Estimates on weight-decay regularization by variable-basis schemes

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Abstract: The optimization problems associated with supervised learning from data by the weight-decay and related techniques are described. Suboptimal solutions expressed by connectionistic models with a given number of computational units are investigated. In the final part of the paper, improvements of some estimates obtained in [1] for the same optimization problems are given.

Key–Words: Learning from data, regularization, weight decay, suboptimal solutions, rates of approximation.

1 Introduction

In supervised learning, an unknown input-output mapping has to be learned on the basis of a sample of input-output data [2]. The problem of approximating a function on the basis of a data sample \( z = \{(x_i, y_i) \in X \times \mathbb{R}, i = 1, \ldots, m\} \) is often ill-posed [3, 4]. Regularization [5] can be used to cope with this drawback.

Among regularization techniques, weight decay (see, e.g., [6]) is a learning technique that penalizes large values of the parameters (weights) of the model to be learned. For linear regression problems, the performance of weight decay was theoretically investigated in [6], where the case of linearization of a nonlinear model was considered, too. As to nonlinear models, a theoretical motivation of the generalization performance of certain neural networks trained through weight decay was given in [7], where the case of binary classification problems was studied using tools from Statistical Learning Theory.

In this paper, we study the optimization problems associated with the weight-decay and other learning techniques. Each problem is formulated as the minimization of a regularized empirical error functional over a suitable hypothesis space. Then, we compare the solution provided to the learning problem by weight-decay regularization with the solution given by the classical Tikhonov’s regularization and a mixed regularization technique (i.e., weight decay combined with Tikhonov’s regularization). When one uses hypothesis spaces spanned by kernel functions implemented by computational units widely used in connectionistic models, the solution to the Tikhonov-regularized learning problem has the form of a linear combination of the \( m \)-tuple of the kernel functions, parameterized by the input data vector \( \mathbf{x} = (x_1, \ldots, x_m) \). The coefficients of the linear combination can be obtained by solving a suitable linear system of equations, and this property can be exploited to develop learning algorithms. In order to simplify the analysis and emphasize the relationships between weight decay and Tikhonov’s regularization, also for the weight-decay learning problem and the mixed weight-decay/Tikhonov one we consider admissible solutions belonging to linear combinations of kernel functions parameterized by the input data vectors. For these problems one can show [1] that the optimal solutions are obtained by solving systems of linear equations, too.

For large data sets, the use of a number of computational units equal to the number \( m \) of data may lead to very complex models and so may be computationally unfeasible. Moreover, practical applications of linear algorithms using \( m \) computational units are limited by the rate of convergence of iterative methods solving the systems of linear equations associated to the three regularization schemes, as such rates depend on the size...
of the condition number of the matrices involved therein. For some methods, the computational requirements of solving such systems grow polynomially with the size $m$ of the data sample (e.g., for the Gaussian elimination and $m$ large enough, they grow at a rate $m^3$ [8, p. 175]). For some data and kernels, keeping the condition number of these matrices small requires a large value of the regularization parameter $\gamma > 0$, which may cause poor fit to the empirical data.

Motivated by these drawbacks, in the last part of the paper we investigate the accuracy of suboptimal solutions to weight-decay learning and to the mixed weight-decay/Tikhonov learning, over hypothesis sets corresponding to models with less computational units than the size of the data sample. We derive upper bounds on the rates of approximation of the optimal solutions, for sequences of suboptimal solutions achievable by minimization over hypothesis sets formed by linear combinations of at most $n < m$ kernel functions with parameters drawn from the data set. The upper bounds improve the ones given in [1] for the same regularized learning problems and are of the form $1/n$ (instead of $1/\sqrt{n}$, as in [1]), times a term that depends on the size $m$ of the data sample, properties of the vector $y = (y_1, \ldots, y_m)$ of output data, properties of the kernel, and the regularization parameter $\gamma$.

2 Notations and definitions

The hypothesis spaces where we set the learning problems are Reproducing Kernel Hilbert Spaces (RKHS). These can be characterized in terms of kernels [9]. A positive-semidefinite (psd) kernel is a symmetric function $K : X \times X \to \mathbb{R}$ such that for all positive integers $m$, all $(w_1, \ldots, w_m) \in \mathbb{R}^m$, and all $(u_1, \ldots, u_m) \in X^m$,

$$\sum_{i,j=1}^{m} w_i w_j K(u_i, u_j) \geq 0. \quad (1)$$

In other words, for every $m$ and every $x = (x_1, \ldots, x_m) \in X^m$, the Gram matrix of the kernel $K$ with respect to $x$, denoted by $K[x]$ and defined as $K[x]_{i,j} := K(x_i, x_j)$, is positive-semidefinite. If, for all positive integers $m$, all $(w_1, \ldots, w_m) \in \mathbb{R}^m$, and all $(u_1, \ldots, u_m) \in X^m$ with no repeated entries $u_i$, the equality in (1) holds only for $w_1 = \ldots = w_m = 0$, then $K$ is called positive-definite (pd) kernel. Every psd kernel $K : X \times X \to \mathbb{R}$ generates an RKHS $\mathcal{H}_K(X)$. Indeed, $\mathcal{H}_K(X)$ can be defined as the completion of the linear span of the set $\{K_u : u \in X\}$ with the inner product $\langle K_u, K_v \rangle \triangleq K(u,v)$. In the following, we denote by $\|\cdot\|_K$ the norm on the RKHS $\mathcal{H}_K(X)$.

In this paper we consider pd kernels $K$; to fix ideas, one can think of the widely-used Gaussian kernel $K(u, v) = e^{-\rho |u - v|^2}$ on $\mathbb{R}^d \times \mathbb{R}^d$, where $\rho > 0$. The corresponding RKHS contains all functions obtainable by Gaussian radial-basis function networks with a fixed “width” $\rho$. One reason for choosing RKHSs as hypothesis spaces is that the norms $\|\cdot\|_K$ on RKHSs defined by a large variety of kernels $K$ play the role of measures of various types of oscillations of functions in those spaces. Thus, the choice of suitable RKHSs as hypothesis spaces allows one to impose a condition on oscillations of admissible solutions to the learning problem. See, e.g., [10] for details.

For a subset $G$ of a linear space and a positive integer $n$, we denote by $\text{span}_n G \triangleq \{\sum_{i=1}^{n} w_i g_i : w_i \in \mathbb{R}, g_i \in G, i \in \mathbb{N}\}$ and $\text{span}_n G \triangleq \{\sum_{i=1}^{m} w_i g_i : w_i \in \mathbb{R}, g_i \in G\}$ the sets of all linear combinations of elements of $G$ and of all linear combinations of $n$-tuples of elements of $G$, resp. We let $G_K \triangleq \{K_x : x \in X\}$ and, for an input data sample $x, G_{K_x} \triangleq \{K_{x_1}, \ldots, K_{x_m}\}$. So, $\text{span}_n G_K$ and $\text{span}_n G_{K_x}$ are the sets of all input/output functions of a computational model with one hidden layer of $n$ computational units implementing functions from $G_K$ and $G_{K_x}$, resp. Given a linear space $\mathcal{H}$, a set $M \subseteq \mathcal{H}$ and a functional $\Phi : M \to \mathbb{R}$, following standard notation from optimization theory we denote by $(M, \Phi)$ the problem of minimizing $\Phi$ over $M$. Every $f^0 \in M$ such that $\Phi(f^0) = \min_{f \in M} \Phi(f)$ is called an (optimal) solution or a minimum point of the problem $(M, \Phi)$. We denote by $\arg\inf (M, \Phi)$ the set of solutions of $(M, \Phi)$, i.e., $\arg\inf (M, \Phi) \triangleq \{f^0 \in M : \Phi(f^0) = \min_{f \in M} \Phi(f)\}$. For $\varepsilon > 0$, $\arg\inf_{\varepsilon} (M, \Phi)$ is the set of $\varepsilon$-near minimum points of $(M, \Phi)$, i.e., $\arg\inf_{\varepsilon} (M, \Phi) \triangleq \{f^0 \in M : \Phi(f^0) \leq \inf_{f \in M} \Phi(f) + \varepsilon\}$.

Given a normed linear space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$, let $M \subseteq \mathcal{H}$ and $\Phi : M \to \mathbb{R}$ be a functional. If $\Phi$ is continuous and $f \in M$, then the function $\omega_f : [0, +\infty) \to [0, +\infty)$ defined as

$$\omega_f(t) = \sup \{\|\Phi(f) - \Phi(g)\| : g \in M, \|f - g\|_{\mathcal{H}} \leq t\}$$

is called the modulus of continuity of $\Phi$ at $f$. By definition, $\omega_f(t)$ is a nondecreasing function of $t$. For a positive integer $d$, by $\|\cdot\|_1$ and $\|\cdot\|_2$ we denote the 1-norm and Euclidean norm on $\mathbb{R}^d$. 

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responsible for the value requirements: i) fitting to the data sample (via controls the trade-off between the following two

\[ \mathcal{E}_f(f) + \gamma \Psi(f) \]

where \( \mathcal{E}_f(f) = \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - y_i)^2 \) is the empirical error functional, and \( \Psi: \mathcal{H} \rightarrow \mathbb{R} \) is a functional called regularizer. The corresponding model for the learning problem is \( (M, \mathcal{E}_f + \gamma \Psi) \), i.e., inf \( \mathcal{E}_f(f) + \gamma \Psi(f) \). The parameter \( \gamma \) controls the trade-off between the following two requirements: i) fitting to the data sample (via the value \( \mathcal{E}_f(f) \) of the empirical error in correspondence of \( f \)); ii) penalizing functions \( f \) that give a large value of the regularizer \( \Psi(f) \). For certain normed hypothesis spaces \( \mathcal{H} \), the choice \( \Psi(\cdot) = \| \cdot \|^2_{\mathcal{H}} \) allows one to enforce certain smoothness properties of the solution. In this case, the parameter \( \gamma \) quantifies the compromise between enforcing closeness to the data sample and avoiding solutions that are not sufficiently smooth.

Given a regularization parameter \( \gamma > 0 \) and a pd kernel \( K \) (e.g., the Gaussian kernel), for \( f \in \text{span} G_K \) we define the weight-decay empirical error functional as

\[ \Phi_{WD,\gamma}(f) = \mathcal{E}_f(f) + \gamma \| c_f \|^2 \]

where the components of the vector \( c_f = (c_{f,1}, \ldots, c_{f,l})^T \) and \( \hat{x}_1, \ldots, \hat{x}_l \in X \) are the parameters in the expansion

\[ f = \sum_{j=1}^{l} c_{f,j} K_{\hat{x}_j} \].

Choosing a pd kernel guarantees that \( f \) has a unique representation of the form (3), thus \( l \) and \( \| c_f \|^2 \) in (2) are defined unambiguously (otherwise one may choose, among all equivalent representations of \( f \) - possibly with different values of \( l \) - the infimum of the squared norms \( \| c_f \|^2 \) of the corresponding coefficient vectors \( c_f \)). Note that in general the functional (2) cannot be continuously extended on \( \mathcal{H}_K(X) \) if \( K \) is continuous on \( X \times X \subseteq \mathbb{R}^d \times \mathbb{R}^d \). Indeed, by varying the number of kernel units in (2), it is easy to construct a sequence \( \{ f_{2l} \} \) such that \( f_{2l} \in \text{span}_\mathcal{H} G_K \), \( \| f_{2l} \|_K \rightarrow 0 \) and \( \| c_{f_{2l}} \|^2 \rightarrow \infty \) as \( l \rightarrow \infty \). One example of such a sequence is given by

\[ f_{2l} = \sum_{j=1}^{2l} (-1)^j K_{\hat{x}_j(l)}, \]

where, for each \( l \), when \( j \) is odd \( \hat{x}_j(l) \) and \( \hat{x}_{j+1}(l) \) are chosen “sufficiently close” to each other such that \( \| f_{2l} \|_K < \frac{1}{l} \).

The number \( l \) of terms in the expression (3) is equal to the dimension of the vector \( c_f \) in (2). In the following, we consider the weight-decay functional corresponding to the choice \( l = m \) (i.e., \( l \) equal to the size of the data sample) and \( \hat{x}_j = x_j \) for \( j = 1, \ldots, m \). In other words, we investigate the minimization of the functional (2) over linear combinations \( \text{span}_m G_K \) of \( m \) kernel functions centered at the \( m \) input data. Hence, we model the weight-decay learning problem as

\[ (\text{span}_m G_K, \Phi_{WD,\gamma}) \].

The next proposition, from [1], expresses the solution to the weight-decay learning problem as a linear combination of kernel functions centered at the input data points, with coefficients obtained by solving a linear system of equations. Its proof given in [1] is based on the theory of regularization of inverse problems; for completeness of exposition, here we sketch a simpler proof.

**Proposition 1** Let \( X \) be a nonempty set, \( K : X \times X \rightarrow \mathbb{R} \) a pd kernel, \( m \) a positive integer, \( x = (x_1, \ldots, x_m) \in X^m \) with no repeated entries \( x_i \), \( y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m \), and \( \gamma > 0 \). Then there exists a unique solution

\[ f_{WD,\gamma}^o = \sum_{j=1}^{m} c_{WD,\gamma,j}^o K_{x_j} \]

to the problem \( (\text{span}_m G_K, \Phi_{WD,\gamma}) \), where

\[ c_{WD,\gamma,j}^o = \left( c_{WD,\gamma,1,j}^o, \ldots, c_{WD,\gamma,m,j}^o \right)^T \]

is the unique solution to the linear system of equations

\[ (K[x] + \gamma m K^{-1}[x]) c_{WD,\gamma}^o = y \].

**Proof.** Let \( F = \{ e_1, \ldots, e_m \} \) be the canonical orthonormal basis of \( \mathbb{R}^m \) and define the \( m \)-variable function \( \Phi_{WD,\gamma} : \mathbb{R}^m \rightarrow \mathbb{R} \) such that, for every \( f = \sum_{j=1}^{m} e_{f,j} K_{x_j} \in \text{span} G_K \), \( \Phi_{WD,\gamma}(f) \triangleq \Phi_{WD,\gamma}(c_f) \). The problems \( (\text{span} G_K, \Phi_{WD,\gamma}) \) and \( (\text{span} F, \Phi_{WD,\gamma}) \) are clearly equivalent, and straightforward computations show that

\[ \Phi_{WD,\gamma}(c_f) = c_f^T \left( \frac{1}{m} \lambda^2 K[x] + \gamma I \right) c_f - \frac{\gamma}{m} c_f^T K[x] y + \frac{\gamma}{m} y^T y \].

Then (6) follows by minimizing (7) w.r.t. \( c_f \). \( \square \)
4 Comparison with Tikhonov regularization

Tikhonov regularization in learning from data can be formalized in terms of the following Tikhonov-regularized empirical error functional:

$$\Phi_{T, \gamma} \triangleq \mathcal{E}_x(f) + \gamma \| f \|^2_K.$$  

(8)

The corresponding Tikhonov-regularized learning problem is

$$(\mathcal{H}_K(X), \Phi_{T, \gamma}).$$  

(9)

Existence, uniqueness, and an explicit formula describing the solution to the learning problem (9) are given by the so-called Representer Theorem (see, e.g., [11, p. 42]). For a nonempty set $X$, $K : X \times X \to \mathbb{R}$ a psd kernel, $m$ a positive integer, $x = (x_1, \ldots, x_m) \in \mathbb{X}^m$, $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$, and $\gamma > 0$, the Representer Theorem states that there exists a unique solution $f^o_{T, \gamma}$ to the problem $(\mathcal{H}_K(X), \Phi_{T, \gamma})$ and it has the form

$$f^o_{T, \gamma} = \sum_{j=1}^m c_{T, \gamma,j} K_{x_j},$$  

(10)

where $c_{T, \gamma} = (c_{T, \gamma,1}, \ldots, c_{T, \gamma,m})^T$ is the unique solution to the linear system of equations

$$(\mathcal{K}[x] + \gamma m I) c_{T, \gamma} = y.$$  

(11)

The optimal solution to the Tikhonov-regularized learning problem described by (10) is an element of span$_m G_{K_x} \subseteq$ span$_m G_K$. Since by the Representer Theorem, the solution to the Tikhonov-regularized learning problem $(\mathcal{H}_K(X), \Phi_{T, \gamma})$ belongs to span$_m G_{K_x}$, one can restate such a problem as (span$_m G_{K_x}, \Phi_{T, \gamma})$ and compare its solution with the solution to the weight-decay learning problem (span$_m G_{K_x}, \Phi_{W D, \gamma})$. This is done in [1] in terms of spectral windows.

5 Combining weight decay and Tikhonov regularization

By considering, for $\gamma_T, \gamma_{W D} > 0$ the minimization of the following mixed regularized functional

$$\Phi_{W D T, \gamma_T, \gamma_{W D}}(f) \triangleq \mathcal{E}_x(f) + \gamma_T \| f \|^2_K + \gamma_{W D} \| f \|^2_{W D}$$  

it is possible to combine weight decay and Tikhonov regularization. For simplicity and without loss of generality, we take $\gamma_T = \gamma_{W D} = \gamma/2$ and we define the mixed regularized learning problem (span$_m G_{K_x}, \Phi_{W D T, \gamma/2})$, where

$$\Phi_{W D T, \gamma/2}(f) \triangleq \Phi_{W D T, \gamma/2}(f).$$

The next proposition, also taken from [1], investigates the problem (span$_m G_{K_x}, \Phi_{W D T, \gamma/2})$ and gives a formula for its solution. Its proof is similar to that of Proposition 1, so is omitted.

**Proposition 2** Let $X$ be a nonempty set, $K : X \times X \to \mathbb{R}$ a psd kernel, $m$ a positive integer, $x = (x_1, \ldots, x_m) \in \mathbb{X}^m$ with no repeated entries $x_i$, $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$, and $\gamma > 0$. Then there exists a unique solution

$$f^o_{W D T, \gamma/2} = \sum_{j=1}^m c^o_{W D T, \gamma,2,j} K_{x_j}$$  

(12)

to the problem (span$_m G_{K_x}, \Phi_{W D T, \gamma/2})$, where $c^o_{W D T, \gamma,2,j} = (c_{W D T, \gamma,2,1}, \ldots, c_{W D T, \gamma,2,m})^T$ is the unique solution to the linear system of equations

$$\left(\mathcal{K}[x] + \frac{\gamma}{2} m (I + K^{-1}[x]) \right) c^o_{W D T, \gamma/2} = y.$$  

(13)

Similarly to Proposition 1, Proposition 2 expresses the solution to the mixed regularized learning problem as a linear combination of kernel functions centered at the data points, with coefficients obtained by solving the linear system of equations (13). The expression $c^o_{f_{W D T, \gamma,2}} = (\mathcal{K}[x] + \frac{\gamma}{2} m (I + K^{-1}[x]))^{-1} y$ can be compared in terms of spectral windows with the expressions $c^o_{f_{W D T, \gamma,2,j}} = (\mathcal{K}[x] + \gamma m K^{-1}[x])^{-1} y$ and $c^o_{f_{T, \gamma,2}} = (\mathcal{K}[x] + \gamma m I)^{-1} y$ for the coefficients of the solutions to the weight-decay and the Tikhonov-regularized learning problems, respectively [1].

6 Accuracy of suboptimal solutions

The expressions (6), (11), and (13) for the coefficients of the linear combinations providing the solutions to the respective problems require to solve linear systems of equations, so, in principle, they can be used to design linear learning algorithms. However, their applications are limited by the rates of convergence of iterative methods solving linear systems of equations.

Recall that the condition number of a nonsingular $m \times m$ matrix $A$ with respect to a norm
\[ \| \cdot \| \text{ on } \mathbb{R}^m \text{ is defined as } \text{cond}(A) = \|A\| \|A^{-1}\|, \]
where \( \|A\| \) denotes the norm of \( A \) as a linear operator on \( (\mathbb{R}^m, \| \cdot \|) \). For a symmetric matrix \( A \), we denote by \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \) its maximum and minimum eigenvalues, respectively. It is easy to check that for every norm \( \| \cdot \| \) on \( \mathbb{R}^m \) and every \( m \times m \) symmetric nonsingular matrix \( A \), \( \text{cond}(A) \geq \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \) and \( \text{cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \), where \( \text{cond}(A) \) denotes the condition number of \( A \) with respect to the \( \| \cdot \|_2 \)-norm on \( \mathbb{R}^m \) [8, p. 35].

To simplify the notation, we write \( \lambda_{\max} \) instead of \( \lambda_{\max}(K|x|) \) and similarly for \( \lambda_{\min} \).

For pd kernels and every \( x \) with no repeated entries \( x_i \), the matrix \( K[x] \) is positive definite, so all its eigenvalues are positive. By simple algebraic manipulations and spectral theory, for the condition numbers of the matrices involved in the solutions of the linear systems of equations (6), (11), and (13), simple calculations give

\[ \text{cond}_2(K[x] + \gamma m K^{-1}[x]) \leq \text{cond}_2(K[x]), \quad (14) \]
\[ \text{cond}_2(K[x] + \gamma m I) = \text{cond}_2(K[x]), \quad (15) \]
\[ \text{cond}_2(K[x] + \gamma m I) \leq 1 + \frac{\lambda_{\max}}{\gamma m}, \quad (16) \]

\[ \text{cond}_2(K[x] + \frac{\gamma}{2} m(I + K^{-1}[x])) \leq \text{cond}_2(K[x]), \quad (17) \]

and

\[ \text{cond}_2(K[x] + \frac{\gamma}{2} m(I + K^{-1}[x])) \leq \frac{\lambda_{\max}}{\gamma m} + \frac{\gamma}{\lambda_{\min}(\lambda_{\min} + \frac{1}{2} m)} . \quad (18) \]

By equations (14), (15), and (17), when \( \text{cond}_2(K[x]) \) is sufficiently small, good conditioning of the respective matrices is guaranteed for every value of \( \gamma \). However, for large values of the size \( m \) of the data sample, the matrix \( K[x] \) might be ill-conditioned. On the other hand, the regularization parameter \( \gamma \) can always be chosen "large enough" such that \( \text{cond}_2(K[x] + \gamma m K^{-1}[x]), \text{cond}_2(K[x] + \gamma m I), \) and \( \text{cond}_2(K[x] + \frac{\gamma}{2} m(I + K^{-1}[x])) \) are close to \( \text{cond}_2(K[x]), 1, \) and \( 1 + \frac{\gamma}{\lambda_{\min}}, \) resp. Unfortunately, good conditioning of the matrices is not the only requirement for \( \gamma \), as its value must also allow a good fit to the empirical data and thus it cannot be too large.

Summing up, when a small condition number of the matrices in (6), (11), and (13) and a good fit to the empirical data cannot be simultaneously guaranteed, one has to consider other learning techniques. In particular, one may be interested in suboptimal solutions depending on a number of computational units smaller than \( m \), as they have smaller memory requirements, better interpretability, and in some cases are easier to find than the optimal ones (e.g., through the so-called greedy algorithms [12]). For instance, in contrast to the respective optimal solutions, which are linear combinations of \( K_{x_1}, \ldots, K_{x_m} \) determined by the sample \( x = (x_1, \ldots, x_m) \) of input data, one may search for suboptimal solutions formed by linear combinations of \( n < m \) such functions, or that depend on arbitrary \( n \)-tuples of elements of \( G_K \triangleq \{ K_x : x \in X \} \). In the following, we investigate the accuracy of suboptimal solutions over (span \( \gamma \)), \( \Phi_{WD, \gamma} \), and (span \( \beta \)), \( \Phi_{WD, \gamma} \) to the solutions \( f_{WD, \gamma} \), \( f_{WD, \gamma} \) provided by Propositions 1 and 2, resp.

The next theorem improves [1, Theorem 4] and estimates for increasing values of \( n < m \) the rates of approximation of \( f_{WD, \gamma} \) which can be obtained by suboptimal solutions to the problem (span \( \gamma \)), \( \Phi_{WD, \gamma} \). Note that, differently from [1, Theorem 4], we do not require the condition \( s_K = \sup_{x \in X} \sqrt{K(x, x)} < +\infty \).

**Theorem 3** Let \( X \) be a nonempty set, \( K : X \times X \to \mathbb{R} \) a pd kernel, \( z \) a data sample of size \( m \) with no repeated entries \( x_i \), \( \lambda_{\min} \) and \( \lambda_{\max} \) the minimum and the maximum eigenvalues of \( K[x] \), resp., and \( \gamma > 0 \). Let \( \alpha(t) \triangleq \left( \frac{\lambda_{\max}}{m} + \gamma \right) t^2, \) and

\[ \Delta_{WD, \gamma} \triangleq \| \zeta_{WD, \gamma} \|_1^2 - \| \zeta_{WD, \gamma} \|_2^2. \]

Then the following holds.

(i) For every positive integer \( n < m \)

\[ \inf_{f \in \text{span}_n G_K} \Phi_{WD, \gamma}(f) - \Phi_{WD, \gamma}(f_{WD, \gamma}) \leq \alpha \left( \sqrt{\frac{\Delta_{WD, \gamma}}{n}} \right). \]

(ii) Let \( \varepsilon_n > 0 \) and \( f_n \in \text{argmin}_{\varepsilon_n} \text{span}_n G_K, \Phi_{WD, \gamma} \). Then

\[ \| f_n - f_{WD, \gamma} \|_K^2 \leq \frac{\lambda_{\max}}{\lambda_{\min} + \gamma} \left[ \alpha \left( \sqrt{\frac{\Delta_{WD, \gamma}}{n}} \right) + \varepsilon_n \right]. \]

**Proof.** (i) Since \( \zeta_{WD, \gamma} \) is optimal for the problem (span \( F, \Phi_{WD, \gamma} \)), by the first-order optimality condition for unconstrained optimization and the specific form of \( \Phi_{WD, \gamma} \) it follows \( \Phi_{WD, \gamma}(\zeta_f) = \Phi_{WD, \gamma}((\zeta_{WD, \gamma} + 1/4 H(\zeta_{WD, \gamma}))(\zeta_f - \zeta_{WD, \gamma})), \) where the Hessian \( H(\zeta_{WD, \gamma}) \) has the expression \( H(\zeta_{WD, \gamma}) = 2 \left( \frac{\lambda_{\min}}{m} + \gamma I \right) \) and is positive definite. So the function \( \alpha(t) \triangleq \left( \frac{\lambda_{\max}}{m} + \gamma \right) t^2 \) is
the modulus of continuity of \( \Phi_{W,D,\gamma} \) on \( (\mathbb{R}^m, \|\cdot\|_2) \). Let \( F = \{e_1, \ldots, e_n\} \) be the canonical orthonormal basis of \( \mathbb{R}^m \). Then by the definitions of \( \Phi_{W,D,\gamma} \) and of modulus of continuity, we have

\[
\inf_{f \in \text{span}_n G_{Kx}} \Phi_{W,D,\gamma}(f) - \Phi_{W,D,\gamma}(f^o_{W,D,\gamma}) = \inf_{c_f \in \text{span}_n F} \Phi_{W,D,\gamma}(c_f) - \Phi_{W,D,\gamma}(c^o_{W,D,\gamma}) \leq \alpha \left( \inf_{c_f \in \text{span}_n F} \|c^o_{W,D,\gamma} - c_f\|_2 \right).
\]

The estimate \( \inf_{c_f \in \text{span}_n F} \|c^o_{W,D,\gamma} - c_f\|_2 \leq \sqrt{\frac{\Delta W_D,\gamma}{n}} \) follows by a particular case of Maurey-Jones-Barron’s Lemma (see [1, Theorem 2]).

(ii) By the proof of (i) and the definition of \( f_n \), we get

\[
\left( \frac{\lambda_{\text{min}}^2}{m} + \gamma \right) \|c_f - c^o_{W,D,\gamma}\|^2_2 \leq \alpha \left( \sqrt{\frac{\Delta W_D,\gamma}{n}} \right) + \varepsilon_n.
\]

The estimate is obtained by applying (19) and \( \|f_n - f^o_{W,D,\gamma}\|^2_K \leq \lambda_{\text{max}} |c_f - c^o_{W,D,\gamma}|^2_2 \) (which follows easily by the definition of the norm in an RKHS).

The next theorem improves [1, Theorem 4] and estimates for increasing values of \( n < m \) the rates of approximation of \( f^o_{W,D,\gamma/2} \), which can be obtained by suboptimal solutions to the problem \( (\text{span}_n G_{Kx}, \Phi_{W,D,\gamma/2}) \). The proof is similar to that of Theorem 3, so it is omitted.

**Theorem 4** Let \( X \) be a nonempty set, \( K : X \times X \to \mathbb{R} \) a pd kernel, \( z \) a data sample of size \( m \) with no repeated entries \( x_i \), \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) the minimum and the maximum eigenvalues of \( K[x] \), resp., and \( \gamma > 0 \). Let \( \beta(t) \triangleq \left( \frac{\lambda_{\text{min}}^2}{m} + \frac{\gamma}{2} (\lambda_{\text{max}} + 1) \right) t^2 \), and \( \Delta W_{D,\gamma/2} \triangleq \|c^o_{W,D,\gamma/2}\|_2^2 - \|c^o_{W,D,\gamma/2}\|_2^2 \). Then the following hold.

(i) For every positive integer \( n < m \)

\[
\inf_{f \in \text{span}_n G_{Kx}} \Phi_{W,D,\gamma/2}(f) - \Phi_{W,D,\gamma/2}(f^o_{W,D,\gamma/2}) \leq \beta \left( \sqrt{\frac{\Delta W_D,\gamma/2}{n}} \right).
\]

(ii) Let \( \varepsilon_n > 0 \) and \( f_n \in \arg\min_{c_n} \{\text{span}_n G_{Kx}, \Phi_{W,D,\gamma/2}\} \). Then

\[
\|f_n - f^o_{W,D,\gamma/2}\|_K \leq \frac{\lambda_{\text{max}}}{\lambda_{\text{min}} m} + \frac{\gamma (\lambda_{\text{min}} + 1)}{2} \left[ \beta \left( \sqrt{\frac{\Delta W_D,\gamma/2}{n}} \right) + \varepsilon_n \right].
\]

**References:**


