

On Simulation Study of Mixture of Two Weibull Distributions

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Abstract: - The aim of this paper is to introduce a method of combining two different Weibull distributions. This method illustrates on how to produce a mixture distribution based on two Weibull distributions by including a mixing parameter, say w , which represents the proportion of mixing associated to the two component models. Weibull distributions. The mixture distribution produced from the combination of two Weibull distributions has a number of parameters which include shape parameters, scale parameters and location parameters in addition to the mixing parameter. In this paper, we focus on the estimation of parameters of the proposed mixture Weibull distribution using maximum likelihood method. In addition, we illustrate the characteristics of this distribution in terms of the probability density function, cumulative distribution function, reliability function and failure rate for a particular value of w .

Key-Words: - Weibull distribution, Mixture Weibull distribution, Mixing parameter

1 Introduction

Mixture Weibull distributions is one of the new areas of research available in the literatures, which is commonly found for applications in the reliability studies. This paper aims to illustrate how to combine two Weibull distributions and produce a mixture distribution by including a mixing parameter which represents the proportions of mixing of the two components of Weibull distributions. The mixture Weibull distribution produced from the combination can have five or more parameters. These are shape parameters, scale parameters, location parameters, in addition to the mixing parameter (w). This type of distribution is particularly suitable to model multiple causes of failure. There are a number of mixture distributions that has been studied. Our study will concentrate on mixture Weibull distribution. This paper will include estimating the parameters of the mixture Weibull distribution using maximum likelihood estimation. Under a specific value of the mixing parameter, we obtain the parameter estimates and find the probability density function, cumulative distribution function, reliability function and the failure rate of the mixture Weibull distribution. We illustrate an

application using a real data with sample size of 20.

2 Mixture Models

Assume that we have n -fold mixture model that involves n sub-populations which can be given by

$$F(x) = \sum_{i=1}^n w_i F_i(x) \text{ with } w_i > 0 \text{ and } \sum_{i=1}^n w_i = 1 \quad (1)$$

where w_i is a mixing parameter and $F_i(x)$, $i=1,2,\dots,n$ are distribution functions either with two or three parameter Weibull distributions. These models involve two or more distributions with one or more being Weibull distribution. The distributions involved are called sub-populations or components, and the model is called finite Weibull mixture model. In the literature the Weibull mixture model has been referred by many other names, for example Murthy *et al.* (2004), such as additive-mixed Weibull distributions, bimodal-mixed Weibull (for a two fold mixture), mixed-mode Weibull distribution, Weibull distribution of the mixed type, multi modal Weibull distribution, and so forth.

2.1 Mixing Parameter (w)

The mixing parameter w , represents the proportions of combining a number of distributions. It takes values from $0 < w_i < 1$, and $\sum w_i = 1$, $i = 1, 2, \dots, n$. One of the earliest studies in which an analysis of a mixture distribution was attempted was that by Pearson (1894) who used the method of moments to estimate the five parameters of a two component univariate normal mixture. This method of moments became one of the most popular ways of estimating the parameters of mixture distributions. Bartholomew (1959) used a system of equations which were very similar to that of Kabir method. Kabir (1968) considers a fairly general method, assuming a mixture of the form

$$f(x) = \sum_{i=1}^c w_i g_i(x, \mu_i) \quad (2)$$

where $w_i \geq 0$, $\sum w_i = 1$, $\mu_1 < \mu_2 < \dots < \mu_c$ and μ_i are the parameters for the Weibull distributions. He then established a system of equations

$$\sum_{i=1}^c A_i \lambda_i (\mu_i)^j = \Phi_j \quad j = 0, 1, \dots, 2c-1 \quad (3)$$

The A_i are non-zero constants, which may be functions of the μ_i and λ_i is a strictly monotonic function of μ_i , ($i = 1, \dots, c$), Kabir then proved that the resulting estimators of μ_i and w_i were both consistent and asymptotically normal. Talis and Light (1968) attempted to improve the efficiency of moment estimators for an exponential mixture, though it seems likely that the approach will have wider applicability. Apolloni and Murino (1979) also considered this problem of mixture samples.

3 Mixture Weibull Distribution

The probability density function (pdf) of a 2-parameter Weibull distribution is,

$$f(x; \alpha, \beta) = \left(\frac{\alpha x^{\alpha-1}}{\beta^\alpha} \right) e^{-(x/\beta)^\alpha} \quad \text{for } 0 \leq x < \infty \quad (4)$$

and for a 3-parameter Weibull distribution is

$$f(x; \alpha, \beta, \gamma) = \left(\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha} \right) e^{-((x-\gamma)/\beta)^\alpha} \quad \text{for } \gamma \leq x < \infty \quad (5)$$

where α , β , and γ are the shape, scale and location parameters respectively. A mixture distribution is a distribution made of combining two or more component distributions. The probability density function of this mixture distribution can be shown as;

$$f(x) = w_1 f_1(x) + \dots + w_n f_n(x) \quad \text{with } w_i > 0 \text{ and } \sum_{i=1}^n w_i = 1 \quad (6)$$

where w_i is the mixing parameter which represents the proportion of mixing of the component distributions.

The function $f_i(x)$ is the probability density function of the component distribution i . While n is the number of component distributions being mixed.

In this paper we will consider a simple case of only two component distributions, and the mixing parameter for each component will be defined simply as w and $(1-w)$. The probability density function of the mixture Weibull distribution of the two distributions in (4) and (5) is as follows;

$$f(x) = w(\alpha_1 \frac{x^{\alpha_1-1}}{\beta_1^{\alpha_1}}) e^{-(x/\beta_1)^{\alpha_1}} + (1-w)(\frac{\alpha_2(x-\gamma)^{\alpha_2-1}}{\beta_2^{\alpha_2}}) e^{-((x-\gamma)/\beta_2)^{\alpha_2}} \quad (7)$$

where, $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $0 \leq \gamma \leq x$ and $0 < w < 1$. In equation (7), α_1 and α_2 are the shape parameters, β_1 and β_2 are the scale parameters, γ is the location parameter, while w is the mixing parameter.

As a result when the two marginal components are given by equations (4) and (5), the mixture distribution (7) is characterised by six parameters; shape, scale and location parameters for the two component distributions in addition to the mixing parameter.

However, other mixture distributions can be produced from (4) and (5) with five and seven parameters. To obtain two-fold Weibull model which is given by,

$$F(x) = wF_1(x) + (1-w)F_2(x) \quad (8)$$

with $F_1(x)$, and $F_2(x)$ are the distribution functions taken from equations (4) or (5). As a result, when the two sub-populations are given by equation (4), the model is characterized by five parameters, the shape and scale parameters for the two sub-populations and the mixing parameter w ; with $0 < w < 1$. The probability density function and failure rate of the two-fold Weibull mixture are given by;

$$f(x) = wf_1(x) + (1-w)f_2(x) \quad (9)$$

and

$$h(x) = \sum_{i=1}^n w_i(x) h_i(x) \quad (10)$$

where

$$w_i(x) = \frac{w_i R_i(x)}{\sum_{i=1}^n w_i R_i(x)}, \quad \text{with } \sum_{i=1}^n w_i(x) = 1 \quad (11)$$

$$\text{and} \quad R_i(x) = 1 - F_i(x) \quad (12)$$

where $R_i(x)$ is the reliability function. Hence

$$h(x) = \frac{wR_1(x)}{wR_1(x) + (1-w)R_2(x)} h_1(x) + \frac{(1-w)R_2(x)}{wR_1(x) + (1-w)R_2(x)} h_2(x) \quad (13)$$

Also, Boes (1966) who assumed that only the mixing proportions are unknown for the two component case;

$$f(x) = wf_1(x) + (1-w)f_2(x) \quad (14)$$

By integrating a family of equations of the form;

$$F(x) = wF_1(x) + (1-w)F_2(x) \quad (15)$$

which leads to the following

$$w = \frac{F(x) - F_2(x)}{F_1(x) - F_2(x)} \quad (16)$$

This gives necessary and sufficient conditions on F_1 and F_2 for the uniform attainment of the Cramer-Rao bound on the variance of w .

3.1 Estimating the Parameters of Mixture Distribution

There are many methods for estimating the parameters of mixture distributions both graphical and analytical approaches have been used. Because the principals and procedures of the analytical methods are the same except for some details. About the analytical methods we know that these methods start from Pearson's (1894) method of moments, through the formal maximum likelihood approaches, general curve fitting, Bayesian and so on. In this paper the maximum likelihood method was used for estimating the parameters of the mixture Weibull distribution.

3.2 Maximum Likelihood Estimation

Mendenhall and Hader (1958) considered an n -fold mixture Weibull distribution. They derived the maximum likelihood estimates for the scale and mixing parameters, assuming that the shape parameters are known. Also Beetz (1982) estimated the parameters of a mixed Weibull distribution by fitting the mixed probability density to the experimental histogram using the maximum likelihood method. Ashour (1987) considered the problem of maximum likelihood estimation with five unknown parameters of the mixed Weibull distribution for multistage censored type I sample.

Jiang and Kececioglu (1992) presented an algorithm for estimating the parameters of a Weibull mixture model with the right censored data using the method of maximum likelihood. Ahmad and Abdulrahman (1994) presented a procedure for finding the maximum likelihood estimates of the parameters of a

mixture of two Weibull distributions. The method of maximum likelihood estimators (MLE) was popularised by Fisher in the 1920's. The definition of likelihood is;

Let y_1, y_2, \dots, y_n be a sample observations taken on the corresponding continuous random variables Y_1, Y_2, \dots, Y_n . Then the likelihood L will be $L(y_1, y_2, \dots, y_n)$ is defined as the joint density evaluated at y_1, y_2, \dots, y_n .

Parameters are selected so that the likelihood function is maximized. This means that the likelihood function is defined as follows;

$$L = f(x_1).f(x_2). \dots .f(x_n) \quad (17)$$

or alternatively ,

$$L = \prod_{i=1}^n f(x_i) \quad (18)$$

is the pdf associated with the observation x_i and n is the sample size. MLE are used because they have useful properties not shared by other parameter estimation techniques. The log likelihood function, where $f(x)$ is the pdf for the mixture of two Weibull distributions and $f_1(x)$ or $f_2(x)$ is the pdf for the single component Weibull distribution is;

$$LL = \ln\left(\prod_{i=1}^n f(x_i)\right) \quad (19)$$

or equivalently;

$$LL = \sum_{i=1}^n \ln(f(x_i)) \quad (20)$$

The derivative of the log likelihood function with respect to each of the six parameters was as follows;

$$\frac{dLL}{d\alpha_1} = \sum_{i=1}^n \frac{w}{f(x_i)} \cdot \frac{df_1}{d\alpha_1} \quad (21)$$

$$\frac{dLL}{d\beta_1} = \sum_{i=1}^n \frac{w}{f(x_i)} \cdot \frac{df_1}{d\beta_1} \quad (22)$$

$$\frac{dLL}{d\alpha_2} = \sum_{i=1}^n \frac{(1-w)}{f(x_i)} \cdot \frac{df_2}{d\alpha_2} \quad (23)$$

$$\frac{dLL}{d\beta_2} = \sum_{i=1}^n \frac{(1-w)}{f(x_i)} \cdot \frac{df_2}{d\beta_2} \quad (24)$$

$$\frac{dLL}{d\gamma} = \sum_{i=1}^n \frac{(1-w)}{f(x_i)} \cdot \frac{df_2}{d\gamma} \quad (25)$$

$$\frac{dLL}{dw} = \sum_{i=1}^n \frac{f_1(x_i, \alpha_1, \beta_1) - f_2(x_i, \alpha_2, \beta_2, \gamma)}{w.f_1(x_i, \alpha_1, \beta_1) + (1-w).f_2(x_i, \alpha_2, \beta_2, \gamma)} \quad (26)$$

MLE can be obtained by equating the above equations to zero and solving for all the parameters. Numerical methods and computer facilities are required to solve these equations and get the required results

4 Applications on Lifetimes of Electronic Components

The above equations were used to estimate the parameters for the mixture Weibull distribution, and the results were obtained using the data for the lifetimes of 20 electronic components that were taken from Murthy et al (2004) and were analysed to find the mixture Weibull distribution and the other functions, as shown in the following tables;

Table 1: Estimation of Mixture Weibull Distribution Parameters

	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\gamma}$
$w=0.10$	0.0804	0.2106	0.7432	0.8981	1.125
$w=0.40$	0.3215	0.8424	0.4955	0.5987	0.7500
$w=0.80$	0.6431	1.6848	0.1652	0.1996	0.2500

Table 2: Two Weibull Distributions with the Parameters $\alpha_1=3, \beta_1=1, \alpha_2=2, \beta_2=2, \gamma=1$

Obs. No.	x_i	$f_1(x)$	$f_2(x)$	$F_1(x)$	$F_2(x)$
1	0.03	0.002700	0.000000	0.000027	0.000000
2	0.12	0.043125	0.000000	0.001727	0.000000
3	0.22	0.143662	0.000000	0.010592	0.000000
4	0.35	0.352076	0.000000	0.041969	0.000000
5	0.73	1.083475	0.000000	0.322277	0.000000
6	0.79	1.143540	0.000000	0.389233	0.000000
7	1.25	0.664829	0.123062	0.858170	0.015504
8	1.41	0.361523	0.196563	0.939385	0.041154
9	1.52	0.206847	0.243005	0.970157	0.065366
10	1.79	0.031046	0.337937	0.996770	0.144462
11	1.8	0.028501	0.340858	0.997068	0.147856
12	1.94	0.007617	0.376845	0.999325	0.198203
13	2.38	0.000024	0.428629	0.999999	0.378799
14	2.4	0.000017	0.428838	0.999999	0.387374
15	2.87	0.000000	0.390068	1.000000	0.582815
16	2.99	0.000000	0.369710	1.000000	0.628433
17	3.14	0.000000	0.340534	1.000000	0.681744
18	3.17	0.000000	0.334324	1.000000	0.691867
19	4.72	0.000000	0.058483	1.000000	0.968558
20	5.09	0.000000	0.031222	1.000000	0.984732

continue

Obs. No.	x_i	$R_1(x)$	$R_2(x)$	$h_1(x)$	$h_2(x)$
1	0.03	0.999973	1.000000	0.00270	0.00000
2	0.12	0.998273	1.000000	0.04320	0.00000
3	0.22	0.989408	1.000000	0.14520	0.00000
4	0.35	0.958031	1.000000	0.36750	0.00000
5	0.73	0.677723	1.000000	1.59870	0.00000
6	0.79	0.610767	1.000000	1.87230	0.00000
7	1.25	0.141830	0.984496	4.68750	0.12500
8	1.41	0.060615	0.958846	5.96430	0.20500
9	1.52	0.029843	0.934634	6.93120	0.26000
10	1.79	0.003230	0.855538	9.61230	0.39500
11	1.80	0.002932	0.852144	9.72000	0.40000
12	1.94	0.000675	0.801797	11.29080	0.47000
13	2.38	0.000001	0.621201	16.99320	0.69000
14	2.4	0.000001	0.612626	17.28000	0.70000
15	2.87	0.000000	0.417185	24.71070	0.93500
16	2.99	0.000000	0.371567	26.82030	0.99500
17	3.14	0.000000	0.318256	29.57880	1.07000
18	3.17	0.000000	0.308133	30.14670	1.08500
19	4.72	0.000000	0.031442	66.83520	1.86000
20	5.09	0.000000	0.015268	77.72430	2.04500

Table 3: Failure rate of a Mixture Weibull Distributions with parameters $\alpha_1=3, \beta_1=1, \alpha_2=2, \beta_2=2, \gamma=1$ and mixing parameter ; $w=0.10, w=0.40$ and $w=0.80$

Obs. No.	x_i	$h(x)$ $w=0.10$	$h(x)$ $w=0.40$	$h(x)$ $w=0.80$
1	0.03	0.000270	0.001080	0.002160
2	0.12	0.004313	0.017262	0.034548
3	0.22	0.014381	0.057709	0.115912
4	0.35	0.035356	0.143235	0.291446
5	0.73	0.111956	0.497527	1.167887
6	0.79	0.118985	0.541765	1.328512
7	1.25	0.196882	0.524796	1.792980
8	1.41	0.245171	0.437905	1.367394
9	1.52	0.283584	0.399048	1.015548
10	1.79	0.398865	0.418140	0.532116
11	1.80	0.403562	0.421331	0.526538
12	1.94	0.471012	0.476066	0.506295
13	2.38	0.690004	0.690024	0.690147
14	2.40	0.700003	0.700018	0.700107
15	2.87	0.935000	0.935000	0.935000
16	2.99	0.995000	0.995000	0.995000
17	3.14	1.070000	1.070000	1.070000
18	3.17	1.085000	1.085000	1.085000
19	4.72	1.860000	1.860000	1.860000
20	5.09	2.045000	2.045000	2.045000

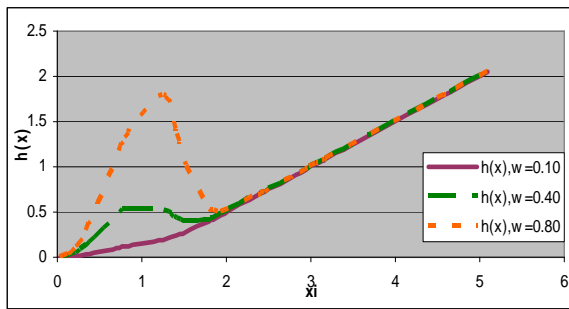


Figure 1: Failure rate of Mixture Weibull Distribution, with $\alpha_1 = 3$, $\beta_1 = 1$, $\alpha_2 = 2$, $\beta_2 = 2$, $\gamma = 1$

5 Conclusion

From this paper we explained the idea of how to combine two Weibull distributions. We gave an example of how to obtain a mixture distribution using a mixing parameter $w=0.1, 0.4, 0.80$. It also explained in detail the estimation of parameters for the mixture Weibull distribution using the maximum likelihood estimation.

Table 1 showed the estimated values of the parameters for the mixture Weibull distribution with the value of the mixing parameter $w=0.1, 0.4, 0.80$. Table 2 present the functions of the two original component of Weibull distributions. Table 3 present the failure rate of the mixture Weibull distribution with mixing parameter as shown in Table 1. We can see from Table 3 and Figure 1 that failure rate increases and this give us a correct result compared to the result in Tables 2. From figure 1, we can see that all the curves increases and the best one is when the mixing parameter $w=0.10$.

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