The Neoclassical Model with Variable Population Change

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Abstract: We extend the neoclassical growth model with logistic population change introduced by Ferrara and Guerrini [5] by considering a more general law for the population growth rate. In this kind of setup, the model is represented by a two dimensional dynamical system, whose non-trivial steady states, in contrast to the neoclassical model, may be node as well as saddle points.

Key-Words: Neoclassical model, Solow-Swan model, Variable population.

1 Introduction

The starting point for any study of economic growth is the neoclassical growth model. This model, constructed by Solow [9] and Swan [10], shows how economic policy can raise an economy’s growth rate by inducing people to save more. It also predicts that such an increase in growth cannot last indefinitely. An economy, regardless of its starting point, converges to a balanced growth path, where long-run growth of output and capital are determined solely by the rate of labor-augmenting technological progress and the rate of population growth (see, for example, Barro and Sala-i-Martin [1]). In the neoclassical model of economic growth, it is usually assumed that labor (population) force grows at a positive constant rate (Malthusian model [8]), an assumption this one which is not a good approximation to reality. In fact, it implies that population exponentially grows without limits, which is clearly unrealistic. To remove the prediction of unbounded population size in the very long run, Verhulst [11] wrote an alternative model, known as the logistic growth model. He claimed the model proposed by Malthus to be too simplistic as it only included linear terms. Hence, he considered an additional quadratic term with a negative coefficient in the Malthusian model. In this way, any population growth rate would essentially follow a bell-shaped curve, starting from zero, steadily increasing to a maximum, and declining once again to zero in a fashion symmetrical to the positive growth phase. The population stock then evolves according to the elongated S-curve, which has a point of inflection at the maximal value of the growth rate, and then levels off at a new but higher plateau, at which point the growth rate declines to zero. Recently, several attempts to analyze how the logistic-type population growth hypothesis might affect the dynamics of some growth models have been done in different directions (see, for example, Ferrara and Guerrini [3 – 6], and Guerrini [7]). In particular, Ferrara and Guerrini [5] have analyzed the role of a logistic population growth law within the Solow-Swan model, and proved the model to have a unique equilibrium, which is globally asymptotically stable. We argue that there is a need to allow for a more general notion of population than that of exponential or logistic growth. The alternative framework we suggest in this paper is based on the notion of population change. More concretely, we assume the population growth rate to be non-constant, but variable over time, subject only to be between prescribed upper and lower limits. The exponential and logistic growth occur as particular cases. With this setup, the model is described by a two dimensional dynamical system. The existence or not of non-trivial steady states for the system now relies on the particular form of the population growth rate function. Contrary to the neoclassical model, asymptotically stable node as well as saddle points may occur.

2 The model

We consider a closed economy where a homogeneous good is produced according to a technology involving two inputs: physical capital and labor. The production function is specified as $Y_t = F(K_t, L_t)$, where $Y_t$ is the flow of output, $K_t$ is the stock of physical capital, and $L_t$ is labor, at time $t$. Let us assume the production
function to exhibit positive and diminishing marginal products with respect to physical capital and labor, to display constant returns to scale, and satisfy the Inada conditions. The homogeneity of $Y_t$ implies that the production function can be written in terms of capital per worker as $y_t = f(k_t)$, where $y_t = Y_t/L_t$ is the per capita output, $k_t = K_t/L_t$ denotes the capital stock per worker, and $f(k_t) = F(k_t, L_t)$. As well, the mentioned properties of the production yield $f'(k_t) > 0$, $f''(k_t) < 0$. Output is assumed to be used for consumption $C_t$ or for investment $I_t$ in physical capital. A constant fraction $\delta > 0$ of the capital stock depreciates every period. Consequently, the net increase in capital stock at any moment in time is equal to the amount of gross investment less the amount of depreciated capital, i.e. $\dot{K}_t = I_t - \delta K_t$, where a dot over a variable denotes differentiation with respect to time. Since the economy is closed, the output of the economy equals total income, whereas investments equal savings. Households save a constant fraction of their income, i.e. the saving rate $s$ satisfies $0 < s < 1$. From the relations $I_t = S_t$, and $S_t = sY_t$, we have that the capital accumulation equation takes the form $\dot{K}_t = sY_t - \delta K_t$. As we have seen before, output per capita depends on the stock of capital per worker. Therefore, we will continue our analysis by writing the previous equation in terms of capital stock per worker. Dividing both sides of the previous capital accumulation equation by $L_t$, and then expressing $\dot{K}_t/L_t$ as a function of $k_t$ by using the relation $\dot{k}_t = d(K_t/L_t)/dt = K_t/L_t - k_t \dot{L}_t/L_t$, yields the non-linear differential equation

$$\dot{k}_t = s f(k_t) - (\delta + \dot{L}_t/L_t)k_t.$$  

(1)

In general, population is assumed to growth according to $L_t = n(L_t)$, where $n > 0$ is the given population growth rate. The main problem of this assumption is that population grows exponentially, and so tends to infinity as time goes to infinity, which is clearly unrealistic. Contrary to the standard literature, we consider a more realistic approach by assuming that $L_t$ at any moment of time is a function of the population size $L_t$ at that moment, i.e. $L_t = g(L_t)$. Since a zero population has a zero growth, $L_t = 0$ is an algebraic root of the function $g(L_t)$. Thus, we may write $L_t = L_t n(L_t)$, where $n(L_t)$ is a function of $L_t$. Following Guerrini [7], we assume the population growth rate $n(L_t)$ to be non-constant, but variable and bounded over time. More precisely, $n(L_t)$ is controllable subject to be between prescribed upper and lower limits, i.e.

$$0 \leq n(L_t) \leq M, \quad M > 0.$$  

(2)

As well, we assume that there exist $\lim_{t \to \infty} n(L_t) = n(L_\infty)$. Normalizing today’s population, i.e. $L_0 = 1$, then, from (2), we get

$$1 \leq L_t \leq e^{Mt},$$  

(3)

for all $t$. In particular, we see that $L_t \geq 1$, for all $t$. Note that there exists $\lim_{t \to \infty} L_t = L_\infty$ as $L_t$ is monotonically increasing. From (2) and (3), we derive that

$$0 \leq n(L_\infty) \leq M, \quad 1 \leq L_\infty \leq \infty,$$

i.e. in the long run $n(L_t)$ is finite, whereas $L_t$ may be finite or infinite.

**Remark 1.** If $L_\infty < \infty$, then $n(L_\infty) = 0$. This statement follows from the following result:

"Let $\varphi : [x_0, +\infty) \to \mathbb{R}$ be a differentiable function such that there exist (finite or infinite) the limits $\lim_{x \to +\infty} \varphi(x) = l$, $\lim_{x \to +\infty} \varphi'(x) = n$. If $l$ is finite, then $n = 0$."

**Proof:** By Lagrange’s theorem, $\varphi(x + 1) - \varphi(x) = \varphi'(\xi_x)$, for some $\xi_x \in (x, x + 1)$. Since $\lim_{x \to +\infty} \xi_x = +\infty$, we have that $\lim_{x \to +\infty} \varphi'(\xi_x) = \lim_{x \to +\infty} \varphi'(x) = n$. Consequently, $\lim_{x \to +\infty} [\varphi(x + 1) - \varphi(x)] = n$. The statement follows noting that $l$ finite also implies $\lim_{x \to +\infty} [\varphi(x + 1) - \varphi(x)] = l - l = 0$.

**Remark 2.** A population growth rate satisfying the assumptions of our model is given by the well-known logistic map: $\dot{L}_t/L_t = n(L_t) = a - bL_t$, with $a > b > 0$. Solving this equation yields $L_t = ae^{at}/(a - b + be^{at})$, and so $n(L_t) = a(a-b)/(a-b+be^{at})$. Therefore, we derive that $n(L_t)$ is a monotone decreasing function from $n(0) = a - b$ to $n(L_\infty) = 0$, and that $L_t$ is a monotone increasing function from $L_0 = 1$ to $L_\infty = a/b$. In particular, we have that $0 \leq n(L_t) \leq a/b$, and $1 \leq L_t \leq a/b$.

In conclusion, we have that the model’s economy is described by the following system of non-linear differential equations

$$\begin{aligned}
\dot{k}_t &= s f(k_t) - [\delta + n(L_t)]k_t, \\
\dot{L}_t &= L_t n(L_t),
\end{aligned}$$  

(4)

with $1 \leq L_t \leq e^{Mt}$, for all $t$, $1 \leq L_\infty \leq \infty$, $0 \leq n(L_t) \leq M$, $0 \leq n(L_\infty) \leq M$, where $n(L_\infty) = 0$ if $L_\infty < \infty$. Given $k_0 > 0$, (4) becomes a Cauchy problem which has a unique solution $(k_t, L_t)$ defined on $[0, \infty)$ (see Birkhoff and Rota [2]).
3 Steady states analysis

A steady state is defined as a situation in which the growth rates of the per capita physical capital and the labor growth rate are equal to zero. Let us denote the steady state equilibrium values of \( k_*, L_ * \), by \( k_s, L_s \), respectively. In studying the steady states of our model, we will confine our analysis to interior steady states only, i.e. we will exclude the economically meaningless solutions such as \( k_s = 0 \) or \( L_s = 0 \).

**Proposition 3.** The existence of interior steady states for the system (4) depends on the function \( n(L_t) \). In particular, there are no steady states if \( n_{L_t}(L_t) \geq 0 \) for all \( L_t \), whereas there exists a unique steady state if \( n_{L_t}(L_t) < 0 \) for all \( L_t \).

**Proof:** To solve the steady states equilibrium we impose the growth rates in system (4) to be zero. This gives the following system of equations: \( \delta k_t = \delta k_t = 0, \) \( n(L_t) = 0 \). The Inada conditions, and the fact that \( f(k_t)/k_t \) is a monotone decreasing function of \( k_t \) (its derivative with respect to \( k_t \) is given by \( f'(k_t) - f(k_t)/k_t < 0 \)) yield that there exists a unique \( k_t > 0 \) which solves the equation \( s f(k_t)/k_t = \delta \). The statement follows noting that, in the \((n(L_t), L_t)\)-plane, the function \( n(L_t) \) is increasing, starting from \( n(0), \) if \( n_{L_t}(L_t) > 0 \) for all \( L_t \), it is constant if \( n_{L_t}(L_t) = 0 \) for all \( L_t \) (in these two cases, the curve \( n(L_t) \) will not cross the \( L_t\)-axes, and so \( n(L_t) \) is never zero), it is decreasing if \( n_{L_t}(L_t) < 0 \) for all \( L_t \) (in this case \( n(L_t) \) will cross the \( L_t\)-axes in a unique point).

**Remark 4.** The number of roots of the equation \( n(L_t) = 0 \) gives the number of interior steady states of the system. These correspond to the number of intersections of the function \( n(L_t) \) with the \( L_t\)-axes in the \((L_t, L_t)\)-plane. The difficulty in our investigation is that an expression for \( n(L_t) \) is not given. It is clear that our study would be easier working with a given \( n(L_t) \), but this would lead our study to some results which are valid for this particular \( n(L_t) \), and, maybe, not true in general.

**Remark 5.** If a steady state \((k_*, L_*)\) exists, then \( k_* \) is the unique solution of the equation \( s f(k_*)/k_* = \delta \). The steady state capital-labor ratio \( k_* \) comes from \( s f(k)/k = \delta \), while in the Solow-Swan model the steady state capital-labor ratio \( k^* \) can be found from \( s f(k)/k = \delta + \eta \), with \( \eta > 0 \) the labor growth rate. Thus, in this latter case, equilibrium occurs with a higher capital-labor ratio, i.e. \( k^* > k_* \). If \( n_{L_t}(L_t) < 0 \) for all \( L_t \), and \( L_{\infty} < \infty \), then \( L_{\infty} = L_* \). This is immediate recalling that Remark 1 gives \( n(L_{\infty}) = 0 \), and proposition 1 says that there exists a unique value \( L_* \) where \( n(L_*) = 0 \) if \( n_{L_t}(L_t) < 0 \) for all \( L_t \). An example where this fact happens is provided by the logistic law (recall it is \( L_{\infty} = a/b, n_{L_t}(L_t) = -b < 0 \)).

The local dynamic around a steady state equilibrium \((k_*, L_*)\) is determined by the signs of the eigenvalues of the Jacobian matrix corresponding to its linearized system, which writes

\[
\begin{bmatrix}
\dot{k}_t \\
\dot{L}_t
\end{bmatrix} = J^* \begin{bmatrix}
k_t - k_* \\
L_t - L_*
\end{bmatrix},
\]

where

\[
J^* = \begin{bmatrix}
J^*_{11} & J^*_{12} \\
J^*_{21} & J^*_{22}
\end{bmatrix}
\]

is the Jacobian matrix of system (4) evaluated at \((k_*, L_*)\). A direct calculation gives

\[
\begin{align*}
J^*_{11} &= (\partial f(k)/\partial k)_{(k_*, L_*)} = s f'(k_*) - \delta, \\
J^*_{12} &= (\partial f(k)/\partial L_t)_{(k_*, L_*)} = -n_{L_t}(L_*)k_*, \\
J^*_{21} &= (\partial L_t/\partial k)_{(k_*, L_*)} = 0, \text{ and } J^*_{22} = (\partial L_t/\partial L_t)_{(k_*, L_*)} = n_{L_t}(L_*)L_*.
\end{align*}
\]

Therefore, we have

\[
J^* = \begin{bmatrix}
J^*_{11} & J^*_{12} \\
J^*_{21} & J^*_{22}
\end{bmatrix}.
\]

**Proposition 6.**

i) If \( n_{L_t}(L_*) < 0 \), the two eigenvalues of \( J^* \) are real and negative.

ii) If \( n_{L_t}(L_*) > 0 \), the two eigenvalues of \( J^* \) are real with opposite signs.

iii) If \( n_{L_t}(L_*) = 0 \), one of the two eigenvalues of \( J^* \) is null, the other is real and negative.

**Proof:** It is immediate to check that the matrix \( J^* \) has the following two eigenvalues:

\[
\lambda_1 = s f'(k_*) - \delta, \quad \lambda_2 = n_{L_t}(L_*)L_*.
\]

Remark 5, and the inequality \( k_t f'(k_t) - f(k_t) < 0 \) imply \( \lambda_1 = s f'(k_*) - f(k_*)/k_* \) \( < 0 \). The statement now follows from (6).

**Theorem 7.** Let \((k_*, L_*)\) be a steady state of system (4). Let \( n_{L_t}(L_*) \neq 0 \). The local stability of the stationary solution is determined by one of the following three possibilities:

i) If \( n_{L_t}(L_*) < 0 \), the steady state is an asymptotically stable node. All the trajectories satisfying (5) which begin in the neighborhood of \((k_*, L_*)\) converge back to the steady state;
ii) if $n_{L_1}(L_*) > 0$, the steady state is a saddle. There is a one-dimensional manifold in the $(k_t, L_1)$-plane with the property that trajectories beginning on this manifold converge to the steady state, but all other trajectories diverge. The stable and unstable manifolds, which are asymptotes to all trajectories, intersect at the point $(k_*, L_*)$.

iii) if $n_{L_1}(L_*) = 0$, no conclusion can be made.

**Proof:** This follows from Proposition 6. In fact, if $n_{L_1}(L_*) < 0$, both eigenvalues of the matrix $J^*$ are negative. The steady state is a stable node. If $n_{L_1}(L_*) > 0$, the eigenvalues come in pairs of opposite sign. The steady state is a saddle point. If $n_{L_1}(L_*) = 0$, then $\lambda_2 = 0$. Since one of the two eigenvalues is equal to zero, the equilibrium point is non-hyperbolic. In this case we cannot make any conclusion concerning the stability or instability of $(k_*, L_*)$ based on linearization. Other methods need to be used. Two of these are center manifold theory and normal form theory. However, since we do not have an explicit expression for $n(L_t)$, these two techniques do not help. □

**Remark 8.** If the population growth rate obeys the logistic law, then $n_{L_1}(L_*) = -b < 0$. Thus, there is a unique non-trivial steady state which is a stable node.

## 4 Speed of convergence

We want now to provide a quantitative assessment of the speed of transitional dynamics. The negative eigenvalues are the analogous to the convergence coefficient in the Solow-Swan growth model. The negative eigenvalue $\lambda_2$ corresponds to the speed of convergence of population. The speed of convergence of capital depends on eigenvalues $\lambda_1$ and $\lambda_2$, while that of population only depends on $\lambda_2$. If $\lambda_2 < -\lambda_1$, then the speed of convergence of $L_t$ is faster than that of $k_t$, while if $\lambda_1 < -\lambda_2$ then all variables converge at the same speed $-\lambda_2$. In any case, the speed of convergence of the system is the absolute value of the higher negative eigenvalue. In case $\lambda_1 < 0$, $\lambda_2 > 0$, i.e. the steady state is a saddle, then there is a unique negative eigenvalue. Let $\hat{\lambda} = -\lambda_1 = -[sf'(k_*) - \delta] = \delta(1 - f'(k_*)k_*/f(k_*))$. We call $\hat{\lambda}$ the rate or speed of convergence to the steady state $(k_*, L_*)$. In case $\lambda_1 < 0$, $\lambda_2 < 0$ (without a saddle), i.e. a node, then the system has two negative eigenvalues, so that the stable manifold is a two-dimensional locus. The existence of two-dimensional transition paths introduces important new characteristics to the transition. First, in contrast to the Solow-Swan model, the transition no longer occurs at a constant speed. In the neoclassical growth model, the stable manifold is a one-dimensional locus, so that the speed of adjustment is parameterized unambiguously by the magnitude of the unique stable eigenvalue. By contrast, in the present model where the stable transitional path is a two-dimensional locus, the speed of convergence in general varies over time and across variables.

**Remark 9.** Let $f(k_t) = k_t^\alpha$, $\alpha \in (0, 1)$, i.e. the production function $f(k_t)$ is of Cobb-Douglas type. In case of the Solow-Swan model, we know that a quantitative measure of the speed of convergence $\beta$ can be written as $\beta = (\delta + n)(1 - f'(k_*)k_*/f(k_*))$, where $n > 0$ is the labor growth rate, and $k_*$ is the unique steady state of the Solow-Swan model. We have $\beta = (\delta + n)(1 - \alpha)$. In our model, when $\lambda_1 < 0$, $\lambda_2 > 0$, we know that the speed of convergence is given by $\hat{\lambda} = -\lambda_1 = \delta(1 - f'(k_*)k_*/f(k_*)) = \delta(1 - \alpha)$. Since $\hat{\lambda} < \beta$, the economy converges more rapidly to the steady state in our model than does in the Solow-Swan model.

## 5 Conclusion

In this paper, we have examined a modified version of the neoclassical Solow-Swan growth model, when the population growth rate is non-constant, but variable and bounded over time. As a result, contrary to the standard neoclassical model, the dynamics of the economy is modeled by a system of two differential equations, which can be analyzed via the usual linearization method around a non-trivial steady state. It is seen that asymptotically stable node as well as saddle points may occur.

**References:**


