Unsteady Boundary Layer Flow and Heat Transfer over a Stretched Surface in a Micropolar Fluid

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Abstract: - The present study deals with the analysis of unsteady boundary layer flow and heat transfer of an incompressible micropolar fluid over a stretching sheet when the sheet is stretched in its own plane. The velocity and temperature are assumed to vary linearly with the distance along the sheet. Two equal and opposite forces are impulsively applied along the \( x \)-axis so that the sheet is stretched, keeping the origin fixed in a micropolar fluid. The transformed unsteady boundary layer equations are solved numerically using the Keller-box method for the whole transient from the initial state flow to the final steady-state flow. Numerical results are obtained for the velocity, microrotation and temperature distributions as well as the skin friction coefficient and local Nusselt number for various values of the material parameter \( K \) and Prandtl number \( Pr \), when \( n = \frac{1}{2} \) (weak concentration particles at the plate).

Key-Words: - Boundary layer, Heat transfer, Micropolar fluid, Stretching surface, Unsteady flow

1 Introduction

The fluid dynamics due to a stretching sheet is important in extrusion processes. The production of sheeting material arises in a number of industrial manufacturing processes and includes both metal and polymer sheets. The quality of the final product depends on the rate of heat transfer at the stretching surface. Since the pioneering study by Crane [1] who presented an exact analytical solution for the steady two-dimensional stretching of a surface in a quiescent fluid, many authors have considered various aspects of this problem and obtained similarity solutions. Many authors presented some mathematical results, and a good amount of references can be found in the papers by Magyari and Keller [2,3], Liao and Pop [4], Nazar et al. [5] and Ishak et al. [6,7]. The studies carried out in these papers deal only with steady-state flow, but the flow and thermal fields may be unsteady due to either impulsive stretching of the surface or external stream and sudden change in the surface temperature. Kumari et al. [8] studied the unsteady free convection flow over a continuous moving vertical surface in an ambient fluid, and Ishak et al. [9] investigated theoretically the unsteady mixed convection boundary layer flow and heat transfer due to a stretching vertical surface in a quiescent viscous and incompressible fluid. Further, Pop and Na [10] and Wang et al. [11] deal with the unsteady boundary layer flow due to impulsive starting from rest of a stretching sheet in a viscous fluid and Nazar et al. [12] studied the problem of unsteady boundary layer flow due to a stretching surface in a rotating fluid. Further, Liao [13] and Xu et al. [14] obtained series solutions of the unsteady boundary layer flows and unsteady three-dimensional MHD flow and heat transfer in the boundary layer over an impulsively stretching plate, respectively.

On the other hand, it is well known that the theory of micropolar fluids has generated a lot of interests and many flow problems have been studied. The theory of micropolar fluids, which display the effects of local rotary inertia and couple stresses, can explain the flow behavior in which the classical Newtonian fluids theory is inadequate. This theory takes into account the microscopic effects arising from the local structure and micromotions of the fluid elements. The theory is expected to provide a mathematical model, which can be used to describe the behavior of non-Newtonian fluids in many practical applications. Since introduced by Eringen [15,16], several researchers have considered various stretching problems in micropolar fluids including the present authors (see Ishak et al. [6,7]). Extensive reviews of the theory and its applications can be found in the review articles by Ariman et al. [17,18] and the recent books by Lukaszewicz [19] and Eringen [20].
Motivated by the above-mentioned investigations and applications, in this present paper, we investigate the behavior of the boundary layer flow and heat transfer of an incompressible micropolar fluid over a stretching sheet when the sheet is stretched in its own plane. The stretching velocity is assumed to vary linearly with the distance along the sheet. The transformed governing parabolic partial differential equations in two variables are solved numerically using the Keller-box method for some values of the physically governing parameters.

2 Mathematical Formulation

Consider the flow of an incompressible micropolar fluid in the region \( y > 0 \) driven by a plane surface located at \( y = 0 \) with a fixed end at \( x = 0 \). It is assumed that the surface is stretched in the \( x \)-direction such that the temperature and \( x \)-component of the velocity vary linearly along it, i.e. \( T_s(x) = T_s + ax \) and \( u_s(x) = cx \), respectively, where \( a \) and \( c \) are arbitrary positive constants. The simplified two-dimensional equations governing the flow in the boundary layer of a steady, laminar and incompressible micropolar fluid are

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left( \frac{\mu + \kappa}{\rho} \right) \frac{\partial^2 u}{\partial y^2} + \kappa \frac{\partial N}{\partial y} - \frac{\partial N}{\partial x}
\]

subject to the initial and boundary conditions

\[
t < 0: \quad u = v = 0, \quad N = 0, \quad T = T_s \quad \text{for any} \quad x, y,
\]

\[
t \geq 0: \quad v = 0, \quad u = u_s(x) = cx, \quad N = -n \frac{\partial u}{\partial y}, \quad T = T_s(x) = T_s + ax \quad \text{at} \quad y = 0,
\]

\[
u \to 0, \quad N \to 0, \quad T \to T_s \quad \text{as} \quad y \to \infty,
\]

where \( u \) and \( v \) are the velocity components along the \( x \)- and \( y \)-axes, respectively, \( t \) is time, \( N \) is the microrotation or angular velocity whose direction of rotation is in the \( x-y \) plane, \( T \) is temperature, \( \mu \) is dynamic viscosity, \( \rho \) is density, \( j \) is microinertia per unit mass, \( \gamma \) is spin gradient viscosity, \( \kappa \) is vortex viscosity and \( \alpha \) is thermal diffusivity. Further, \( n \) is a constant where \( 0 \leq n \leq 1 \). The case \( n = 0 \), which indicates \( N = 0 \) at the wall represents concentrated particle flows in which the microelements close to the wall surface are unable to rotate (Jena and Mathur [21]). This case is also known as the strong concentration of microelements (Guram and Smith [22]). The case \( n = \frac{1}{2} \) indicates the vanishing of antisymmetric part of the stress tensor and denotes weak concentration of microelements (Ahmadi [23]). The case \( n = 1 \) as suggested by Peddieson [24] is used for the modeling of turbulent boundary layer flows. In this paper we consider only the case of \( n = \frac{1}{2} \).

We introduce the new variables

\[
\psi = (c\nu)^{1/2} \xi^2 \frac{\partial}{\partial \xi} \left( \xi, \eta \right), \quad N = (c/\nu)^{1/2} \xi \frac{\partial}{\partial \xi} \left( \xi, \eta \right), \quad \theta(\xi, \eta) = (T - T_s) / (T_e - T_s), \quad \eta = (c/\nu)^{1/2} \xi^{1/2} y,
\]

\[
\xi = 1 - e^{-t}, \quad \tau = ct,
\]

where \( \psi \) is the stream function defined in the usual way as \( u = \partial \psi / \partial \eta \) and \( v = -\partial \psi / \partial x \), and identically satisfy Eq. (1). Substituting variables (6) into Eqs. (2) and (3) gives

\[
(1 + K) f'' + (1 - \xi) \frac{\eta}{2} f'' + \xi \left( f'' - f''\right) + Kg' = \xi \left(1 - \xi \right) \frac{\partial^2 f''}{\partial \xi^2}
\]

\[
\left( 1 + \frac{K}{2} \right) g'' + (1 - \xi) \frac{1}{2} g' + \xi \left( f'' - f''\right) = \xi \left(1 - \xi \right) \frac{\partial g''}{\partial \xi}
\]

\[
\frac{1}{\Pr} \theta'' + (1 - \xi) \frac{\eta}{2} \theta' + \xi \left( f'' - f''\right) = \xi \left(1 - \xi \right) \frac{\partial \theta'}{\partial \xi}
\]

where \( K = \frac{\kappa}{\mu} \) is the material parameter. Here \( \gamma \) and \( j \) are assumed to be given by Rees and Pop [25]

\[
\gamma = (\mu + \kappa / 2), \quad j = \mu (1 + K / 2) \quad \text{and} \quad j = \nu / c,
\]

respectively. The boundary conditions (5) become

\[
f(\xi, 0) = 0, \quad f'(\xi, 0) = 1, \quad g(\xi, 0) = -nf'(\xi, 0), \quad \theta(\xi, 0) = 1,
\]

\[
f'(\xi, \infty) = 0, \quad g(\xi, \infty) = 0, \quad \theta(\xi, \infty) = 0.
\]

The physical quantities of interest in this problem are the skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_y \), which are defined as

\[
C_f = \frac{\tau_s}{\rho u_s}, \quad Nu_y = -\frac{q_x}{k(T_e - T_s)}
\]

where \( \tau_s \) is the surface shear stress and \( q_x \) is the heat flux from the surface of the sheet and they are given by

\[
\tau_s = \left[ (\mu + \kappa) \frac{\partial u}{\partial y} + \kappa N \right]_{y=0}, \quad q_x = -k \frac{\partial T}{\partial y}_{y=0}
\]

Using variables (6) in Eqs. (11) and (12), we obtain
\[ C_f \operatorname{Re}_s^{1/2} = \xi^{-1/2} \left[ 1 + \frac{(1-n)K}{f''(\xi, 0)} \right], \quad (13) \]

\[ Nu_s / \operatorname{Re}_s^{1/2} = \xi^{-1/2} \left[ -\theta'(\xi, 0) \right]. \quad (14) \]

3 Results and Discussion

The transformed equations (7)-(9), satisfying the boundary conditions (10) were solved numerically using the Keller box-method described in the book by Cebeci and Bradshaw [26] for several values of parameters \( \Pr \) and \( K \). The results for skin friction coefficient, local Nusselt number, velocity distribution, microrotation distribution and temperature distribution are illustrated in Figs. 1-8, while the values of skin friction coefficient and local Nusselt number for final steady-state flow \( (\xi = 1) \) are tabulated in Tables 1 and 2, respectively.

Table 1 presents the values of the skin friction coefficient \( C_f \operatorname{Re}_s^{1/2} \) when \( \xi = 1 \) (final steady-state flow) for various values of \( K \). It can be observed that the values of \( C_f \operatorname{Re}_s^{1/2} \) are negatives and decrease with \( K \) for all values of \( K \) considered in this study, which means that the magnitude of \( C_f \operatorname{Re}_s^{1/2} \) increases as \( K \) increases. It is worth mentioning that the skin friction coefficient is independent of \( \Pr \).

Table 1  Values of skin friction coefficient \( C_f \operatorname{Re}_s^{1/2} \) of final steady-state flow \( (\xi = 1) \) for various \( K \)

<table>
<thead>
<tr>
<th>( K )</th>
<th>( C_f \operatorname{Re}_s^{1/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1.0000</td>
</tr>
<tr>
<td>1</td>
<td>-1.2247</td>
</tr>
<tr>
<td>2</td>
<td>-1.4142</td>
</tr>
<tr>
<td>4</td>
<td>-1.7320</td>
</tr>
</tbody>
</table>

Table 2 presents the values of local Nusselt number \( Nu_s / \operatorname{Re}_s^{1/2} \) when \( \xi = 1 \) (final steady-state flow) for various values of \( \Pr \) and \( K \). It can be seen that the values of \( Nu_s / \operatorname{Re}_s^{1/2} \) are positives for all values of \( \Pr \) and \( K \) considered in this study. For a particular value of \( \Pr \), increasing of \( K \) is to increase the local Nusselt number and the same trend can be observed for a particular value of \( K \), that is, \( Nu_s / \operatorname{Re}_s^{1/2} \) increases as \( \Pr \) increases.

Figures 1 and 2 present the variation of the skin friction coefficient \( C_f \operatorname{Re}_s^{1/2} \) and the local Nusselt number \( Nu_s / \operatorname{Re}_s^{1/2} \), respectively, as a function of \( \xi \) for various values of \( K \). It is noticed that due to impulsive motion, both the skin friction coefficient and the local Nusselt number have large magnitude (absolute value) for small time \( (\tau \approx 0 \text{ or } \xi \approx 0) \) after the start of the motion, and it decreases monotonically and reach the steady-state values at \( \xi = 1 \) \( (\tau \to \infty) \). There is, therefore, a smooth transition from small-time solution to the large-time solution. From both figures, it is observed that the absolute values of \( C_f \operatorname{Re}_s^{1/2} \) and \( Nu_s / \operatorname{Re}_s^{1/2} \) increase as \( K \) increases.

Table 2  Values of local Nusselt number \( Nu_s / \operatorname{Re}_s^{1/2} \) of final steady-state flow \( (\xi = 1) \) for various \( K \) and \( \Pr \)

<table>
<thead>
<tr>
<th>( K )</th>
<th>( \Pr )</th>
<th>0.7</th>
<th>1</th>
<th>3</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7937</td>
<td>1.0000</td>
<td>1.9237</td>
<td>3.0725</td>
<td>3.7211</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.8420</td>
<td>1.0484</td>
<td>1.9706</td>
<td>3.1182</td>
<td>3.7665</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.8709</td>
<td>1.0771</td>
<td>1.9982</td>
<td>3.1453</td>
<td>3.7934</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.9048</td>
<td>1.1106</td>
<td>2.0306</td>
<td>3.1772</td>
<td>3.8252</td>
<td></td>
</tr>
</tbody>
</table>
Figure 3 shows the velocity profiles of final steady-state flow ($\xi = 1$) for various values of $K$. This figure shows that the velocity boundary layer thickness increases with increasing values of $K$. For a particular value of $K$, the velocity decreases monotonically with $\eta$, and becomes zero at the outside of the boundary layer. This property satisfies the boundary condition $f'(\infty) = 0$. Therefore, this figure supports the validity of the present results.

Figure 4 presents the velocity profiles of fully developed unsteady flow and final steady-state flow. This figure shows that the velocity profiles corresponding to increasing of $\xi$ ($0 < \xi < 1$) approach the steady profile corresponding to $\xi = 1$. It can be observed that there is a smooth transition from small time solution ($\xi \approx 0$) to large time solution ($\xi = 1$). Again, it is observed that the velocity decreases monotonically with $\eta$, and becomes zero far away from the surface, which satisfies the boundary conditions (10).

The angular velocity or microrotation distributions are shown in Figs. 5 and 6. It is observed from Fig. 5 that the microrotation continuously decreases with $\eta$ and becomes zero far away from the plate.

As expected, the microrotation effects are more dominant near the wall. Also, the microrotation decreases as $K$ increases in the vicinity of the plate but the reverse happens as one moves away from it. Figure 6 represents the microrotation distribution of fully developed flow ($0 < \xi \leq 1$) when $K = 1$. From this figure, it is observed that the microrotation profiles continuously decreasing with $\eta$. It is evident from this figure that, there is a smooth transition from small time solution ($\xi \approx 0$) to large time solution ($\xi = 1$).

The temperature distribution of final steady-state flow for various $K$ is shown in Fig. 7. It is observed that the temperature decreases with an increase in $K$, which results in decreasing manner of the thermal boundary layer thickness. For a particular value of $K$, the temperature decreases continuously with $\eta$, the same trend as in Pr.
Figure 8 presents the temperature profiles of fully developed unsteady flow and final steady-state flow. This figure shows that the temperature profiles corresponding to increasing of $\xi$ ($0 < \xi < 1$) approach the steady profile corresponding to $\xi = 1$. It can be observed that there is a smooth transition from small time solution ($\xi \approx 0$) to large time solution ($\xi = 1$). Again, it is observed that the temperature decreases monotonically as the distance from the surface increases until it achieves a constant value, namely zero, which satisfies the boundary conditions (10).

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