Numerical Modeling of Extended Mild Slope Equation with Mac Cormack Method

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Abstract The transformation of waves is one of the important subjects in coastal engineering field. Refraction, diffraction, shoaling, reflection can be analysed with the mild slope equation over mild sloped topographies. But the extended mild slope equation can be applied to the rapidly varying topographies since it includes higher order bottom effects such as square of bottom slope and bottom curvature. In this study, extended mild slope equation has been solved with finite difference method using Mac Cormack and Point Gauss Seidel Methods together. The nonlinear wave celerity and group velocity have been used. The numerical model has been tested on elliptic shoaling area and compared with the physical experiment measurements given in literature. Numerical model has been applied to a coastal area in the Kocaeli Bay in the Marmara Sea in Turkey.

Key-words: Extended mild slope equation, Mac Cormack Method, Point Gauss Seidel Method, wave refraction, diffraction

1 Introduction
Since seventies mild slope equation has been used to simulate wave transformations. Berkhoff proposed the equation called mild slope equation in literature [1].

\[ \frac{\partial}{\partial x}(CCx \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(CCy \frac{\partial \phi}{\partial y}) + \sigma^2 \frac{C_x}{C} \phi = 0 \]  

(1)

\( \phi \) is velocity potential function, \( C \) is wave celerity, \( C_g \) is group velocity, \( \sigma \) is wave frequency. \( x \) and \( y \) are horizontal coordinates. The major characteristic of mild slope equation is analysing wave refraction and diffraction together. The mild slope equation is valid when \( \nabla h/ kh << 1 \). \( h \) is the water depth, \( k \) is the wave number and \( V \) is horizontal operator. This assumption means that the bottom slope or the change of water depth per wave length is very small. The mild slope equation includes wave refraction, diffraction, shoaling and reflection.

In recent years, the researchers have worked on the decrease of the limitations of the mild slope equation and adding the other phenomena like wave breaking, harbor resonance and bottom friction effects in the mild slope equation. In this study, the extended mild slope equation suggested by Maa et al. [2] has been solved numerically. The nonlinear wave celerity and group velocity have been used in the calculations. The numerical model has been tested on the elliptical shoaling area given in literature and applied to a coastal region in Kocaeli Bay of Marmara Sea in Turkish coasts.

2 Theory
The extended mild slope equation proposed by Maa et al. has been given in the equation (2) [2]. It includes harbor resonance and dissipations due to wave breaking & bottom friction beside wave refraction, diffraction,
shoaling and reflection. Furthermore it is applicable to the rapidly varying topographies since it includes higher order bottom effects such as bottom curvature and square of the bottom slope.

\[ \nabla \left( C \nabla \phi \right) + k^2 C \left( 1 + if_{bd} \right) \phi + \left[ f_1 g \nabla^2 h + f_2 \left( \nabla h \right)^2 gk \right] = 0 \]  

(2)

\( \nabla h \) is bottom slope, \( \nabla^2 h \) is bottom curvature, \( f_{bd} \) is the sum of bottom friction dissipation factor and energy dissipation factor after breaking. \( g \) is the gravitational acceleration. \( f_1 \) is the bottom curvature coefficient and \( f_2 \) is the coefficient of square of bottom slope. They are functions of wave number and water depth.

\[ f_1 = \frac{4kh \cos(kh) + \sinh(3kh) + \sinh(kh) + 8(kh)^2 \sinh(kh)}{8 \sinh[2kh + \sinh(2kh)]} \]  

(3)

\[ f_2 = \frac{\sec^2(kh)}{\bar{d}[2kh + \sinh(2kh)]} \left[ 8(kh)^2 + 16(kh)^2 \sinh(2kh) - 9 \sinh^2(2kh) \cos(2kh) + 12(kh)^2 + 2 \sinh^2(kh) \sinh(2kh) \right] \]  

(4)

Bottom friction dissipation factor \( (f_w) \) has been calculated with the equation (5). Here, \( f_w \) is wave friction factor [2].

\[ f_w = \frac{4f_w}{3 \pi ne^{\frac{\pi}{3} \sinh^3 kh}} \]  

(5)

\[ n = \frac{1}{2} \left[ 1 + \frac{2kh}{\sinh 2kh} \right] \]  

(6)

Jonsson and Carlsen [3] recommended the equation (7) to calculate wave friction factor.

\[ \frac{1}{4f_w} + \log_{10} \frac{1}{4\sqrt{f_w}} = m_f + \log_{10} \frac{a_1m}{k} \]  

(7)

\( a_1m \) is semi distance of the movement of the fluid particle on bottom and \( k \) is the Nikuradse roughness coefficient. After empirical studies, \( m_f \) had been calculated as -0.08 by Jonsson and Carlsen [3]. If \( a_1m/k \) is less than 2, wave friction factor \( f_w \) is 0.24. Otherwise the value calculated in the equation (7) is used in the numerical solution [4].

Wave breaking dissipation factor \( (f_d) \) has been calculated with the equation (8) [2]. \( \Gamma \) and \( K \) are empirical constants and \( \Gamma = 0.4, \ K = 0.15 \) [4].

\[ f_d = \frac{\Gamma}{kh} \left( 1 - \frac{K^2}{4 \gamma^2} \right) \]  

(8)

Breaker index \( (\gamma_b) \) is calculated with the formulation [5] given in the equation (9).

\[ \gamma_b = 0.53 - 0.3 \exp \left( -3 \frac{h}{L_0} \right) + 5 \tan \beta^{1/2} \exp \left[ -45 \left( \frac{h}{L_0} - 0.1 \right)^2 \right] \]  

(9)

\( \gamma \) is the ratio between wave amplitude and water depth \( (\gamma = a/h) \). \( \gamma \) and \( \gamma_b \) are calculated in each step and compared. If \( \gamma \) is less than \( \gamma_b \), \( f_d \) is equalized to zero. Otherwise \( f_d \) is calculated with the equation (9) [4].

### 2.1 Nonlinear Wave Celerity and Group Velocity

Nonlinear effects are especially important in shallow regions where refraction is dominant. Nonlinearity should be taken into account to obtain more accurate results in wave propagation problems. Kirby and Dalrymple recommended a dispersion relationship given in the equation (10) to determine nonlinear wave celerity and group velocity [6,7]. It is valid either in deep sea or in shallow regions.

\[ \sigma^2 = gk \left[ 1 + f_1 \left( kh \right)e \right] \tan \left( kh + f_2 \left( kh \right) \right) \]  

(10)

\[ e = \frac{ka}{k} \]  

(11)
\[ D = \frac{\cosh(4kh) + 8 - 2 \tanh^2(kh)}{8 \sinh^4(kh)} = \frac{9 - 12 \tanh^2(kh) + 13 \tanh^4(kh) - 2 \tanh^6(kh)}{8 \tanh^6(kh)} \]  
(12)

\[ f_1'(kh) = \tanh^5(kh) \]  
(13)

\[ f_2'(kh) = \left( \frac{kh}{\sinh(kh)} \right)^4 \]  
(14)

After the calculation of \( \sigma \), the nonlinear wave celerity and group velocity can be obtained simply.

\[ C = \frac{\sigma}{k} \quad \text{and} \quad C_g = \frac{d\sigma}{dk} \]  
(15)

### 2.2 Boundary Conditions

General boundary condition used in coastal engineering problems referring radiation, partial and full reflection conditions is given in the equation (16) [8].

\[ \frac{\partial \phi}{\partial n} + \alpha^* k \phi = 0 \]  
(16)

\( \alpha^* (= \alpha_1 + i \alpha_2) \) is complex transmission coefficient and depend on energy transfer on boundary, wave height, wave phase and reflection coefficient. \( \alpha_1 \) and \( \alpha_2 \) are calculated with the equations (17) and (18), respectively.

\[ \alpha_1 = \frac{2K_R \sin \beta \cos \theta}{1 + K_R^2 + 2K_R \cos \beta} \]  
(17)

\[ \alpha_2 = \frac{[1 - K_R^2] \cos \theta}{1 + K_R^2 + 2K_R \cos \beta} \]  
(18)

Here, \( K_R \) is reflection coefficient, \( \beta \) is the phase difference between incident and approaching waves, \( \theta \) is the angle between boundary normal and incident wave. Since \( \beta \) is so small, it is assumed to be zero in the calculations.

Total potential function on the boundary of incident and reflected waves is given in the equation (19).

\[ \phi = A \left\{ \exp[i(k \cos \theta - y \sin \theta)] + \exp[-ik(x \cos \theta - y \sin \theta) + i\beta] \right\} \]  
(19)

Velocity potential of a wave with the height \( H \) and period \( T \) is calculated using the equation (20) with linear theory. They have been used as input values.

\[ \phi_g = \frac{igH}{2\sigma} e^{i\sigma} \]  
(20)

\[ k_x = k \cos \theta \]  
(21)

\[ k_y = k \sin \theta \]  
(22)

\[ s = k_o \cos \theta_0 x - k_o \sin \theta_0 y - \sigma t \]  
(23)

Wave number vector is related to wave phase.

\[ \tilde{k} = \nabla s \]  
(24)

Phase function (s) is determined with the equation (21) and wave angle is a function of wave phase. The details can be found in the study of Hsu and Wen [9].

\[ s = \tan^{-1} \left( \frac{\text{Im}(\phi)}{\text{Re}(\phi)} \right) \]  
(25)

\[ \theta = \tan^{-1} \left( \frac{\partial s / \partial y}{\partial s / \partial x} \right) \]  
(26)

### 3 Numerical Model

Mac Cormack Method and Point Gauss Seidel Method have been used together in the numerical model. Since Mac Cormack Method is a multistep method, more stable results can be obtained. In Mac Cormack method, predictor and corrector values are calculated, respectively. In predictor step, forward finite difference method in the order of \( O(\Delta x) \) and \( O(\Delta y) \) has been used for the first order derivatives and central finite difference method in the order of \( O(\Delta x^2) \) and \( O(\Delta y^2) \) has been used for the second order derivatives. Backward finite difference method in the order of \( O(\Delta x) \) and \( O(\Delta y) \) has been applied to the first order derivatives in corrector step. The second order derivatives have been calculated with
the same method used in predictor step. The calculated nodes have been taken into account rapidly because of the use of Point Gauss Seidel Method. So iteration number has been minimized. Mac Cormack Method has been applied to the governing equation (2).

**Predictor Step**

\[
\begin{align*}
\frac{\phi_{k+1/2}^{i,j} - \phi_{k}^{i,j}}{\alpha_{k}^{i,j}} &= (C_{i,j}^{x} - C_{i,j}^{x+1}) \cdot \frac{C_{i,j}^{x+1} - C_{i,j}^{x}}{\alpha_{k}^{i,j}} + C_{i,j}^{x} \\
\frac{\phi_{k+1/2}^{i,j} - \phi_{k}^{i,j}}{\beta_{k}^{i,j}} &= (C_{i,j}^{y} - C_{i,j}^{y+1}) \cdot \frac{C_{i,j}^{y+1} - C_{i,j}^{y}}{\beta_{k}^{i,j}} + C_{i,j}^{y}
\end{align*}
\]

\[
C_{i,j}^{x} = \frac{2\phi_{i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k}^{i,j} \Delta x^2} + \frac{2\phi_{i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k}^{i,j} \Delta x^2} + \frac{2\phi_{i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k}^{i,j} \Delta x^2}
\]

\[
C_{i,j}^{y} = \frac{2\phi_{i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k}^{i,j} \Delta y^2} + \frac{2\phi_{i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k}^{i,j} \Delta y^2} + \frac{2\phi_{i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k}^{i,j} \Delta y^2}
\]

\[
\frac{k_{x,j} \phi_{x,i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k}^{i,j} \Delta x^2} + \frac{k_{y,j} \phi_{y,i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k}^{i,j} \Delta y^2} + \frac{k_{x,j} \phi_{x,i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k}^{i,j} \Delta x^2} + \frac{k_{y,j} \phi_{y,i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k}^{i,j} \Delta y^2} = 0
\]

\[
\phi_{k+1/2}^{i,j} = \frac{1}{2} \left( \phi_{k}^{i,j} + \phi_{k}^{i,j} \right)
\]

**Corrector Step**

\[
\begin{align*}
\frac{\phi_{k+2}^{i,j} - \phi_{k+1/2}^{i,j}}{\alpha_{k+1/2}^{i,j}} &= (C_{i,j}^{x} - C_{i,j}^{x+1}) \cdot \frac{C_{i,j}^{x+1} - C_{i,j}^{x}}{\alpha_{k+1/2}^{i,j}} + C_{i,j}^{x} \\
\frac{\phi_{k+2}^{i,j} - \phi_{k+1/2}^{i,j}}{\beta_{k+1/2}^{i,j}} &= (C_{i,j}^{y} - C_{i,j}^{y+1}) \cdot \frac{C_{i,j}^{y+1} - C_{i,j}^{y}}{\beta_{k+1/2}^{i,j}} + C_{i,j}^{y}
\end{align*}
\]

\[
C_{i,j}^{x} = \frac{2\phi_{i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k+1/2}^{i,j} \Delta x^2} + \frac{2\phi_{i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k+1/2}^{i,j} \Delta x^2} + \frac{2\phi_{i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k+1/2}^{i,j} \Delta x^2}
\]

\[
C_{i,j}^{y} = \frac{2\phi_{i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k+1/2}^{i,j} \Delta y^2} + \frac{2\phi_{i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k+1/2}^{i,j} \Delta y^2} + \frac{2\phi_{i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k+1/2}^{i,j} \Delta y^2}
\]

\[
\frac{k_{x,j} \phi_{x,i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k+1/2}^{i,j} \Delta x^2} + \frac{k_{y,j} \phi_{y,i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k+1/2}^{i,j} \Delta y^2} + \frac{k_{x,j} \phi_{x,i,j}^{x+1} - \phi_{i,j}^{x}}{\alpha_{k+1/2}^{i,j} \Delta x^2} + \frac{k_{y,j} \phi_{y,i,j}^{y+1} - \phi_{i,j}^{y}}{\beta_{k+1/2}^{i,j} \Delta y^2} = 0
\]

\[
\phi_{k+2}^{i,j} = \frac{1}{2} \left( \phi_{k+1/2}^{i,j} + \phi_{k+1/2}^{i,j} \right)
\]

**4 Applications of the Numerical Model**

Numerical model has been tested on the elliptical shoaling area given in literature. After obtaining acceptable results, the numerical model has been applied to a coastal region in Kocaeli Bay of Marmara Sea in Turkey.

**4.1 Elliptic Shoaling Area**

Berkhoff et al. [1] prepared a topography model to model wave propagation physically. The bottom has a slope 1/50 and elliptical shoaling area. Wave period (T) is 1 sec. Incident wave amplitude (a0) at x=0 is 0.0232m. The bathmetry of the elliptical shoaling area has been shown in the Fig. 1.
In Fig. 2, the increase of the wave amplitude due to shoaling can be observed. In Fig. 3, the shoaling and diffraction effects are dominant when the wave leaves the elliptical shoaling area. Diffraction at this caotic region can not be determined by linear theory. The RMSE and BIAS values obtained from the physical experiment and numerical model have been given in Table 1.

Table 1: RMSE and BIAS values

<table>
<thead>
<tr>
<th></th>
<th>RMSE</th>
<th>BIAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>x=11m</td>
<td>0,1588</td>
<td>0,0550</td>
</tr>
<tr>
<td>x=13m</td>
<td>0,2434</td>
<td>-0,1264</td>
</tr>
</tbody>
</table>

4.2 A Coastal Region in Kocaeli Bay

Kocaeli is located in Marmara Region and one of the important industrial cities of Turkey. There are industrial harbor, many factories and oil pipelines under the sea. Therefore it is critical region point of view of coastal engineering. The location of studied region has been shown in Fig. 4.

Bathymetry of the computed coastal region has been given in Fig. 5. The dominant wave direction is W. The significant wave height is $H_0=3m$ and wave period is $T=8\sec$ under the effect of the wind velocity $U=13m/\sec$. The wave height distributions obtained by the numerical model have been shown in Fig. 6.
5 Conclusion

Extended mild slope equation can be applied to rapidly varying topographies because higher order bottom effects such as the square of bottom slope and bottom curvature. It includes wave refraction, diffraction, shoaling, reflection, harbor resonance and dissipations due to wave breaking and bottom friction. Extended mild slope equation has been solved numerically using Mac Cormack Method and Point Gauss Seidel together. It provides stable and rapid solution. Furthermore, the nonlinear wave celerity and group velocity have been taken into account in the numerical model. It is an advantage especially in shallow areas where refraction is important. The numerical model has been compared with the physical experiment given in literature and obtained acceptable results. The numerical model can be used for the simulation of wave transformation in coastal engineering studies.

References