A = \{a, b\} \) is a pseudo-atom, but it is not an atom of \( \mu \).

III) If \( \mu : C \to \mathcal{P}_0(X) \) is a multimeasure, then \( A \subseteq C \) is an atom of \( \mu \) if and only if it is a pseudo-atom. Conversely, in this case, \( \mu \) is (finitely) purely atomic if and only if it is (finitely) purely pseudo-atomic.

IV) If \( \mu : C \to \mathcal{P}_0(X) \) is finitely purely (pseudo)atomic, then it is also purely (pseudo)atomic.

V) If \( \mu, \nu : C \to \mathcal{P}_0(X), \mu \ll \nu \) and \( A \subseteq C \), with \( \mu(A) \not\supseteq \{0\} \) is an atom of \( \nu \), then \( A \) is an atom of \( \mu \), too.

**Proposition 3.4.** Let \( m : C \to \mathbb{R}_+ \) be a finitely additive set function and \( \mu : C \to \mathcal{P}_0(L^\infty(m)) \) defined by \( \mu(A) = [0, \mathfrak{N}_A] \), for every \( A \subseteq C \), where \( \mathfrak{N}_A \) is the characteristic function of \( A \). Then \( \mu \) is countably additive if and only if \( m \) is finitely purely atomic.

**Proof.** We observe that \( \nu : C \to L^\infty(m) \), defined for every \( A \subseteq C \) by \( \nu(A) = \mathfrak{N}_A \), is finitely additive. Then, by [1,2], \( \nu \) is countably additive if and only if \( m \) is finitely purely atomic. We also remark that \( \mu \) is countably additive if and only if \( \nu \) is countably additive, so the conclusion follows.

4 Measurability and integrability considerations for finitely purely atomic special set multifunctions

In what follows, without any special assumptions, suppose \( \mathcal{A} \) is an algebra of subsets of an abstract space \( T, X \) is a Banach space, \( \mu : A \to \mathcal{P}_f(X) \) is a set multifunction of finite variation, with \( \mu(\emptyset) = \{0\} \) and \( f : T \to \mathbb{R} \) is a bounded function. We recall from [22, 23] the following notions and results.

**Definition 4.1.** I) A partition of \( T \) is a finite family \( P = \{A_i\}_{i=1}^m \subset \mathcal{A} \) such that \( A_i \cap A_j = \emptyset, i \neq j \) and \( \bigcup_{i=1}^m A_i = T \).

II) Let \( P = \{A_i\}_{i=1}^m \) and \( P' = \{B_j\}_{j=1}^m \) be two partitions of \( T \). \( P' \) is said to be finer than \( P \), denoted \( P \leq P' \) (or \( P' \geq P \)), if for every \( j = 1, m \), there exists \( i_j = 1, n \) so that \( B_j \subseteq A_{i_j} \).

III) The common refinement of two partitions \( P = \{A_i\}_{i=1}^m \) and \( P' = \{B_j\}_{j=1}^m \) is the partition \( P \wedge P' = \{A_i \cap B_j\}_{i=1}^m, j=1, m \).

Obviously, \( P \wedge P' \geq P \) and \( P \wedge P' \geq P' \).

We denote by \( \mathcal{P} \) the class of all partitions of \( T \) and if \( A \subseteq \mathcal{A} \) is fixed, by \( \mathcal{P}_A \), the class of all partitions of \( A \).

For a set multifunction \( \mu : A \to \mathcal{P}_0(X) \), we consider the set function \( \tilde{\mu} \) defined by:

\[
\tilde{\mu}(A) = \inf \{\overline{\mu}(B) : A \subseteq B, B \in \mathcal{A} \}, \text{ for every } A \subseteq T.
\]

**Remark 4.2.** Since \( \overline{\mu} \) is monotone, then \( \tilde{\mu}(A) = \overline{\mu}(A) \), for every \( A \in \mathcal{A} \). Consequently, \( \tilde{\mu}(A) \geq |\mu(A)| \), for every \( A \in \mathcal{A} \) and this implies: \( \tilde{\mu} \) is o-continuous on \( \mathcal{P}(T) \Rightarrow \mu \) is o-continuous on \( \mathcal{A} \).

**Definition 4.3.** I) \( f \) is said to be \( \tilde{\mu}\)-totally-measurable on \( (T, \mathcal{A}, \mu) \) if for every \( \varepsilon > 0 \) there exists a partition \( P_\varepsilon = \{A_i\}_{i=1}^\infty \) of \( T \) such that:

\[
\begin{cases}
\mu(A_i) \leq \varepsilon \\
\sup_{t,s \in A_i} |f(t) - f(s)| = osc(f, A_i) \leq \varepsilon, \text{ for every } i = 1, n.
\end{cases}
\]

II) \( f \) is said to be \( \tilde{\mu}\)-totally-measurable on \( B \in \mathcal{A} \) if the restriction \( f|_B \) of \( f \) to \( B \) is \( \tilde{\mu}\)-totally measurable on \( (B, \mathcal{A}_B, \mu_B) \), where \( \mathcal{A}_B = \{A \cap B : A \in \mathcal{A} \} \) and \( \mu_B = \mu|_B \).

**Remark 4.4.** If \( f \) is \( \tilde{\mu}\)-totally-measurable on \( T \), then \( f \) is \( \tilde{\mu}\)-totally-measurable on every \( A \in \mathcal{A} \).

In what follows, \( \sigma_{f\mu}(P) \) (or, if there is no doubt, \( \sigma_f(P) \), \( \sigma_{\mu}(P) \) or \( \sigma(P) \)) denotes \( \sum_{i=1}^n f(t_i)\mu(A_i) \), for every \( P = \{A_i\}_{i=1}^\infty \in \mathcal{P} \) and every \( t_i \in A_i, i = 1, n \).
Definition 4.5. [22, 23] I) $f$ is said to be $\mu$-integrable on $T$ if the net $(\sigma(P))_{P \in (P, \leq)}$ is convergent in $(\mathcal{P}_{f}(X), h)$, where $\mathcal{P}$ is ordered by the relation "$\leq$" given in Definition 4.1.

If $(\sigma(P))_{P \in (P, \leq)}$ is convergent, then its limit is called the integral of $f$ on $T$ with respect to $\mu$, denoted by $\int_{T} f \, d\mu$.

II) If $B \in \mathcal{A}, f$ is said to be $\mu$-integrable on $B$ if the restriction $f|_{B}$ of $f$ to $B$ is $\mu$-integrable on $(B, A_{B}, \mu_{B})$.

Remark 4.6. [22, 23] I) $f$ is $\mu$-integrable on $T$ if and only if there exists a set $I \in \mathcal{P}_{bf}(X)$ such that for every $\varepsilon > 0$, there exists a partition $P_n$ of $T$, so that for every other partition of $T$, $P = \{A_i\}_{i=1}^{n}$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i$, $i = 1, \ldots, n$, we have $h(\sigma(P), I) < \varepsilon$.

II) If $\mu$ is $\mathcal{P}_{kc}(X)$-valued, then $\int_{T} f \, d\mu \in \mathcal{P}_{kc}(X)$.

III) According to [13], if $m : \mathcal{A} \rightarrow \mathbb{R}_+$ is a sub-measure of finite variation and $f : T \rightarrow \mathbb{R}$ is bounded, then $f$ is $\tilde{m}$-totally-measurable if and only if it is $m$-integrable.

Theorem 4.7. Suppose $(T, \rho)$ is a compact metric space, $\mathcal{B}$ is the Borel $\delta$-ring generated by the compact subsets of $T$, $f : T \rightarrow \mathbb{R}$ is continuous on $T$ and $\mu : \mathcal{B} \rightarrow \mathcal{P}_{f}(X)$ is finitely purely atomic, null-additive and monotone. Then $f$ is $\mu$-totally-measurable on every atom $A_i, i = 1, \ldots, p$ (where $T = \bigcup_{i=1}^{p} A_i$).

Proof. Since $\mu$ is monotone and null-additive, by Theorem 5.2-[3], $\exists!a_1 \in A_1$ so that $\mu(A_1 \setminus \{a_1\}) = \{0\}$.

Consider an arbitrary partition $\{B_1, B_2, \ldots, B_n\}$ of $A_1$. Because $A_1$ is an atom, then, without loss of generality we may suppose that $\mu(B_1) = \mu(A_1)$ and $\mu(B_2) = \ldots = \mu(B_n) = \{0\}$.

Since $f$ is continuous in $A_1$, then for every $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ so that for every $t \in A_1$, with $\rho(t, a_1) < \delta_\varepsilon$, we have $|f(t) - f(a_1)| < \varepsilon/2$.

Let $B_0 = \{t \in A_1; \rho(t, a_1) < \delta_\varepsilon\} \subset A_1$. We observe that $B_0 \in \mathcal{B}$. Because $A_1$ is an atom, we have $\mu(B_0) = \{0\}$ or $\mu(A_1 \setminus B_0) = \{0\}$.

I) If $\mu(B_0) = \{0\}$, then since $A_1 \setminus B_0$, we get $\mu(\{a_1\}) = \{0\}$. But $\mu(A_1 \setminus \{a_1\}) = \{0\}$, so $\mu(A_1) = \{0\}$, a contradiction.

II) If $\mu(A_1 \setminus B_0) = \{0\}$, then $\mu(B_0 \setminus B_1) = \ldots = \mu(B_0 \setminus B_n) = \{0\}$. The partition $P_{A_1} = \{B_1, B_2, \ldots, B_n\}$ assures the $\mu$-totally-measurability of $f$ on $A_1$.

We make similar considerations for any $A_i, i = \frac{1}{2}, \ldots, p$.

Proposition 4.8. Let $\mu : \mathcal{A} \rightarrow \mathcal{P}_{f}(X)$ be a multi(sub)measure and $A, B \in \mathcal{A}$. Then $f$ is $\tilde{\mu}$-totally-measurable on $A \cup B$ if and only if it is $\tilde{\mu}$-totally-measurable on $A$ and $B$.

Proof. According to Remark 4.4, the if part is straightforward. For the only if part, suppose first that $A \cap B = \emptyset$. By the $\tilde{\mu}$-totally-measurability of $f$ on $A$ and $B$, there are $P_A = \{A_i\}_{i=1}^{n} \in \mathcal{P}_{A}$ and $P_B = \{B_j\}_{j=1}^{m} \in \mathcal{P}_{B}$ satisfying condition (M). Since $\mathcal{P}$ is additive on $A$, then $P_{A, B} = \{A_0 \cup B_0, A_1, \ldots, A_n, B_1, \ldots, B_m\} \in \mathcal{P}_{A \cup B}$ also satisfies condition (M), so $f$ is $\tilde{\mu}$-totally-measurable on $A \cup B$.

If $A$ and $B$ are not disjoint, since $A \cup B = (A \setminus B) \cup B$ and $\tilde{\mu}$-totally-measurability is hereditary, the statement is proved.

Remark 4.9. Under the assumptions of Proposition 4.8, let $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$. Then $f$ is $\tilde{\mu}$-totally-measurable on $\bigcup_{i=1}^{n} A_i$ if and only if the same is $f$ on every $A_i, i = 1, \ldots, n$.

By Remark 4.9 and Theorem 4.7, we immediately get:

Corollary 4.10. Suppose $T$ is a compact metric space, $f : T \rightarrow \mathbb{R}$ is continuous on $T$ and $\mu : \mathcal{B} \rightarrow \mathcal{P}_{f}(X)$ is a finitely purely atomic multisubmeasure. Then $f$ is $\tilde{\mu}$-totally-measurable on $T$.

Theorem 4.11. Suppose $\mu : \mathcal{A} \rightarrow \mathcal{P}_{f}(X)$ is monotone and null-additive. If $f$ is $\tilde{\mu}$-totally-measurable on $T$ and $A \in \mathcal{A}$ is an atom of $\mu$, then $f$ is $\mu$-integrable on $A$.

Proof. First, we observe that, if $A$ is an atom of $\mu$ and if $\{A_i\}_{i=1}^{n} \in \mathcal{P}_{A}$, then, there exists only one set, for instance, without any loss of generality, $A_1$, so that $\mu(A_1) \subseteq \{0\}$ and $\mu(A_2) = \ldots = \mu(A_n) = \{0\}$.

Let $A \in \mathcal{A}$ be an atom of $\mu$. Since $f$ is $\tilde{\mu}$-totally-measurable on $A$, then for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=1}^{n}$ of $A$ such that:

(i) $\tilde{\mu}(A_0) < \frac{\varepsilon}{2M}$ (where $M = \sup_{t \in T}|f(t)|$)

(ii) $\sup_{t,s \in A_i} |f(t) - f(s)| < \frac{\varepsilon}{\mu(T)}$, \forall $i = 1, \ldots, n$.

Let $\{B_j\}_{j=1}^{k}$, $\{C_p\}_{p=1}^{s}$ be two arbitrary partitions which are finer than $P_\varepsilon$ and consider $s_j \in B_j, j = 1, \ldots, k, \theta_p \in C_p, p = 1, \ldots, s$.

We prove that

$h(\sum_{j=1}^{k} f(s_j)\mu(B_j), \sum_{p=1}^{s} f(\theta_p)\mu(C_p)) < \varepsilon$. 

We have two cases:

I. \( \mu(A_0) \geq \{0\} \). Then \( \mu(A_1) = ... = \mu(A_n) = \{0\} \).

Suppose, without any loss of generality that \( \mu(B_1) \geq \{0\}, \mu(C_1) \geq \{0\} \) and \( \mu(B_2) = ... = \mu(B_k) = \{0\}, \mu(C_2) = ... = \mu(C_s) = \{0\} \). Then \( B_1 \subseteq A_0 \text{ and } C_1 \subseteq A_0 \). Consequently,

\[
\begin{align*}
    h\left( \sum_{j=1}^{k} f(s_j) \mu(B_j), \sum_{p=1}^{s} f(\theta_p) \mu(C_p) \right) &= \mu(B_1) \mu(C_1).
\end{align*}
\]

Because, generally, \( h(\alpha M, \beta M) \leq |\alpha - \beta||M| \), for every \( \alpha, \beta \in \mathbb{R} \) and every \( M \in \mathcal{P}(X) \), we have

\[
\begin{align*}
    h\left( \sum_{j=1}^{k} f(s_j) \mu(B_j), \sum_{p=1}^{s} f(\theta_p) \mu(C_p) \right) &\leq |\mu(B_1)||f(s_1) - f(\theta_1)| \leq p(T) \frac{\varepsilon}{M(T)} = \varepsilon.
\end{align*}
\]

Therefore, \( (\sigma(P))_{P \in \mathcal{P}A} \) is a Cauchy net in the complete metric space \( (\mathcal{P}(X), h) \), hence \( f \) is \( \mu \)-integrable on \( A \).

By [21, 22, 23] and Theorem 4.11, we obtain:

**Corollary 4.12.** Suppose \( \mu : A \rightarrow \mathcal{P}(X) \) is monotone, null-additive and finitely purely atomic. If \( f : T \rightarrow \mathbb{R} \) is \( \bar{\mu} \)-totally-measurable on \( T \), then \( f \) is \( \mu \)-integrable on \( T \).

By Corollaries 4.10 and 4.12, we immediately have:

**Corollary 4.13.** If \( T \) is a compact metric space, \( \mathcal{B} \) is the Borel \( \delta \)-ring generated by the compact subsets of \( T \), \( f : T \rightarrow \mathbb{R} \) is continuous on \( T \) and \( \mu : \mathcal{B} \rightarrow \mathcal{P}(X) \) is a finitely purely atomic multisubmeasure, then \( f \) is \( \mu \)-integrable on \( T \).

In what follows, we introduce \( \mathcal{L}^p \) spaces and point out that under suitable assumptions, \( \mathcal{L}^p \) is a Banach space. In the sequel, \( A \) will be a \( \sigma \)-algebra of subsets of \( T \) and \( m : A \rightarrow \mathbb{R}_+ \) a submeasure of finite variation.

**Lemma 4.14.** [12] Let \( m : A \rightarrow \mathbb{R}_+ \) be o-continuous and finitely purely atomic. Suppose for every \( n \in \mathbb{N} \), \( f_n : T \rightarrow \mathbb{R} \) is bounded, \( \bar{m} \)-totally-measurable on \( T \). If there is \( K > 0 \) so that \( \int_T f_n dm \leq K \), for every \( n \in \mathbb{N} \), then there exists \( C \in A \) so that \( m(C) = 0 \) and \( (f_n)_n \) is uniformly bounded on \( C \) (here, \( \bar{C} = T \setminus C \)).

**Proposition 4.15.** (Minkowski inequality) [11] Let \( f, g : T \rightarrow \mathbb{R} \) be \( m \)-integrable functions on \( T \). Then for every \( p \in (1, \infty) \), \( |f|^p, |g|^p, |f + g|^p \) are \( m \)-integrable on \( T \) and

\[
\begin{align*}
    \int_T |f + g|^p dm &\leq (\int_T |f|^p dm)^{\frac{1}{p}} + (\int_T |g|^p dm)^{\frac{1}{p}}.
\end{align*}
\]

Now, we consider \( \mathcal{L}^p \) as a linear space and \( || \cdot || : \mathcal{L}^p \rightarrow \mathbb{R}_+ \), defined for every \( f \in \mathcal{L}^p \) by

\[
    ||f|| = (\int_T |f|^p dm)^{\frac{1}{p}}, \text{ is a semi-norm}.
\]

**Proposition 4.16.** [11] \( \mathcal{L}^p \) is a linear space and \( \| \cdot \| : \mathcal{L}^p \rightarrow \mathbb{R}_+ \), defined for every \( f \in \mathcal{L}^p \) by \( \|f\| = (\int_T |f|^p dm)^{\frac{1}{p}} \), is a semi-norm.

**Proposition 4.17.** [11] Suppose \( m : A \rightarrow \mathbb{R}_+ \) is o-continuous. If for every \( n \in \mathbb{N}^* \), \( f_n : T \rightarrow \mathbb{R} \) is \( \bar{m} \)-totally-measurable on \( T \) and \((f_n)_n\) is uniformly bounded and pointwise converges to \( f : T \rightarrow \mathbb{R} \), then \( f \) is \( \bar{m} \)-totally-measurable on \( T \).

**Proposition 4.18.** (Fatou lemma) [11] Suppose \( \bar{m} \) is o-continuous on \( \mathcal{P}(T) \). Let \((f_n)_n\) be a sequence of uniformly bounded, \( \bar{m} \)-totally-measurable functions \( f_n : T \rightarrow \mathbb{R} \). Then

\[
    \int_T \liminf f_n dm \leq \liminf \int_T f_n dm.
\]

Following [20] and Propositions 4.15-4.18, we get:
**Theorem 4.19.** [12] Let \( m : A \to \mathbb{R}_+ \) be finitely purely atomic, so that \( \tilde{m} \) is o-continuous on \( \mathcal{P}(T) \). Then \( L^p \) is a Banach space.

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