Discrete Signal Induced Unitary Ttransforms

Artyom M. Grigoryan
University of Texas at San Antonio
Electrical and Computer Engineering
One UTSA Circle, TX 78249
USA

Merughan M. Grigoryan
Yerevan State University
Department of Radiophysics
1 Alex Manoogian Str., 375049 Yerevan
Armenia

Abstract: This paper describes a new class of unitary discrete transforms which are induced by input signals. These transforms have a simple form of composition by Givens rotations and fast algorithms for any length of processed signals. In the wavelet theory, the signal decomposition is performed by the complete set of scaled and slide versions of an analyzing function, or wavelet. When working with heap transforms, we achieve the decomposition of signals by a complete system of functions, which represent themselves a family of interactive waves moving in the field generated by the signal-generators. Such waves compose a complete and periodic system of states, when each waves passes special states of the system with different periods. Examples of basis functions of heap transforms, which we call *discrete signal-induced heap transforms, (DsiHT),* are described. *

Key–Words: Discrete transforms, wavelet transforms.

1 Introduction

In the wavelet theory, fully scalable modulated windows for frequency localization are used [1]-[4]. The window is sliding (or shifted in the discrete case), and the transform of a part of the signal is calculated for every position. The result of the wavelet transform is a collection of time-scaling representation of the signal with different resolutions. The wavelet transform $\mathcal{T}[f]$ of the function $f(t)$ is defined as the family of cross-correlations of the function with wavelet functions $a^{-1/2}\psi\left(t/a\right)$ scaled by the time transformation $t \rightarrow t/a$,

$$\mathcal{T} : f(t) \rightarrow T(a,b),$$

where $a > 0$, and $b \in (-\infty, +\infty)$. The parameter a is referred to as a *dilation* parameter and b as a *location* parameter. The function $f(t)$ can be recovered from its magnified wavelet transform $a^{-2}T(a,b)$ by integrating its values over all locations b and dilations a. The function is not a sum of moving but sliding (or shifted) and non-interacting waves.

We here begin the discussion of the discrete unitary heap transforms generated by vector-signals which represent themselves specific waves [5]. Matrices of heap transforms are completely defined by the generators and their unique motions in the space of functions. We stand only on the case, when each heap transform is defined by one signal-generator, and the heap is located at the first component of the transform.

*This work is supported by the National Science Foundation (NSF) under Grant 0551501.

2 Discrete Heap Transforms

In this section, we describe heap transforms in the framework of the moving waves. The row-vectors, or the basis functions of the transformation represent certain waves which are propagated in the ”field” which is associated with the generator [6]. We describe in detail the matrix of the heap transformation. The analytical formulas for calculating coefficients of the matrix are given in [5]. Those formulas express the coefficients through the correlations of the transformed signal with the generator.

The composition of the discrete heap transformation by a given vector-generator $\mathbf{x} = (x_0, x_1, x_2, ..., x_{N-1})'$ is performed by the sequential calculation of basic 2-D transforms, such as for instance, Givens rotations of vectors on the horizontal $Y = 0$. In matrix form, this composition can be written as $\mathbf{H} = \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\cdots\mathbf{T}_{N-1}$, where each transformation T_k, $k = 1 : (N-1)$, changes only two components of the input. Transformations T_k are parameterized, and values of their parameters are defined by the vector-generator. This is the main point, the generator $\mathbf{x}$ determines all transformations T_k which then are applying on an input $\mathbf{z}$.

Figure 1 shows the transform-network with the signal-flow graph of the transform of the signal $\mathbf{z} = (z_0, z_1, z_2, ..., z_{N-1})'$. The parameters (angles) of the transformation are generated by the signal-generator $\mathbf{x}$. In the 1st level and the kth stage of the flow-graph, the angle φ_k is calculated by inputs $\{x_0^{(k-1)}, x_k\}$, where $k = 1 : (N-1)$ and $x_0^{(0)} = x_0$. These angles are used in the incomplete transforms $T_k = T_{\varphi_k}$ to define the next components $x_0^{(k)}$, as well as to perform the transform of the input signal $\mathbf{z}$, in the 2nd level. The full graph itself represents a co-ordinated network of transformation of the vector $\mathbf{z}$, under the action on $\mathbf{x}$.

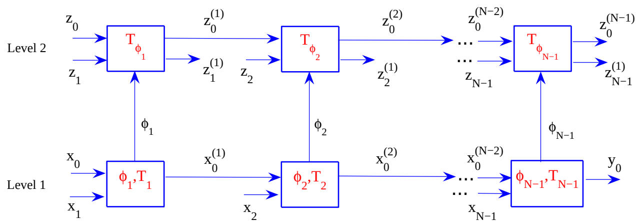

Figure 1: Network of the $\mathbf{x}$-induced heap transform of the signal $\mathbf{z}$.

Example 1: Let $\mathcal{H}$ be the heap transformation generated by the five-dimensional vector $\mathbf{x} = (1,2,3,2,1)'$. The generator itself is transformed into the scaled unit vector

$$\begin{aligned} H(\mathbf{x}) = ||\mathbf{x}||\mathbf{e}_1 &= (||\mathbf{x}||, 0, 0, ..., 0)' \\ &= (\sqrt{19}, 0, 0, ..., 0)'. \end{aligned} \tag{1}$$

The matrix $\mathbf{H}$ of the heap transform generated by this vector can be represented as the product of two matrices $\mathbf{D}^{-1}\mathbf{M}$,

$$\mathbf{H} = \mathbf{D}^{-1} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ -2 & 1 & 0 & 0 & 0 \\ -1 & -2 & \frac{5}{3} & 0 & 0 \\ -1 & -2 & -3 & 7 & 0 \\ -1 & -2 & -3 & -2 & 18 \end{bmatrix}, \tag{2}$$

where

$$\mathbf{D} = diag\{4.3589, 2.2361, 2.7889, 7.9373, 18.4932\},$$

and the angels of rotations equal $\varphi_1 = -63.43°$, $\varphi_2 = -53.30°$, $\varphi_3 = -28.13°$, and $\varphi_4 = -13.26°$.

The five basis functions of this heap transformation are shown in Fig. 2 in part a, along with the basis functions of another heap transformation generated by the vector $(1,2,3,4,5)'$ in b. The first three basis functions are the same, since the vector-generators coincide at three first points. We can see that the basis functions represents themselves the moving waves that change in their movement from the left to right. The first basis function $h_1(n)$ of the DsiHT coincides with the generator, up to the normalized coefficient $||\mathbf{x}||$. This property holds for the considered above examples, as well as in the general $N > 1$ case.

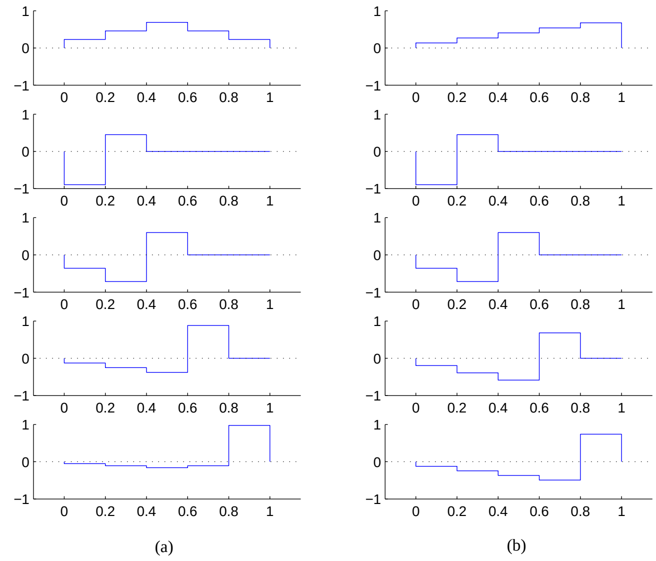

Figure 2: Basis functions of the five-point DsiHTs generated by the vectors (a) $(1,2,3,2,1)'$ and (b) $(1,2,3,4,5)'$.

There are three stages which can be separated during the process of motion and transformation of one basis function into another one, when starting from the wave-generator. In the first stage, *the statical stage*, the generator itself is lying as the basis function. The second stage, *the evolution stage*, is related to the formation of a new wave. The last stage is *the dynamical stage*, when the new established wave is moving to the end of the path. This wave is composed by two parts, the first part resembles the generator and the second part, or a splash, is a static wave increasing by amplitude. If we consider the normalized coefficients, then the amplitude of the mentioned splash tends to 1 when the generator vanishes in amplitude. This can be seen better, when analyzing examples with large N. For instance, in the $N = 8$ case, when $\mathbf{x} = (1,1,1,1,-1,-1,-1,-1)'$, the integer matrix of the heap transformation, $\mathbf{H} = \mathbf{D}^{-1}\mathbf{M}$, generated by this vector equals

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & 5 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & 6 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 7 \end{bmatrix}. \tag{3}$$

The eight normalized waves composing the matrix $\mathbf{H}$ of the eight-point heap transformation are shown in Fig. 3. Part a of this figure shows the generator in its statical stage. The evaluation stage is defined from (b) through (e). The following parts show the process of reconstruction of the generator (but in small scales) and the simultaneous process of development of the splash. In general case of generators composed

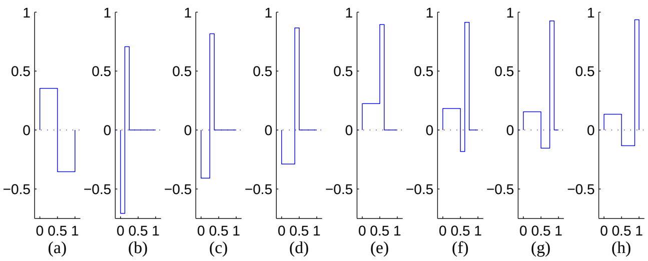

Figure 3: Eight normalized basis functions of the DsiHT generated by the vector $(1,1,1,1,-1,-1,-1,-1)'$.

by units, the waves can be described as compositions of cyclicly shifted versions of two elementary waves

$$\begin{aligned} f_0(t) &= [1,1,0,...,0] \\ m_0(t) &= [1,-1,0,...,0] \end{aligned} \qquad (4)$$

which are step functions with the step $\Delta = 1/N$, where N is the dimension of the vectors. The wave $m_0(t)$ describes the process of division and separation from the wave $f_0(t)$ and future motion following by interactions with its previous states. In general, the wave $m_0(t)$ together with $f_0(t)$ participates in this process. For the above example, the interaction of the elementary waves m_0 and f_0 in the entire process of composition of these row-waves can be described as shown in Table 1. In the first column, the coefficients c_k of the linear combination of waves $m_0(t)$ with $f_0(t)$ are given. In the evaluation stage, the wave

c_t	1	1	1	1	−1	−1	−1	−1	t
±1	−1	1							1
±2		−1	1						2
±3			−1	1					3
4				1	1				4
5					−1	1			5
6						−1	1		6
7							−1	1	7
	1	1	1	1	−1	−1	−1	7	

Table 1: Composition of waves for the matrix $\mathbf{M}$.

$m_0(t)$ is moving and interacting with previous waves m_t, when passing states $t = 1,2,$ and 3. For these states, the coefficients $c_t = t$. In the last state of evaluation, $t = 4$, the wave m_3 in rotated (i.e. all coefficients c_1, c_2, and c_3 change the signs) and the wave $f_0(t)$ enters into the interaction with wave $m_3(t)$ as

$$m_4(t) = -m_3(t) + 4f_0(t-3\Delta).$$

This is the break point of the motion, which occurs because of the change of the sign in the generator. After this point, the wave $m_0(t)$ again as before is moving and interacting with waves obtained on previous stages. The motion and interaction of row-waves m_k can be described as follows:

$$\begin{aligned} m_k(t) &= \alpha m_{k-1}(t) + kw(t-(k-1)\Delta), \\ \alpha &= (-1)^{h_k/h_{k-1}}, \quad k = 2:7, \end{aligned} \qquad (5)$$

where $\Delta = 1/8$ and the wave $w(t) = m_0(t)$ or $f_0(t)$, depending on state number k. The wave of the system at each state is defined by the previous wave plus the narrow radiated wave. This second wave is of two types and its amplitude increases linearly.

The mentioned above three stages in composition of the basis waves of the heap transformation can be observed for any vector generator. As an example, Fig. 4 shows a signal $\mathbf{x}$ composed by 127 random numbers in the interval $[0,1]$ in part a, along with a few normalized basis functions of the 127-point heap transformation generated by $\mathbf{x}$. Namely, seven waves with numbers $4, 11, 20, 40, 60, 100$, and 107 have been plotted together.

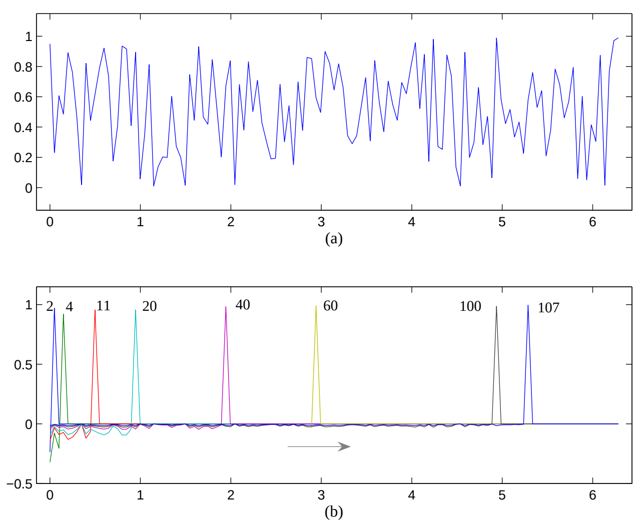

Figure 4: (a) The random signal-generator, and (b) eight basis functions with numbers $2, 4, 11, 20, 40, 60, 100,$ and 107 (from left to right) of the heap transformation generated by the random signal.

Analyzing matrices of the heap transformations, one can notice that the last row coincides with the first one at each point, except the last one, where the value $(N-1)$ is written. We call the vector composed by this row *the dual vector*. Thus, $m_{N-1}(n) = m_1(n), \quad n = 0:(N-2)$. The coincidence is within the sign. This property, or $m_{N-1}(n) = -m_1(n)$, holds for the heap transformation generated by any vector $\mathbf{x}$. Therefore, the first and last basis functions of the heap transformation are similar, i.e.

$$h_{N-1}(n) = \alpha^{-1} h_1(n), \quad n = 0:(N-2), \qquad (6)$$

where $\alpha = h_1(0)/h_{N-1}(0)$ is the coefficient of similarity (which equals $\sqrt{N-1}$ for considered above examples). The information about the generator is lies completely in the first row of the heap transformation, as well as in the last one. This property also tells us the following. If we consider the new heap transformation H_2 generated by the dual-vector h_{N-1} (or m_{N-1}), then the matrix of this transformation will be similar to the original heap transformation H generated by the vector-generator h_1 (or m_1).

3 Heap transform: System of states

We here briefly describe the full circle of the movement of the basis-waves of the heap transform; each of these waves passes special states of the system with a different period. To illustrate this process, we consider the $N = 5$ case. Let x be the generator $\mathbf{x} = (1,1,2,3,4)'$, and let $\mathcal{H}$ be the heap transformation generated by $\mathbf{x}$. The matrix of this transform can be described in the form $\mathbf{H} = \mathbf{D}^{-1}\mathbf{M}$, where the matrix $\mathbf{M}$ is almost integer matrix

$$\mathbf{M} = \mathbf{M}_{[1,1,2,3,4]} = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & -2 & 0 \\ 1 & 1 & 2 & 3 & -3.75 \end{bmatrix}$$

and $\mathbf{D}$ is the following diagonal matrix

$$\text{diag}\{5.5678, -1.4142, -1.7321, -3.1623, -5.3910\}.$$

The dual vector

$$\hat{\mathbf{x}} = (-0.1855, -0.1855, -0.3710, -0.5565, 0.6956)'$$

generates the same heap transformation as the last row of the matrix $\mathbf{M}$. Therefore, we consider $\hat{\mathbf{x}} = (1,1,2,3,-3.75)'$. The heap transformation generated by the dual vector has the matrix $\hat{\mathbf{H}}$, which can be written in the form $\hat{\mathbf{H}} = \hat{\mathbf{D}}^{-1}\hat{\mathbf{M}}$, where

$$\hat{\mathbf{M}} = \mathbf{M}_{[1,1,2,3,3.75]} = \begin{bmatrix} 1 & 1 & 2 & 3 & -3.75 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & -2 & 0 \\ 1 & 1 & 2 & 3 & 4 \end{bmatrix}$$

and the diagonal matrix

$$\begin{aligned} \hat{\mathbf{D}} &= \mathbf{D}_{[1,1,2,3,-3.75]} \\ &= \text{diag}\{-5.3910, -1.4142, \\ &\qquad -1.7321, -3.1623, 5.5678\}. \end{aligned}$$

coincides with $\mathbf{D}$, except the first and last coefficients which are replaced in these matrices.

We now compose from these two matrices of the heap transformations the following block matrix:

$$\mathbf{S} = \begin{bmatrix} \mathbf{M} \\ \hat{\mathbf{M}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & -2 & 0 \\ 1 & 1 & 2 & 3 & -3.75 \\ 1 & 1 & 2 & 3 & -3.75 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & -2 & 0 \\ 1 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

One dual vector can be removed from this matrix. Considering the obtained block matrix as a system of waves which are referred to as waves moving in the field generated by $\mathbf{x}$, we obtain the following picture of this dynamic system in time:

$$\begin{array}{ll} t=0: & [1 \quad 1 \quad 2 \quad 3 \quad 4] \\ t=1: & [1 \quad -1 \quad 0 \quad 0 \quad 0] \\ t=2: & [1 \quad 1 \quad -1 \quad 0 \quad 0] \\ t=3: & [1 \quad 1 \quad 2 \quad -2 \quad 0] \\ t=4: & [1 \quad 1 \quad 2 \quad 3 \quad -3.75] \\ t=5: & [1 \quad -1 \quad 0 \quad 0 \quad 0] \\ t=6: & [1 \quad 1 \quad -1 \quad 0 \quad 0] \\ t=7: & [1 \quad 1 \quad 2 \quad -2 \quad 0] \\ t=8: & [1 \quad 1 \quad 2 \quad 3 \quad 4] \end{array} \qquad (7)$$

The first state at time $t = 0$ and the last state at time $t = 8$ coincides, and we can write that $s(t) = s(t+8)$, and the period of motion is $p = 8$. Thus, the vector-generator and dual vector induce a periodical process in time with period p. In other words, in that time the system returns to its original state. In general, this period is equal to $p = 2N-2$.

Let us consider now the system generated by the second row of the heap transformation, $m_1 = (1,-1,0,0,0)$, which is considered as the two-dimensional vector $m_1 = (1,-1)$. The heap transformation generated by this vector and the following heap transformations generated by dual vectors can be considered together and in the form of transitions in one circle as follows:

$$\mathbf{H}_{[1,-1]} \to \hat{\mathbf{H}}_{[1,1]} \to \hat{\mathbf{H}}_{[-1,1]} \to \hat{\mathbf{H}}_{[-1,-1]} \to \mathbf{H}_{[1,-1]} \tag{8}$$

where these matrices equal

$$\mathbf{H}_{[1,-1]} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} = -\mathbf{H}_{[-1,1]},$$
$$\hat{\mathbf{H}}_{[1,1]} = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} = -\mathbf{H}_{[-1,-1]}.$$

Using the integer matrices corresponding to these heap matrices, we can describe this process of transitions by the following diagram:

$$\begin{bmatrix} \mathbf{1} & \mathbf{-1} & & & & & & & & & \\ 1 & 1 & \to & 1 & 1 & & & & & & \\ & & & -1 & 1 & \to & -1 & 1 & & & \\ & & & & & & -1 & -1 & \to & -1 & -1 \\ & & & & & & & & & \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

The system generated by the minimal vector $(1,-1)'$ returns to its original state in time $p_2 = 4$, not $2 \cdot 2 - 2 = 2$. In period equal 2, the system returns to its negative state.

Thus the whole system generated by the vector $(1,1,2,3,4)'$ returns to its original state in time equal to eight units, and its subsystem generated by vector $(1,-1)'$ returns to its original state in time twice faster. We now continue the description of the system, by considering the next third row of the heap transformation, i.e. the state at time $t = 2$, when $m_2 = (1,1,-1,0,0)$, or $(1,1,-1)$. The heap transformation generated by this vector has the following matrix:

$$\mathbf{H}_{[1,1,-1]} = \begin{bmatrix} 0.5774 & 0 & 0 \\ 0 & -0.7071 & 0 \\ 0 & 0 & 0.4082 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

The integer matrices of this transform and the heap transformation generated by the dual vector $(0.4082, 0.4082, 0.8165)'$, or $(1,1,2)'$ are equal, respectively

$$\mathbf{M}_{[1,1,-1]} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix},$$
$$\mathbf{M}_{[1,1,2]} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

We obtain the following subsystem of waves generated by m_2 :

$$\begin{array}{ll} t=0: & [1 \quad 1 \quad -1] \\ t=1: & [1 \quad -1 \quad 0] \\ t=2: & [1 \quad 1 \quad 2] \\ t=3: & [1 \quad -1 \quad 0] \\ t=4: & [1 \quad 1 \quad -1] \end{array} \tag{9}$$

and the period of this periodic motion in the subsystem is $p_3 = 4$.

We now consider the fourth row of the heap transformation, i.e. the state at time $t = 3$, when $m_3 = (1,1,2,-2,0,0)$, or $(1,1,2,-2)$. The heap transformations generated by this vector and its dual have the following matrices, respectively:

$$\mathbf{H}_{[1,1,2,-2]} \quad \to \quad \mathbf{M}_{[1,1,2,-2]} = \begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$
$$\mathbf{H}_{[1,1,2,3]} \quad \to \quad \mathbf{M}_{[1,1,2,3]} = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 2 & -2 \end{bmatrix}.$$

We obtain the following picture in time for the subsystem (with period $p_4 = 6$) of waves generated by the wave m_3 :

$$\begin{array}{ll} t=0: & [1 \quad 1 \quad 2 \quad -2] \\ t=1: & [1 \quad -1 \quad 0 \quad 0] \\ t=2: & [1 \quad 1 \quad -1 \quad 0] \\ t=3: & [1 \quad 1 \quad 2 \quad 3] \\ t=4: & [1 \quad -1 \quad 0 \quad 0] \\ t=5: & [1 \quad 1 \quad -1 \quad 0] \\ t=6: & [1 \quad 1 \quad 2 \quad -2] \end{array} \tag{10}$$

It is interesting to note, that each subsystem passes through the states that are states of the entire system. For instance, the subsystem generated by m_2 can be described in internal time as well as in external time of the entire system by the following diagram:

system time	subsystem time	wave		
2	0	1	1	−1
5	1	1	−1	0
7	2	1	1	2
5	3	1	−1	0
2	4	1	1	−1

For the subsystem generated by m_3, the following diagram describes the motion of waves with respect to the entire system:

system time	subsystem time	wave			
3	0	1	1	2	-2
5	1	1	-1	0	0
6	2	1	1	-1	0
4	3	1	1	2	3
5	4	1	-1	0	0
6	5	1	1	-1	0
3	6	1	1	2	-2

Let us conclude the results obtained above. The entire system generated by the five-dimensional vector $\mathbf{x} = (1, 1, 2, 3, 4)'$ passes seven states at time-points $t = 1, 2, \ldots, 7$, to come back to the original state. The period of the system is $p = 8$. The periods of three subsystems that are generated by the row-vectors m_k, $k = 1, 2$, and 3, equals $4, 4$, and 6, respectively. The system at time $t = 4$ turns into the dual state. At next time-points $t = 5, 6$, and 7, there are the same three subsystems with periods equal $4, 4$, and 6. The full circle for the system is performed by the period $P = 2(4 + 4 + 6) = 28$. Since the system has seven states, an average period for state is $28/7 = 4$. The general picture of states of this system can be described by the following diagram:

system time	1	2	3	wave				
0	$+$			1	1	2	3	4
1	$+$	$+$	$+$	1	-1	0	0	0
2	$+$	$+$	$+$	1	1	-1	0	0
3	$+$	$+$	$+$	1	1	2	-2	0
4	$+$	$+$		1	1	2	3	-3.75
5	$+$	$+$	$+$	1	-1	0	0	0
6	$+$	$+$	$+$	1	1	-1	0	0
7	$+$	$+$	$+$	1	1	2	-2	0
8	$+$			1	1	2	3	4

where in the columns entitled 1,2, and 3, the sign plus indicates that the state with the number 1,2, and/or 3, pass this state.

Conclusion

The described above heap transformations represent a class of discrete unitary signal-induced transformations (DsiHT) which are defined by systems of moving functions. Unlike the theory of discrete wavelets, these wave-functions move in the field associated with the generator, and during this movement they interact. During the full circle of the movement, each wave passes special states of the system with a different period. The matrices of the discrete heap transforms are triangle from the second row, and the first one represents the generator itself. The transforms are fast for any length of the signal and can be used in signal and image processing, and communication. We also can analyze and describe states of systems for other classes of DsiHT, for instance the so-called Haar-type heap transformation [7]. These transformations are also fast and use elementary rotations with the path corresponding to the fast Haar transformation (which is the particular case of the proposed transformations, when the generator $\mathbf{x}$ is the constant sequence $\{1, 1, 1, ..., 1\}$).

References:

[1] O. Rioul and M. Vetterli, ”Wavelets and signal processing,” *in IEEE Signal Processing Magazine,* vol. 8, pp. 11-38, Oct. 1991.

[2] Y. Meyer, *Wavelets. Algorithms and Applications,* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1993.

[3] S.G. Mallat, *A Wavelet Tour of Signal Processing,* San Diego: Academic Press, second edition, 1998.

[4] J.C. Goswami and A.K. Chan, *Fundamentals of Wavelets: Theory, Algorithms, and Applications,* New York: Wiley, 1999.

[5] A.M. Grigoryan and M.M. Grigoryan, ”Nonlinear approach of construction of fast unitary transforms,” *Proc. of the 40th Annual Conference on Information Sciences and Systems, CISS 2006,* Princeton University, pp. 1073-1078, March 22-24, 2006, Princeton.

[6] A.M. Grigoryan and M.M. Grigoryan, ”Discrete unitary transforms generated by moving waves” [6701-25], *Proc. of the International Conference: Wavelets XII, SPIE: Optics+Photonics 2007,* San Diego, CA, August 27-29, 2007.

[7] A.M. Grigoryan and M.M. Grigoryan, ”New discrete unitary Haar-type heap transforms” [6701-79], *Proc. of the International Conference: Wavelets XII, SPIE: Optics+Photonics 2007,* San Diego, CA, August 27-29, 2007.