Mixed formulations (formulations using mixed elements) in electromagnetics via the finite element method: The case of modeling tokamak plasmas in a computational electromagnetic environment

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Abstract: This paper tends to extend the scope of this conference towards the direction of continuum physics. Ordinary matter consists of material points. On the one hand, each material point is so small, that it has no further structure. On the other hand, each material point is made up of so many particles, that the laws of thermodynamics for infinitely large systems apply. It makes sense to speak of content, flow and production of certain quantities like mass, charge, momentum, energy and entropy. The corresponding balance equations are valid irrespective of the special material under consideration. They include precise versions of the first and second law of thermodynamics.

Additional constitutive equations define entire branches of physics like fluid dynamics, electromechanics, heat transport, diffusion, reaction kinetics and so forth. The Navier – Stokes equation for Newtonian fluids, Hooke’s law for elastic media, Fourier’s law of heat and Ohm’s law of charge conduction are examples. More precisely the paper aims to demonstrate the usefulness of mixed formulations (formulations using mixed elements) in electromagnetics via the finite element method.

Key-Words: computational electromagnetism, Maxwell's equations, differential forms, Hodge operator, finite element method, modelling, tokamak, plasma.

1 Introduction
Geometry is the science of space and of the properties of shapes in space. Dating back to Euclid, models of our world have been formulated using simple, geometric metaphors, formalizing evident symmetries and experimental invariants. Consequently, geometry is at the foundation of many current physical theories: general relativity, electromagnetism, gauge theory as well as solid and fluid mechanics. Unfortunately, the inherent geometric nature of such models is often hindered by their formulation in vectorial or tensorial notations. The traditional use of a coordinate system, in which the equations of those models are articulated, often obscures the underlying structures by an awesome usage of indices. In addition, such complex expressions entangle the topological and metrical content of the model. After the publication of the celebrated Maxwell’s "Treatise on Electricity and Magnetism" [1], the laws governing electromagnetic fields were commonly written in differential formulation. Since that time the electromagnetic field equations were identified with the “Maxwell equations”, i.e. with partial differential equations. In the archetype formed by three centuries of differential formulation of physical laws, it became so prevalent that we are led to think that it is the only possible formulation. Maxwell equations link local field variables such as charge density ($\rho$), current density ($\mathbf{J}$), electric and magnetic field intensities ($\mathbf{E}$ and $\mathbf{H}$), electric and magnetic flux densities ($\mathbf{D}$ and $\mathbf{B}$) using vectorial analysis notation are usually in the form:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0, \ \nabla \cdot \mathbf{D} = \rho \quad (2)$$

This formulation belongs to O. Heaviside and J. W. Gibbs, who eliminated vector and scalar potentials in Maxwell's original set of equations. Maxwell’s equations are extremely rich in symmetries and (consequently) conservation laws. In the continuum, many conservation laws follow directly from invariances of the Lagrangian (Noether symmetries) such as energy or momentum conservation, while others have an inherent topological aspect, such as magnetic charge. When Maxwell’s equations are discretized
on a mesh, a number of symmetries of the continuum theory are modified or even broken, like in any other model reduction. The most important task in numerical analysis is to develop compatible (mimetic) algebraic models that yield stable, accurate and physically consistent approximate solutions. Traditionally, finite element, finite volume and finite difference methods achieved necessary compatibility by following different paths that reflected their particular approaches to the discretization process. However, some conservation laws may be preserved on a discrete setting. This is because they often relate a quantity on certain region of space to an associated quantity on the boundary of the region.

Because the boundary is a topological invariant, such conservation laws should not depend on the metric of the space. A natural mathematical language that explores this aspect is the calculus of exterior differential forms and associated algebraic topological structures for its discrete counterpart [2, 3]. This geometry based calculus was developed and refined over the twentieth century to become the foundation of modern differential geometry. The calculus of differential forms allows one to express differential and integral equations on smooth and curved spaces in a consistent manner, while revealing the geometrical invariants at play. For example, the classical operations of gradient, divergence, and curl as well as the theorems of Green, Gauss and Stokes can all be expressed concisely in terms of differential forms and an operator on these forms called the exterior derivative - hinting at the generality of this approach. Compared to classical vectorial calculus, this exterior calculus has several advantages. First, it is often difficult to recognize the coordinate independent nature of quantities written in vectorial notation. Local and global invariants are hard to notice by just looking at the indices. On the other hand, invariants are easily discovered when expressed as differential forms by invoking either Stokes’ theorem, Poincare’s lemma, or by applying exterior differentiation. The large amount of our scientific wisdom relies on a deeply-rooted smooth (differential) comprehension of the world. The abstraction of differentiability allows scientists to model complex physical systems via concise sets of differential equations. The appearance of computers as an essential tool in scientific research has changed the very foundations of scientific modelling since abstraction of smoothness and differentiability are not consistent with a computer’s ability of storing only finite sets of numbers. In order to overcome this obstacle, a first set of computational techniques (e.g., finite difference or particle methods) focused on satisfying the continuous equations at a discrete set of spatial and temporal samples are devised. Alas, focusing on accurately discretizing the local laws often fails to respect important global structures and invariants. Afterward methods such as Finite Elements (FEM), based on the calculus of variations, remedied this defectiveness to some extent by satisfying local conservation laws on average and preserving some important invariants. Together with a much finer ability to deal with geometrically complicated boundaries, FEM became the de facto computational tool of choice for real world problems.

2 Electromagnetism and Differential Forms

The use of the calculus of differential forms in electromagnetism to represent the local field variables has been explored in many important papers and texts, including Deschamps [4], and Burke [5], Warnick [6]. Cartan [7] and others developed the calculus of differential forms in the early 1900’s. Since Cartan’s time, the use of forms has spread to many fields of pure and applied mathematics, from differential topology to the theory of differential equations. Differential forms are used in mechanics, thermodynamics, general relativity, quantum field theory, as well as in electromagnetism. As a language for studying electromagnetism, differential forms offer several important advantages over vector analysis. Vector analysis allows only two types of quantities: scalar fields and vector fields. In a three-dimensional space, differential forms of four different types are available. This allows flux density and field intensity to have distinct mathematical expressions and graphical representations, providing the mental pictures that clearly reveal the different properties of each type of quantity. The basic derivative operators of vector analysis are the gradient, curl, and divergence become special cases of a single operator, the exterior derivative. The Stokes theorem and the Green's divergence theorem have an obvious connection in that they relate integrals over a boundary to integrals over the region inside the boundary, but in the language of vector analysis they appear very different. These theorems are special cases of the generalized Stokes theorem for differential forms, which also
has a simple graphical interpretation.

2.1 Integral and Differential Forms
In order to make this chapter self-sufficient some basic concept from differential geometry necessary for further exposition will be reviewed (in an informal manner, but it will suffice for the purpose of this chapter). First, it is necessary to define framework in which to model electromagnetic phenomena, i.e. the model of the physical space. The three-dimensional vector space (R^3 for example) is not convenient choice since adding of points in physical space and multiplying them by scalars make no sense. Also, the idea of privileged point like origin O (0, 0, 0) has no physical counterpart. It is much better to start with the concept of affine space. Let A be a set with members P called points, V an associated vector space, and + an operation defined by +: A × V→A such that (V, +) acts on A as an Abelian group of translations (with the meaning that if points P_1, P_2 ∈ A the unique x exists in V such that P_2 = P_1 + x ). Then the triple (A, V, +) is called affine point space associated with vector space V, or short affine space. Affine space inherits the dimension from the associated vector space. The affine space A(R^1) = (A, R^1, +) is a suitable arena for electromagnetic fields since in it is possible to deal with points, curves and volumes, which we will denote as space elements, in an intrinsic manner independent of a specific choice of coordinate system. On the other hand it is possible to establish an isomorphism between vector space R^3 and physical space A by designating an arbitrary chosen point O as origin, and associating to any point P∈ A a spatial vector r_P such that P = O + r_P. Spatial vectors are usually represented in three dimensions by directed line segments, and can in turn be associated with triples of real numbers \( r_P \rightarrow (x^1_P, x^2_P, x^3_P) \), the coordinates of point P. Another important concept is differentiable manifold. We shall not give the formal definition of this concept here; rather we shall use this word as a generic term for lines, surfaces, or regions in an affine space (manifolds of order p = 1, 2, 3 respectively). A 0-manifold is a collection of isolated points. Euclidian space R^n, the set of n- tuples of real numbers endowed with the scalar product, metric and the right-screw orientation (remember that orientation of the vector space is determined by the order of its basis vectors), is an affine manifold also. Locally manifolds look like Euclidian space and general manifolds can be built by "patching" sets of locally Euclidian regions. In differential geometry, one can attach to every point P of a differentiable manifold M a tangent space T_M(P), a real vector space which intuitively contains the possible "directions" in which one can pass through P. The elements of the tangent space are called tangent vectors at P. The pairs (P, x) where and P ∈ M and x ∈ T_M(P) are known as bounded vectors. Physical vector fields are usually defined as mappings from manifolds to its tangent spaces. All the tangent spaces have the same dimension, equal to the dimension of the manifold. All the tangent spaces can be "glued" together to form a new differentiable manifold of twice the dimension, the tangent bundle T_M = \( \bigcup_{P \in M} T_M(P) \) of the manifold M.

Also, one can attach to every point P of a differentiable manifold a vector space of linear functionals on T_M(P) called the cotangent space T^*_M(P) at P. That is, every element \( \varphi \in T^*_M(P) \) is a linear map: \( \varphi \in T^*_M(P) \rightarrow R \). The cotangent space is defined as the dual space of the tangent space at P: T^*_M(P) = (T_M(P))^*. The elements of the cotangent space are called tangent co - vectors. The space of multi - linear mappings \( T_M(P) \times \ldots \times T_M(P) \rightarrow R \) we shall denote with \( \Lambda^l(T^*_M(P)) \). An integral form \( \omega \) of degree \( l \) (0 ≤ l ≤ n), n ∈ N ) on a piecewise smooth n - dimensional manifold M is a continuous additive mapping from the set \( S_l(M) \) of compact, oriented, piecewise smooth, l-dimensional sub-manifolds of M into real numbers. Thus, an integral form is a mapping from a space element to a real number. The integral forms defined on manifold M make vector space \( \Lambda^l(M) \). In this definition, word additive mean that the integral form assigns the sum of the respective numbers to the union of disjoint sub-manifolds. Changing the orientation of a sub-manifold changes the sign of the assigned real number. This definition includes continuous functions as the special case of 0-forms. The evaluation of the integral form \( \omega(\Sigma) \), \( \omega \in \Lambda^l(M) \), \( \Sigma \in S_l(M) \), stands for 'integrating \( \omega \) over \( \Sigma \)', or symbolically \( \int_{\Sigma} \omega \). If one looks carefully at the integral form of Maxwell's equations (9 - 12), it is obvious that E and H can be identified as integral 1 - form, while J, B and D should be treated as integral 2 - forms. The local variables
(the values of the fields at particular points in the space) have no sense in the above interpretation. A meaningful transition to the local picture can be performed with the help of the virtual work principle recalling (virtual) measurement procedure. For example, the electric field at the point \( x \in \mathbb{R}^3 \), \( E(x) \), can be measured through the infinitesimal work \( \delta w \) that is necessary to move a test charge \( q \) by \( \delta x \). 

\[ \delta w = qE(x) \cdot \delta x. \quad (3) \]

The magnetic flux density \( B(x) \) can be measured through the Lorenz force, so the work \( \delta w \) is:

\[ \delta w = q(B(x) \times v) \cdot \delta x, \quad (4) \]

where \( v \) is the velocity of the test charge, and \( \times \) is usual cross product in \( \mathbb{R}^3 \). From this perspective \( E \) and \( B \) are continuous differential forms of degree 1 and 2, respectively, in accordance with the definition:

A continuous differential form \( \omega^l(P) \), on a smooth \( n \)-manifold \( M \) of degree \( \Lambda^l \) (\( 0 \leq l \leq n \), \( n \in \mathbb{N} \)), is a continuous mapping assigning to each \( P \in M \) an element of the space \( \Lambda^l(T_M(P)) \). So, a differential form is a mapping from a point to a real number:

\[ \omega^l(P) : P \rightarrow TM(P) \times \ldots \times TM(P) \rightarrow \mathbb{R}. \quad (5) \]

These mappings form vector space \( \text{DF}^l(M) \).

Integrating differential forms is a simple matter since they are geometrical objects which are supposed to be integrated. Any piecewise smooth oriented manifold can be covered and approximated arbitrarily well by tiny flat "tangential" tiles of sub - manifolds \( \Sigma = \bigcup_i \Sigma_i \).

Let \( P_i \in \Sigma_i \) and

\[ \langle \omega(P_i) \rangle_{i} = P_i \rightarrow T_{\Sigma_i} \big( P_i \big) \times \ldots \times T_{\Sigma_i} \big( P_i \big) \rightarrow \mathbb{R}. \quad (6) \]

be the real numbers obtained as result of acting of \( \omega^l(P_i) \) on \( \Sigma_i \), as it is explained in details in [2]. Through the Riemann summation every differential l-form \( \omega^l(P_i) \), \( P \in \Sigma \) spawns an integral l-form \( \omega(\Sigma) \)

\[ \omega(\Sigma) = \sum_i \langle \omega(P_i) \rangle_{i}. \quad (7) \]

It is important to notice that in the integral sum (7) there is no integration measure since it is already included in \( \langle \omega(P_i) \rangle_{i} \) implicitly. This relation defines a close link between integral and differential forms. If \( M = \Lambda (\mathbb{R}^3) \) the tangent space for all \( P \in M \) is \( T_M(P) = \mathbb{R}^3 \), and it can be endowed with the structure of Euclidian space (scalar product, metric and orientation can be defined). Then the isomorphisms \( Y_i \) between differential l-forms and the objects from classical vector calculus (their vector proxies [2]), continuous scalar and vector functions \( u(x) \) and \( u(x) \), can be established:

\[ \int_{x} \omega^o \left( \left[ \begin{array}[]{c} Y_0 \omega^o \end{array} \right](x) \right) \equiv \left[ \begin{array}[]{c} u \end{array} \right](x). \quad (8) \]

\[ \int_{L} \omega^l \left( \left[ \begin{array}[]{c} Y_0 \omega^l \end{array} \right](t) \right) dL \equiv \left[ \begin{array}[]{c} u \end{array} \right](x) \int_{L} dt, \quad (9) \]

\[ \int_{S} \omega^2 \left( \left[ \begin{array}[]{c} Y_0 \omega^2 \end{array} \right](n) \right) dS \equiv \left[ \begin{array}[]{c} u \end{array} \right](x) \int_{S} dn, \quad (10) \]

\[ \int_{V} \omega^3 \left( \left[ \begin{array}[]{c} Y_0 \omega^3 \end{array} \right](n) \right) dV \equiv \left[ \begin{array}[]{c} u \end{array} \right](x) \int_{V} dV. \quad (11) \]

Here with symbols \( \partial S \) and \( \partial V \) borders of space elements (surface \( S \) and volume \( V \)) are denoted, \( t \) is a unit vector tangential to oriented curve \( L \) (or \( \partial S \)), and \( n \) is a unit vector normal to oriented surface \( S \) (or \( \partial V \)). It is supposed that the orientations of the space elements and its borders are consistent, of course.

3 On modelling tokamak plasmas in a computational electromagnetic environment

Next to the experimental approach of phenomena concerning plasma and plasma facing components of a tokamak another approach was adopted at this paper, the numerical modelling one. The study was done using CARIDDI, which is a 3D integral code for the eddy current density \( J \), the problem is solved by computing at every edge of the finite element mesh the projection of a vector, the current vector potential \( T \), whose uniqueness can be demonstrated mathematically by appropriately using the notions of tree and co - tree formed by branches of a graph, according to graph theory. Plasma is considered being a toroidal conductor with an elliptical or rectangular cross - section. The section is sub - divided into elements and the plasma current is carried in each single element by a filament, placed in its center.

3.1 Introduction

The Controlled Thermonuclear Fusion technology advances rapidly during the last years. More precisely: The progress in the design and R&D activity undertaken in the Russian Federation in the period
after completion of the ITER EDA is considerable. For the last 3 years, since 2001 FDR, the ITER design has been evolved in details to resolve issues, to reduce costs, to define details and to prepare the procurement specifications for long-term items. During the ITER Transitional Arrangements (ITA) phase (2003–2004) the Design and R&D activity in the Russian Federation has been redirected and mainly concentrated on items in accordance with the prospective procurement allocation. During the last 2 years 33 Task Agreements were allocated to the Russian Federation Participant Team (RF PT). The list of Tasks includes the qualification of Nb₃Sn strands, manufacturing of NbTi cable for the PF (poloidal field) insert coil, qualification tasks for the FW (first wall) and PFC/Divertor and design and manufacturing studies related to various areas: magnets, vacuum vessel, blanket, divertor, plasma – facing components, assembly, thermal shield, cryoplant, nuclear analysis, materials and safety [8].

The experimental study of liquid metals (Ga, Li) as tokamak Plasma Facing Component (PFC) was undertaken in Russian T-3M and T-11M tokamaks (Iₚ ≤ 100 k A, B₉ ≈ 1 T). In T-3M droplet stream and film flow Ga limiters were tested. In T-11M the experiments with Li capillary pore systems (CPS) as rail limiter for investigation of real Li losses in tokamak boundary condition were performed. It was shown, that a liquid metal (Ga, Li) PFC can be used in tokamak as droplet and CPS structures. The main channel of lithium erosion looks like, as ion sputtering. The motion towards the tokamak reactor with Li PFC seems possible and has no serious physical obstacles [9].

Morphology of carbon plasma facing components retrieved from the TEXTOR tokamak after long operation periods and exposure to total particle doses exceeding 7 × 10²⁶ m⁻² was determined too. Emphasis was on the composition and structure of the erosion zones. Tiles from two limiters—the main toroidal belt pump ALT-II and auxiliary inner bumper—were examined using high-resolution microscopy, surface profilometry, ion beam analysis techniques and energy dispersive X-ray spectroscopy. The essence of results regarding the net-erosion zones is following: (i) microstructure of surfaces is significantly smoother than on a non-exposed graphite, whereas carbon fibre composites show similar appearance prior to the exposure and after; (ii) deuterium retention is 2–5 × 10¹¹ m⁻²; (iii) the presence of plasma impurity atoms (e.g. metals) is detected predominantly in small cavities acting as local shadowed areas on the surface. The results are discussed in terms of processes of material erosion/re-deposition and tokamak operation conditions influencing the morphology of wall components [10].

On the other hand, the development, design, manufacture and testing of actively cooled high heat flux (HHF) plasma facing components (PFCs) has been an essential part of the Tore Supra programme towards long powerful tokamak operation. The Tore Supra PFC programme has culminated in the installation and operation of a toroidal pump limiter, since 2002, which already allowed to reach new world records in steady state operation (1 GJ injected in 6 min discharge). The HHF PFCs development and manufacturing was achieved through a long lead development and industrialisation programme (about 10 years) marked out with a number of challenges. The major technical topics cope with bonding technology analysis involving an adequate material selection and procurement, repair processes development and implementation, development of destructive and non-destructive testing methods, and more generally industrialisation assessment. All these lessons are relevant to the ITER divertor PFCs manufacturing, although the technical solution adopted for Tore Supra (flat tiles concept) is different from the one proposed for the ITER divertor (monoblock concept).

The routine operation of the actively cooled toroidal pumped limiter (TPL), capable to sustain up to 10 MW m⁻² of nominal convected heat flux, was obtained. Up to now, the limiter of Tore Supra fulfills its objectives in terms of heat exhaust. However, the thermo - graphic monitoring exhibits unexpected behaviour of the surface temperature. Particle exhaust control displays a complex pattern, due to the high fraction of the injected deuterium, which remains in the wall. The first experimental results with a full actively cooled wall gives access to ITER relevant information on wall conditioning, hydrogen plasma density and vacuum vessel inventory control, carbon erosion and redeposition and capability of in situ monitoring in a completely actively cooled environment [11].

Last but not least the Mega Amp Spherical Tokamak (MAST) at Culham is one of the leading world machines studying the spherical tokamak (ST) concept. At the time of the initial construction in 1998 little was known about the sort of divertor structures that would be required
in an ST. The machine was therefore provided with relatively rudimentary structures that were designed mostly to protect important components from the hot plasma. While these have served the machine well it was accepted that they might not be suitable when operating MAST to its full potential. The years of experience of operating MAST have led to the design, manufacture and now installation of a new divertor, the MAST improved divertor (MID), that should be able to cope with the full performance of the machine. The design is based on imbricated (fan-shaped) disks of tiles at the top and bottom of the machine for the outer strike points, giving an excellent compromise between power handling and diagnostic access, with substantial new centre column strike point armour and a shaped plate in between. High purity graphite is chosen as the plasma facing material in preference to CFC since in this case it has a better balance of performance and cost. The lower imbricated disk is insulated in alternate sectors for studies of divertor biasing and extensive diagnostics and additional inboard gas injection are included [12].

3.2 Models describing plasma behavior in a tokamak device

Sophisticated computer modeling must be undertaken taking into account the whole spectrum of aspects of plasma modeling in tokamaks: MagnetoHydroDynamic (MHD) description of plasma interacting with the plasma facing components (pfc) of a tokamak structure. However, a cornerstone on developing such an approach is a correct, although simplified, description of plasma interacting with the plasma facing material in preference to CFC since in this case it has a better balance of performance and cost. The lower imbricated disk is insulated in alternate sectors for studies of divertor biasing and extensive diagnostics and additional inboard gas injection are included [12].

When a dynamic disruption is considered, CARIDDI enables the user to adequately model a vertical displacement of the plasma column by defining a set of axi-symmetric (toroidal) conductors one above the other. The current switches on in one conductor, only when it has switched off in the preceding one (this simulates a current moving up or down). The (rectangular) section of the various conductors can be changed. If plasma current decays, (this is the case during VDEs), the cross-section of the (rectangular) conductors must be reduced by the user. Results presented in the next paragraph are obtained by using the code CARIDDI as modeling tool.

3.3 Modeling eddy currents in ITER’s blanket Plasma in next fusion reactors is foreseen, to be surrounded by a blanket.

For the ITER case, first wall (the blanket’s part facing plasma) and shield-blanket system will be divided into distinct inboard and outboard and subdivided toward the toroidal direction, so as to allow automatic installation and removal by using a robot, whose access to the vacuum vessel is foreseen, to be achieved by creating in the model vertical maintenance ports. The global structure has to be designed in such a way, that mechanical forces, appearing during normal operational conditions plus estimated global and local forces due to static disruptions and VDEs, be not excessive to withstand. Essentially, modeling the phenomenon in EM terms is based on the quasi-static simplification of Maxwell equations:

\[ \nabla \times E = -\frac{\partial B}{\partial t} \quad (13) \]

\[ \nabla \times H = J \quad (14) \]

\[ \nabla \times J = 0 \quad (15) \]

\[ \nabla \cdot B = 0 \quad (16) \]

combined with the constitutive equations:

\[ B = \mu H \quad (17) \]

\[ \frac{J}{\sigma} = E + E_{ext} \quad (18) \]

In these equations, \( J \) is the eddy current density, \( E \) the associated electric field intensity, \( T \) the eddy current vector potential, \( E_{ext} \) the externally applied electric field, which here is caused by the plasma current, \( B \) the magnetic flux density, \( H \) the magnetic field intensity, \( \mu \) the magnetic permeability and \( \sigma \) the electric conductivity of the
Detailed numerical formulation of CARIDDI is given in refs. [13, 14]. Briefly it can be said that, the CARIDDI formulation is based on the following hypotheses:

\[ \mu = \mu_0 \text{ in } \mathbb{R}^3 \]  
\[ \nabla J = 0 \]  
\[ n \cdot J = 0 \text{ on } \partial V_c \]  
\[ J = \sigma E \]  
\[ E = -\frac{\partial A}{\partial t} - \nabla \phi \]

By using the method of weighted residuals:

\[ \int_{V_c} (w(x) \left( \eta f(x, t) + \frac{\mu_0}{4\pi} \int_{V_c} \frac{J(x', t)}{|x-x'|} dV' + \frac{\partial A(x, t)}{\partial t} \right) dV = 0 \]

\[ \forall w \in S, J \in S \]
\[ S = \left\{ s / \int_{\Sigma} s \cdot n dS = 0 \text{ closed in } V_c \right\} \]

If \( J \) is approximated by a linear combination of shape functions \( J_k \in S \), the Galerkin method:

\[ w_i = J_k \]  
\[ L \frac{dI}{dt} + RI = V \]

with:

\[ L_{ik} = \frac{\mu_0}{4\pi} \int_{V_c} \int_{V_c} \frac{J_i(x) \cdot J_k(x')}{|x-x'|} dV dV' \]

full matrix:
\[ R_k = \int_{V_c} J_i(x) \cdot \eta J_k(x) dV \]

sparse matrix and:
\[ V_i = \int_{V_c} J_i(x) \cdot \frac{\partial A(x, t)}{\partial t} dV \]

The solenoidality of \( J_k \) can be enforced by using the electric vector potential:

\[ J = \text{curl} \cdot T \]

The gauge enforces the uniqueness of \( T \):
\[ T \cdot w = 0 \]

### 3.4 Vector finite elements [15]

Several serious problems are encountered, when the usual node-based elements, obtained by interpolating the nodal values, are employed to represent vector electric or magnetic fields.

- First, the occurrence of nonphysical or so-called spurious solutions is observed, which is generally attributed to the lack of enforcement of the divergence condition.
- Second, the inconvenience of imposing the boundary conditions at material interfaces as well as at conducting surfaces is noted.
- Third, difficulty in treating conducting and dielectric edges and corners due to field singularities associated with these structures is noted.

Therefore, it is necessary to explore other possibilities or other approaches, beyond just modifications, to lead the finite element analysis of electromagnetics to a new era. Fortunately, a revolutionary approach has been discovered recently. This approach uses so-called vector basis or vector elements, which assign degrees of freedom to the edges rather than to the nodes of the elements. For this reason, they are also called edge elements. Although these types of elements were described by Whitney as early as 45 years ago, their use and importance in electromagnetics was not realized until recently. In the early 1980s, Nedelec discussed the construction of edge elements on tetrahedral and rectangular bricks. Bossavit and Verite applied tetrahedral edge elements to three-dimensional eddy-current problems. Hano, apparently in an independent attempt, introduced rectangular edge elements for the analysis of dielectric-loaded wave-guides. Mur and de Hoop considered the problem of electromagnetic fields in inhomogeneous media. Van Welij and Kameari, using hexahedral edge elements, considered the application of edge elements to eddy-current calculations further. More recently, Barton and Cendes employed the tetrahedral edge elements for three-dimensional magnetic field computations, while Crowley developed a more sophisticated element type, the so-called covariant projection elements, which permit elements with curved edges.

In all of these works, edge elements have been shown to be free of all the previously mentioned shortcomings, which have bothered electromagnetics researchers for years. As can be imagined, the importance of edge elements has quickly been realized, and consequently, extensive investigations as well as some successful
applications have been carried out within the past thirteen years. It is interesting to note, that vector basis functions of a similar nature were also used, independently, by Glisson and Wilton, Rao et al., and Schaubert et al. for electric field integral equation formulations of electromagnetic scattering, at around the same time that they were used in the finite element solution for the first time. In this section the edge elements in two and three dimensions are introduced, while their applications in integral eddy current formulations are described in the previous section.

3.5 Two-dimensional elements

At the beginning the two - dimensional formulations are presented, first for rectangular edge elements and then for triangular and quadrilateral edge elements. Although rectangular elements are restricted to a limited class of geometries, they are simple and therefore best for introducing the concept of the edge elements.

3.6 Rectangular elements

Consider the rectangular element given in Fig. 1, whose side length is $l^e_x$ in the $x$ - direction and $l^e_y$ in the $y$ - direction and whose centre is at $(x^e_C, y^e_C)$. If each side of the element is assigned a constant tangential field component, the field within the element can be expanded as:

$$E^e_x = \frac{1}{l^e_y} \left( y^e_C + \frac{l^e_y}{2} - y \right) E_{x1} + \frac{1}{l^e_x} \left( y - y^e_C + \frac{l^e_x}{2} \right) E_{x2} \quad (33)$$

$$E^e_y = \frac{1}{l^e_y} \left( x^e_C + \frac{l^e_y}{2} - x \right) E_{y1} + \frac{1}{l^e_x} \left( x - x^e_C + \frac{l^e_x}{2} \right) E_{y2} \quad (34)$$

From these expressions, it is not difficult to see that $E^e_{x1}$ represents the field, component $E_x$ along the edge segment $(1,2)$ and similarly, $E^e_{y2}$ is associated with $E_y$ along the edge segment $(4,3)$. Similar identifications can be made for $E^e_{y1}$ and $E^e_{y2}$. Now, if it is defined:

- the edge $(1,2)$ as edge 1,
- the edge $(4,3)$ as edge 2,
- the edge $(1,4)$ as edge 3 and
- the edge $(2,3)$ as edge 4 then (33) and (34) can be written as:

$$E^e = \sum_{i=1}^{4} N^e_i E^e_i \quad (35)$$

where $E^e_i$ denotes the tangential field along the $i^{th}$ edge and $N^e_i$ are the vector interpolation or basis functions given by:

$$N^e_1 = \frac{1}{\ell^e_y} \left( y^e_C + \frac{\ell^e_y}{2} - y \right) \hat{x} \quad (36)$$

$$N^e_2 = \frac{1}{\ell^e_y} \left( y - y^e_C + \frac{\ell^e_y}{2} \right) \hat{x} \quad (37)$$

$$N^e_3 = \frac{1}{\ell^e_x} \left( x^e_C + \frac{\ell^e_x}{2} - x \right) \hat{y} \quad (38)$$

$$N^e_4 = \frac{1}{\ell^e_x} \left( x - x^e_C + \frac{\ell^e_x}{2} \right) \hat{y} \quad (39)$$

An important feature of these basis functions is that $N^e_i$ has a tangential component only along the $i^{th}$ edge and none along all the other edges. Thus, the continuity of the tangential field across all element edges is guaranteed. Another unique feature of these functions is, that each satisfies the divergence condition $\nabla \cdot N^e_i = 0$ within the region of the element. Therefore, they are ideal for representing the vector fields in source free regions. Further, the curl of these can easily be found as:

$$\nabla \times N^e_1 = \frac{1}{\ell^e_y} \hat{z}, \nabla \times N^e_2 = -\frac{1}{\ell^e_y} \hat{z}$$

$$\nabla \times N^e_3 = \frac{1}{\ell^e_x} \hat{z}, \nabla \times N^e_4 = -\frac{1}{\ell^e_x} \hat{z}$$

which obviously are nonzero constants.

In passing, it is noted, that these vector basis functions were proposed in order to build tangential continuity into the field representation. By taking the cross product of $\hat{z}$ with these
functions, it is obtained another set of vector basis functions, \( \hat{z} \times N_i^e \), which guarantee normal continuity. In contrast to \( N_i^e \), the functions \( \hat{z} \times N_i^e \) have zero curl and nonzero divergence and they are ideal for representing surface current densities. In electromagnetics, these are known as rooftop basis functions and have been employed extensively in the moment method solution of integral equations.

### 3.7 Triangular Elements

As mentioned above, the major disadvantage of the rectangular elements is, that they are restricted to a limited class of geometries. When one deals with problems having irregular geometries, it is necessary to use triangular elements and for this reason, the formulation for the triangular edge elements is considered here.

Refer to the triangular element illustrated in Fig. 2.

![Triangular edge element](image)

Unlike the rectangular elements considered above, the edges of the triangular elements are not necessary parallel to the x or y-axis. Therefore, one cannot easily guess the form of the vector expansion functions as done for the rectangular elements. Instead one must derive them according to their properties. Here, a different approach is followed. First, the area coordinates are considered, \( (L_1^e, L_2^e, L_3^e) \). There are also the linear interpolation functions of the element. Consider the vector function given by:

\[
W_{12} = L_1^e \nabla L_2^e - L_2^e \nabla L_1^e
\]  

(40)

First, it is not difficult to see that:

\[
\nabla \cdot W_{12} = \nabla \cdot \left( L_1^e \nabla L_2^e \right) - \nabla \cdot \left( L_2^e \nabla L_1^e \right) = 0
\]  

(41)

and

\[
\nabla \times W_{12} = \nabla \times \left( L_1^e \nabla L_2^e \right) - \nabla \times \left( L_2^e \nabla L_1^e \right) = 2 L_1^e \times \nabla L_2^e
\]  

(42)

Second, let \( e_1 \) be the unit vector pointing from node 1 to node 2. Since \( L_1^e \) is a linear function, that varies from one at node 1 to zero at node 2 and \( L_2^e \) is a linear function that varies from one at node 2 to zero at node 1, it is valid that

\[
e_1 \cdot \nabla L_1^e = -\frac{1}{\ell_1^e} \quad \text{and} \quad e_1 \cdot \nabla L_2^e = \frac{1}{\ell_2^e}
\]

where \( \ell_1^e \) denotes the length of the edge connecting nodes 1 and 2. Therefore:

\[
e_1 \cdot W_{12} = \frac{L_1^e + L_2^e}{\ell_1^e} = \frac{1}{\ell_1^e} = e_1^e
\]  

(43)

or in other words, \( W_{12} \) has a constant tangential component along the edge \((1,2)\). Further, since \( L_1^e \) vanishes along the edge \((2,3)\) and \( L_2^e \) vanishes along the edge \((1,3)\), \( W_{12} \) has no tangential component along these two edges. Thus, \( W_{12} \) possesses all the necessary properties for being a vector basis function for the edge field associated with the edge segment \((1,2)\). If this edge is defined as edge 1, it is valid:

\[
N_1^e = W_{12} e_1^e = \left( L_1^e \nabla L_2^e - L_2^e \nabla L_1^e \right) e_1^e
\]  

(44)

where \( e_1^e \) is included to normalize \( N_1^e \) as well as to make it dimensionless. Similarly, it can be shown, that the appropriate vector basis functions for edges \((2,3)\) and \((3,1)\) are:

\[
N_2^e = W_{23} e_2^e = \left( L_2^e \nabla L_3^e - L_3^e \nabla L_2^e \right) e_2^e
\]  

(45)

\[
N_3^e = W_{31} e_3^e = \left( L_3^e \nabla L_1^e - L_1^e \nabla L_3^e \right) e_3^e
\]  

(46)

Therefore, the vector field within the element can be expanded as:

\[
E^e = \sum_{i=1}^{3} N_i^e E_i^e
\]  

(47)

where \( E_i^e \) denotes the tangential field along the \( i^{th} \) edge. Unlike the vector basis functions for the rectangular elements, vector plots of the vector basis functions for the triangular elements are not easy to imagine. In Fig. 3 the vector plots of these functions on a typical element are displayed.
Fig. 3: Vector basis functions for a triangular element. (a) $N_1^e$, (b) $N_2^e$, (c) $N_3^e$

As expected, a vector basis functions has a tangential component only at the associated edge. From these plots, one may get a strong impression that if a cylindrical coordinate system $(\rho, \phi)$ is defined, with its origin located at node 3 and its angular coordinate $\phi$ increasing as one moves from node 1 to node 2, then $N_1^e$ looks like:

$$N_1^e = \frac{\rho}{h_3^e} \hat{\phi}$$  \hspace{1cm} (48)

where $h_3^e$ denotes the perpendicular distance from node 3 to edge 1. The other two functions exhibit a similar characteristic. This expression is indeed true and can be verified through a coordinate transformation. Finally, similar to the rectangular elements, one can easily verify that $\hat{z} \times N_1^e$ are the basis functions that guarantee normal continuity. They have zero curl and nonzero divergence. In the cylindrical coordinate system defined above, $\hat{z} \times N_1^e$ can be written as:

$$\hat{z} \times N_1^e = -\frac{\rho}{h_3^e} \hat{\rho}$$  \hspace{1cm} (49)

These vector basis functions can be used in electromagnetics to represent surface currents on arbitrarily shaped conducting bodies.

### 3.8 Quadrilateral Elements

As pointed out previously, the major advantage of the triangular elements lies in their capability to model irregular geometries. However, given a specific area and the maximum length of element edges, triangular elements contain about one-third more edges (thus unknowns) than rectangular elements. This element, called a quadrilateral element, has four element edges and can be viewed as a distorted rectangle.

To construct vector basis functions for a quadrilateral element, first a coordinate transformation is introduced, that transforms a quadrilateral element in the $x y$-plane into a square element in a new $\xi \eta$-coordinate plane as illustrated in Fig. 4.
of the quadrilateral element:
\[ x = a + b \xi + c \eta + d \xi \eta \]
\[ y = a' + b' \xi + c' \eta + d' \xi \eta \]  \( (50) \)
This yields:
\[ x_1 = a - b - c + d \quad y_1 = a' - b' - c' + d' \]
\[ x_2 = a + b - c - d \quad y_2 = a' + b' - c' - d' \]
\[ x_3 = a + b + c + d \quad y_3 = a' + b' + c' + d' \]
\[ x_4 = a - b + c - d \quad y_4 = a' - b' + c' - d' \]
Solving for the unknown coefficients, then substituting them into (50), it is obtained:
\[ x = \sum_{i=1}^{4} N_i^e (\xi, \eta) x_i^e \]  \( (51) \)
\[ y = \sum_{i=1}^{4} N_i^e (\xi, \eta) y_i^e \]
where:
\[ N_1^e = \frac{1}{4} (1 - \xi)(1 - \eta) \]
\[ N_2^e = \frac{1}{4} (1 + \xi)(1 - \eta) \]
\[ N_3^e = \frac{1}{4} (1 + \xi)(1 + \eta) \]
\[ N_4^e = \frac{1}{4} (1 - \xi)(1 + \eta) \]
These can be written uniformly as:
\[ N_i^e = \frac{1}{4} (1 + \xi_i)(1 + \eta_i) \]
where \((\xi_i, \eta_i)\) denote the coordinates of the \(i\)th node. From (51), it can be conversely expressed \(\xi\) and \(\eta\) as a function of \(x\) and \(y\). However, this is found to be unnecessary because, as it will be seen, it is much simpler to express the basis functions and perform the required numerical integration in terms of \(\xi\) and \(\eta\) directly.
Now, if it is defined:
the edge connecting node 1 and node 2 as edge 1,
the edge (4,3) as edge 2,
the edge (1,4) as edge 3 and
the edge (2,3) as edge 4.
Further, \(\xi\) is considered as a function of \(x\) and \(y\).
At edge 1, \(\eta=1\) and from (51)) it is obtained:
\[ x = \frac{1}{2} (1 - \xi) x_1^e + (1 + \xi) x_2^e \]  \( (52) \)
Solving for \(\xi\), it is obtained:
\[ \xi = \frac{2x - x_1^e - x_2^e}{x_2^e - x_1^e} \]  \( (53) \)
Let \(s\) be the normalized distance measured along the edge from node 1 to node 2, thus
\[ x = (1 - s) x_1^e + sx_2^e \]. When this is substituted into (53), it is obtained:
\[ \xi = 2s - 1 \]
which indicates that \(\xi\) varies linearly along edge 1. Similarly, it can be shown, that \(\xi\) also varies linearly along edge 2. Since \(\xi\) is a constant along the other two edges, edges 3 and 4, the vector function obtained by taking the gradient of \(\xi\), that is, \(\nabla \xi\), will have a constant tangential component along edges 1 and 2 but no tangential component along edges 3 and 4. Therefore, it may be constructed the vector basis functions for edge 1 as:
\[ N_1^e = \frac{\ell_1^e}{4} (1 - \eta) \nabla \xi \]  \( (54) \)
and for edge 2 as:
\[ N_2^e = \frac{\ell_2^e}{4} (1 - \eta) \nabla \xi \]  \( (55) \)
where \(\ell_i^e\) denotes the length of edge \(i\). Obviously, basis functions so constructed satisfy the basic properties of the vector basis – a constant tangential component along the associated edge and no tangential component along the rest of the edges. Similarly, it can be constructed the vector basis function for edge 3 as:
\[ N_3^e = \frac{\ell_3^e}{4} (1 - \xi) \nabla \eta \]  \( (56) \)
and for edge 4 as:
\[ N_4^e = \frac{\ell_4^e}{4} (1 - \xi) \nabla \eta \]  \( (57) \)
The vector basis functions, constructed above, are displayed in Fig. 5.
Here, again, it is considered the simple rectangular brick elements first and then follows the more complicated tetrahedral and hexahedral elements.

3.9 Brick Elements
Consider a brick element given in Fig. 6.

Its side length is denoted by $\ell_x^e$, $\ell_y^e$ and $\ell_z^e$ in the x, y and z directions, respectively, and whose center is located at $(x_C^e, y_C^e, z_C^e)$. Similar to the two-dimensional case, by assigning a constant tangential field component to each edge of the element, the field components within the element can be expressed as:

$$E_x^e = \sum_{j=1}^{4} N_{xj}^e E_{xi}^e$$
$$E_y^e = \sum_{j=1}^{4} N_{yj}^e E_{yi}^e$$
$$E_z^e = \sum_{j=1}^{4} N_{zj}^e E_{zi}^e$$

(58)

The vector basis functions for the x, y and z directions can be constructed:

$$N_{x1}^e = \frac{1}{\ell_y^e \ell_z^e} \left( y_C^e + \frac{\ell_y^e}{2} - y \right) \left( z_C^e + \frac{\ell_z^e}{2} - z \right)$$
$$N_{x2}^e = \frac{1}{\ell_y^e \ell_z^e} \left( y - y_C^e + \frac{\ell_y^e}{2} \right) \left( z_C^e + \frac{\ell_z^e}{2} - z \right)$$
$$N_{x3}^e = \frac{1}{\ell_y^e \ell_z^e} \left( y_C^e + \frac{\ell_y^e}{2} - y \right) \left( z - z_C^e + \frac{\ell_z^e}{2} \right)$$
$$N_{x4}^e = \frac{1}{\ell_y^e \ell_z^e} \left( y - y_C^e + \frac{\ell_y^e}{2} \right) \left( z - z_C^e + \frac{\ell_z^e}{2} \right)$$

Three – dimensional edge elements
The formulation described above for the two-dimensional case can be extended to the three-dimensional case in a straightforward manner.
If the edge numbers are defined as in Table 1, the expansion can be written in vector notation as:

\[
E^e = \sum_{i=1}^{12} N^e_i E^e_i \tag{59}
\]

where:

\[
N^e_i = N^e_{ii}\hat{x} \quad N^e_{i1} = N^e_{i1}\hat{y} \quad N^e_{i2} = N^e_{i2}\hat{z}
\]

for \( i = 1, 2, 3, 4 \).

It is not difficult to see that the vector basis functions defined in (60) have a zero divergence, that is, \( \nabla \cdot N^e_i = 0 \), and a nonzero curl. It is also not difficult to see that the tangential fields at the edges that form the facet determine the tangential field at an element facet. Therefore, the expansion given in (59) not only guarantees the tangential continuity across the edges, but also guarantees the tangential continuity across the surfaces of the elements.

### Table 1: Edge Definition for a Rectangular Brick Element

<table>
<thead>
<tr>
<th>EDGE I</th>
<th>NODE ( i_1 )</th>
<th>NODE ( i_2 )</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>11</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

### 3.10 Tetrahedral Elements

Linear interpolation functions are derived for the tetrahedral elements, which can be denoted as \( \{L^e_1, L^e_2, L^e_3, L^e_4\} \). Similar to the triangular element, the vector function:

\[
W^e_{12} = L^e_1 \nabla L^e_2 - L^e_2 \nabla L^e_1
\]

is examined. First, it is easy to see that:

\[
\nabla \cdot W^e_{12} = 0 \quad \nabla \times W^e_{12} = 2\nabla L^e_1 \times \nabla L^e_2 \tag{61}
\]

Second, let \( e_1 \) be the unit vector pointing from node 1 to node 2 (Fig. 7).

Fig. 7: Tetrahedral element

Since \( L^e_i \) is a linear function that varies from one at node 1 to zero at node 2 and \( L^e_i \) is a linear function that varies from one at node 2 to zero at node 1, \( e_1 \cdot \nabla L^e_i = -\frac{1}{\ell_i^e} \) and \( e_1 \cdot \nabla L^e_i = -\frac{1}{\ell_i^e} \),

where \( \ell_i^e \) denotes the length of the edge connecting nodes 1 and 2. Therefore:
\[ e_i \cdot W_{12} = \frac{L_i^e + L_2^e}{\ell_1^e} = \frac{1}{\ell_1^e} \]  
(63)

which indicates that \( W_{12} \) has a constant tangential component along edge \((1,2)\). Further, since \( L_i^e \) vanishes along edges \((2,3), (2,4)\) and \((3,4)\) and \( L_2^e \) vanishes along edges \((1,3), (1,4)\) and \((3,4)\), \( W_{12} \) has no tangential component along these five edges. Furthermore, since \( L_1^e \) vanishes on the element facet defined by \((2,3,4)\) and \( L_2^e \) vanishes on the element facet defined by \((1,3,4)\), \( W_{12} \) has no tangential component on either of these facets as well. Its tangential component appears only on the element facets that contain the edge \((1,2)\).

Table 2: Edge Definition for a Tetrahedral Element

<table>
<thead>
<tr>
<th>EDGE I</th>
<th>NODE ( i_1 )</th>
<th>NODE ( i_2 )</th>
</tr>
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<tbody>
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</tbody>
</table>

That is, the element facets \((1,2,3)\) and \((1,2,4)\). Thus \( W_{12} \) possesses all the necessary properties to be a vector basis function for the edge field associated with the edge segment \((1,2)\). If this edge is defined as edge 1, it is obtained:

\[ N_i^e = W_{12} \ell_1^e = (L_i^e \nabla L_1^e - L_2^e \nabla L_1^e) \ell_1^e \]  
(64)

Similarly, the vector basis functions for edge 1 is obtained as:

\[ N_i^e = W_{i,rr} \ell_1^e = (L_i^e \nabla L_2^e - L_2^e \nabla L_1^e) \ell_1^e \]  
(65)

where the edge numbers and the associated nodes \( i_1 \) and \( i_2 \) are defined in Table 2.

Alternative expressions for \( N_i^e \) they are given by:

\[ N_i^e = \begin{cases} f_{\gamma i} + g_{\gamma i} \times r, & r \text{ within the tetrahedron} \\ 0, & \text{otherwise} \end{cases} \]  
(66)

with:

\[ f_{\gamma i} = \frac{\ell_{2,rr}^e r_{i,rr}^e}{6V^e}, g_{\gamma i} = \frac{\ell_{2,rr}^e r_{i,rr}^e - e_i}{6V^e} \]  
(67)

in which \( I=1, 2, 3, \ldots, 6, V^e \) is the volume of the tetrahedral element, \( e_i = \left| r_{i,rr}^e - r_{i,rr}^e \right| / \ell_1^e \) is the unit vector of the \( i^{th} \) edge, and \( \ell_1^e \) is the length of the \( i^{th} \) edge, with \( r_{i,rr}^e \) and \( r_{i,rr}^e \) denoting the location of the \( i_1 \) and \( i_2 \) nodes of the \( e^{th} \) element.

**3.11 Hexahedral Elements**

In addition to the rectangular brick and tetrahedral elements, another type of element worth mentioning is the hexahedral element, the so-called distorted brick element. Similar to quadrilateral elements in two dimensions, a hexahedral element in the \( x y z \) coordinate system can be transformed into a cubic element in a new \( \xi \eta \zeta \) coordinate system (Fig. 8).

![Hexahedral element in the xy coordinate system transformed into a cubic element in the \( \xi \eta \zeta \) coordinate system](image)

Fig. 8: Hexahedral element in the \( xyz \) coordinate system (a) transformed into a cubic element in the \( \xi \eta \zeta \) coordinate system (b).

The required transformation can easily be found to be:

\[ x = \sum_{i=1}^{8} N_i^e(\xi, \eta, \zeta)x_i^e \quad y = \sum_{i=1}^{8} N_i^e(\xi, \eta, \zeta)y_i^e \]  
(68)

where

\[ N_i^e(\xi, \eta, \zeta) = \frac{1}{8}(1 + \xi, \eta, \zeta)(1 + \eta, \eta, \zeta)(1 + \zeta, \zeta, \zeta) \]  
(69)

with \((\xi, \eta, \zeta)\) denoting the coordinates of the \( i^{th} \) node.

Since \( \xi \) varies linearly along the edges corresponding to those parallel to the \( \xi - \) axis and is a constant along the other edges, the vector function defined by \( \nabla \xi \) has nonzero tangential components only along those edges parallel to the \( \xi - \) axis. Therefore, the vector basis functions for those edges may be constructed as:

\[ N_i^e = \frac{\ell_1^e}{8}(1 + \eta, \eta, \zeta)\nabla \xi \]  
(70)

where \((\eta, \zeta)\) denotes the coordinate value of \((\eta, \zeta)\) at edge \( i \). Similarly, the vector basis functions for the edges parallel to the \( \eta - \) axis may be constructed as:

\[ N_i^e = \frac{\ell_1^e}{8}(1 + \xi, \xi, \zeta)\nabla \eta \]  
(71)

and for the edges parallel to the \( \zeta - \) axis as:
\[ N_e^c = \frac{\ell e}{8} (1 + \xi \eta \zeta)(1 + \zeta \xi \eta) \nabla \zeta \quad (72) \]

The vector basis functions, so constructed, possess all the properties that guarantee the tangential continuity across the element edges as well as across element surfaces. However, unlike the vector basis functions for the rectangular brick and tetrahedral elements, these are not divergence-free. Since the divergence is small for a slightly distorted element and large for a drastically distorted element, one should try to avoid the hexahedra, which deviate greatly in shape from a rectangular brick.

### 3.12 Modeling hypotheses and results

In a static disruption scenario, the plasma current decays to zero without any changes in the plasma ring position.

On the contrary, the plasma appears a vertical displacement and the plasma current may switch off slowly during the displacement, or very rapidly at the end of it, when a dynamic disruption occurs.

Simulation of a downward VDE was carried out, the current density being considered constant. This means, that in this case the user of the CARIDDI code was not constrained to change the cross-section of the conductors carrying plasma current at every step of the resolution phase. The plasma cross-section was considered to have a rectangular shape.

According to studies, carried out in the USA [16], the displacement of the plasma has been by an exponential movement:

\[ z = z_{eq} - z_0 e^{\gamma t} \]

in the ITER case the following data for constants being valid:

- \( z_{eq} = 1.30 \) m equilibrium position
- \( z_0 = 0.05 \) m initial offset
- \( \gamma = 150 \) s\(^{-1}\) growth rate.

Given a displacement time of 25 ms, the growth rate is chosen to have about 2 m of total vertical motion, which is compatible with ITER’s size. For the simulations, the following parameters have been taken:

- plasma current : 25 MA
- plasma center position \( r = 7.75 \) m in radial direction and \( Z = 1.3 \) m in vertical direction
- elliptical plasma: minor semi-axis at equilibrium \( a = 3.0 \) m
- major semi axis at equilibrium \( b = 4.7 \) m

section \( S = 44.27 \) m\(^2\)

picked \((\alpha = 3)\) or uniform \((\alpha = 0)\) radial plasma current density profile

- rectangular plasma: (dimensions at equilibrium)
  - \( D_x = 6.00 \) m; \( D_y = 9.40 \) m; \( D_x / D_y = \text{const.} = 1.6 \)
  - section \( S = 56.4 \) m\(^2\)
- flat radial plasma current density profile
- magnetic field at plasma center: 7.8 T

In the modeling of the blanket module, four materials have been considered: materials 1 and 2 for the BP, and materials 3 and 4 for the FW (see figs. 1 and 2). Materials 1 and 2 (BP) are stainless steel, with a resistivity of \( 1.0 \times 10^{-6} \) Ω m, whereas 3 and 4 present a higher resistivity, fixed to the value \( 1.5 \times 10^{-6} \) Ω m, because it is taken into account the segmented curved configuration of the FW.

The following reference disruption scenarios have been considered:

- model \( P1 \) : linear current decay in 25 ms for an elliptical cross-section plasma (slow current quench).
- model \( P2 \) : linear current decay in 25 ms for a rectangular cross-section plasma (slow current quench).
- model \( P3 \) : vertical displacement over 2 m in 25 ms (pure VDE).
- model \( P4 \) : vertical plasma motion of 2 m in 25 ms superposed to a slow current quench (1 MA/ms).
- model \( P5 \) : VDE (2 m in 25 ms) followed by a fast current quench (5 MA/ms).
- model \( P6 \) : linear current decay in 5 ms for an elliptical cross-section plasma (fast current quench).
- model \( P7 \) : linear current decay in 5 ms for a rectangular cross-section plasma (fast current quench).

Models \( P1, P2, P6 \) and \( P7 \) concern plasma disruptive instabilities known as static disruptions, whereas \( P3, P4 \) and \( P5 \) simulate plasma VDE accidental events known as dynamic disruptions.

Detailed results may be found in [17]. Some additional results are contained in figs. 9, 10 and 11 presenting ohmic power (P) dissipated and magnetic energy (W) stored in the blanket sector under study, due to eddy currents, for models \( P2, P3, P4, P5, P7. \)
Figure 9: Graphic comparison of Ohmic power measured in MW and magnetic energy measured in MJ for models P2 and P3 (time is measured in ms).

Figure 10: Graphic comparison of Ohmic power measured in MW and magnetic energy measured in MJ for models P2 and P4 (time is measured in ms).

Figure 11: Graphic comparison of Ohmic power measured in MW and magnetic energy measured in MJ for models P5 and P7 (time is measured in ms).

4 Conclusions

The results clearly show, that the effect of a pure VDE (model P3) is globally quite different from that of a pure (slow) current decay (model P2), having as consequence appearance of lower levels of ohmic power and magnetic energy (fig. 4). On the other hand, if models P2 and P4 are compared, it is observed, once again, that, as far as blanket is concerned, vertical displacement of plasma does not produce essentially new effects, due to the fact that the slow current quench accidental event is pre-dominant (fig. 5). When the current quench rate is increased, as it is done in models P5 and P7, higher levels of ohmic power are observed (fig. 6). However, a shift of the P and W patterns is noticed, when current decays at the end of VDEs (as it is apparent in model P5). This is an indication element, once more, showing that current quench is pre-dominant in these accidental scenarios too. Furthermore, higher values of P, when eddy currents increase, is an indication element showing that, in blankets, which are continuous, as far as it concerns the toroidal direction, resistive effects are much more
important than the inductive ones.

References: