Comparison between the MHFEM formulation and a 2\textsuperscript{nd} spatial order FV formulation of the linear groundwater flow problem

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Abstract: - Mixed and Mixed Hybrid Finite Elements (MHFE) methods have been widely used in the last decade for simulation of groundwater flow problem, petroleum reservoir problems, potential flow problems, etc. The main advantage of these methods is that, unlike the classical Galerkin approach, they guarantee local and global mass balance, as well the flux continuity between inter-element sides. The simple shape of the control volume, where the mass conservation is satisfied, makes also easier to couple this technique with a Finite Volume technique in the time splitting approach for the solution of advection-dispersion problems. In the present paper a new second spatial approximation order Finite Volume formulation (FV2) for triangular irregular meshes, is proposed for the solution of the linear groundwater flow problem and the numerical results are compared with the corresponding one given by a MHFE method. It can be shown that the FV2 approach is equivalent to the MHFE approach in the case of isotropic medium and regular or mildly irregular mesh, but has a smaller number of unknowns and better matrix properties. In the case of irregular mesh, an approximation is proposed to maintain the superior matrix properties of the FV2 approach, with the consequent introduction of a small error in the computed solution.

Key-Words: - groundwater, finite elements method, mixed hybrid finite elements method, finite volumes method, positive-definite matrix, M-property, Raviart-Thomas basis function

1 Introduction
The numerical solution of a groundwater flow problem, according to the Eulerian approach, yields a set of discrete values of potential head or pressure and velocity, referred to different points of a mesh dividing the computational domain in elements.

The most common solutions of the governing equation system are based on Finite Difference (FD), Finite Volume (FV) and Finite Elements (FE) methods. The FD method is the oldest one, is based on a homogeneous and regular domain decomposition and is easy to implement. The FV and FE methods are more flexible in discretizing complex geometry domains as well as complex boundary conditions.

For a correct solution of the transport problem, the velocity field needs to preserve mass conservation, both locally and globally. Unlike FD and FV methods, in FE approach, the velocity field is calculated by differentiation of the potential solution inside the elements, the discrete normal fluxes are discontinuous across inter-element boundaries and the local mass conservation is not warranted.

One way to circumvent this problem is to proper exploit the local subdomains where the mass conservation property is satisfied. For example, in the Galerkin technique applied on 2D triangulation, the subdomains can be defined as the Voronoi (or Thiessen) polygons. Application of this approach in 3D is very complicated and provides a strong increment of the computational time. On the other hand, the Finite Volume (FV) approach is mass conservative because the subdomain where the mass balance is applied coincides with the mesh element.

The Mixed Finite Element (MFE) methods provide and attractive framework for these types of problems: by simultaneously approximating the potential head and normal fluxes, the computed normal fluxes are continuous across inter-element edges and the local and global mass balance is automatically achieved. MFE methods have been extensively used for the solution of parabolic problem in many application fields (groundwater flow problem, petroleum reservoir problems, potential flow problems, …), but in elliptic problems (i.e. steady state problems) the matrix of the system becomes ill-conditioned [see Bergamaschi and Putti, 1999; Younes et al., 2004].

An improved technique is the Mixed Hybrid Finite Element (MHFE) method. The MHFE discretizes spatially the flow equations in a set of continuity equations across all the edges of the mesh, using the average potentials along these edges as unknowns. If the mesh is triangular and the element parameters are isotropic, it can be proved [Younes et al., 2006] that the MHFE is algebraically equivalent to a FV approach, where the potential gradients are computed between the circumcenters of the triangles. In the FV approach the unknowns are the potentials in the circumcenters, that are less than the mesh edges and, if the mesh is regular or mildly irregular, the final system matrix is symmetric positive-definite with good conditioning properties.
In time splitting techniques for the simulation of advection-dispersion problems, the solution of the dispersive-dispersive component by means of MHFE methods can be easily coupled with the solution of the convective components obtained by means of other formulations (Aricò and Tucciarelli, 2008).

In the present paper, a 2nd spatial approximation order FV formulation is presented for the solution of the linear groundwater flow problem. The results are shown to be equivalent to the results obtained by the MHFE method in the case of regular and mildly regular meshes. In the case of irregular meshes, a simple approximation is proposed; the approximation is aimed to maintain the good properties of the final matrix, even if this simplification is paid with a reduction of the result accuracy provided by the second order spatial approximation.

2 The physical problem

Let’s assume a saturated porous medium and consider the following evolutionary problem on the domain \( \Omega \in \mathbb{R}^2 \):

\[
S_0 \frac{\partial h}{\partial t} + \nabla \cdot \mathbf{G} = Q
\]  
(1a),

\[
\mathbf{G} = -K \nabla h
\]  
(1b),

with the following boundary conditions:

\[
h = h_{ij} \quad \text{on} \quad \Gamma_D
\]  
(2a),

\[
q_N = -K \left. \frac{\partial h}{\partial n} \right|_{\Gamma_N}
\]  
(2b),

where \( n \) is the outward unit normal to \( \Omega \). The initial condition are:

\[
h(x,0)=h_0(x)
\]  
(2c).

Eq. (1a) represents the mass balance equation of the fluid in the porous medium, where the velocity field \( \mathbf{v} \) is expressed by means of the Darcy law:

\[
\mathbf{v} = -K \nabla h
\]  
(3).

In Eqn. (1-3) \( \mathbf{G} \) is the diffusive flux, equivalent to the Darcy velocity \( \mathbf{v} \) provided by Eqn. (3), \( h \) is the potential head, \( K \) the hydraulic conductivity, assumed constant in time, \( S_0 \) is the storativity coefficient, assumed constant in space and time, \( Q \) is the source or sink term. \( \Gamma, \Gamma_D \) and \( \Gamma_N \) are respectively the boundary of \( \Omega \), the part of \( \Gamma \) with given piezometric head \( h_{ij} \) (Dirichlet condition) and the part of \( \Gamma \) with given flux \( q_N \) (Neumann condition).

3 Formulation of the problem

3.1 The Mixed Hybrid Finite Element method formulation

Let \( T \) be a triangulation of \( \Omega \) with \( m \) triangles \( T_j \) (\( j = 1, \ldots, m \)), \( N \) nodes and \( L \) edges \( e_j \) (\( j = 1, \ldots, L \)). The piezometric head is approximated as:

\[
h = \sum_{j=1,m} h_j \psi_j
\]  
(4),

where \( \psi_j \) are \( P_0(T_j) \) scalar basis function, taking on the value one on the triangle \( T_j \) and zero elsewhere. The velocity \( \mathbf{v} \) in element \( l \) is approximated as:

\[
\mathbf{v}_l = \sum_{j=1,3} g_{lj} \mathbf{w}_j \quad l = 1, \ldots, m
\]  
(5),

where \( g_{lj} \) is the flux leaving element \( l \) from side \( j \) and \( \mathbf{w}_j \) are basis of the lowest-order Raviart-Thomas space \( \mathbf{X}_T \), defined as:

\[
\mathbf{w}_j = \frac{1}{2|\Omega|} \begin{pmatrix} x - x_j \\ y - y_j \end{pmatrix} \quad j = 1,2,3
\]  
(6),

with \( x_j \) and \( y_j \) the coordinates vertices (see figure 1).

Important properties of the \( \mathbf{X}_T \) space are [for example Chavent and Roberts, 1991; Bergamaschi and Putti, 1999; Younes et al., 2006]:

\[
\nabla \cdot \mathbf{v}_l \quad \text{is constant on} \quad T_j
\]  
(7a),

\[
\mathbf{v}_l \cdot \mathbf{n}_l \quad \text{is constant over each edge} \quad e_i
\]  
(7b).

Introducing the approximated functions expressed in Eqns. (4)-(7) in Eqn. (1-3) and integrating in space and time, the implementation of the MHFE is [see for example Mazzia et al., 2000]:

\[
\int_{T_j} K^{-1}_{lj} \mathbf{v}_l \cdot \mathbf{w}_j d\Omega - \int_{T_j} h \nabla \cdot \mathbf{w}_j d\Omega + \int_{T_j} \lambda \mathbf{w}_j \cdot \mathbf{n}_j d\Gamma = 0
\]  
(8),

\[
\frac{S_0}{\Delta t} \int_{T_j} \mathbf{h}_j \cdot \mathbf{n}_j d\Omega + \int_{\partial T_j} \mathbf{v}_l \cdot \mathbf{n}_j d\Gamma + \int_{\partial T_j} \mathbf{v}_j \cdot \mathbf{n}_j d\Gamma = 0 \quad e_j \in \Gamma
\]  
(9),

\[
\int_{\partial T_j} \mathbf{v}_l \cdot \mathbf{n}_j d\Gamma = b_{lj} \quad e_j \in \Gamma
\]  
(10),

\[
\lambda_j = \frac{h_D}{b_{lj}} \quad e_j \in \Gamma_D
\]  
(12),

where \( i = 1, 2, 3 \), \( l = 1, \ldots, m \), \( j = 1, \ldots, L \). Eqn. (8) is the MHFE discretization of the dispersive flux (1,b) and \( \lambda \) are the unknown Lagrange multipliers further defined, Eq. (9) is the discretization of the mass balance equation (1,a), Eq. (10) is the equivalence of the element fluxes across the inter-element edges and Eqn. (11)-(12) represent the Neumann and Dirichlet boundary conditions.

The Lagrange multipliers in Eqn. (8) are expressed as:

\[
\lambda = \sum_{j=1,L} \lambda_j \mu_j
\]  
(13),

where \( \mu_j \) is a piecewise constant basis function with value equal to 1 on the edge \( e_j \) and zero elsewhere and \( \lambda_j \) represent the value of the potential head on \( e_j \).

According to the properties expressed in Eqn. (7), the flux law (1,b) can be written in variational form as:

\[
\int_{T_j} \mathbf{v}_l \cdot \mathbf{w}_j = \sum_{j=1,3} g_{lj} \mathbf{w}_j \cdot \mathbf{w}_j = K_j \left[ \nabla h_{lj} \cdot \mathbf{w}_j \right] = K_j \left( h_{lj} - \lambda_j \right)
\]  
(14).

Eq. (14) can be written as:

\[
g_{lj} = K_j \sum_{j=1,3} A_{lj}^{-1} \left( h_{lj} - \lambda_j \right)
\]  
(15).
where \( A = [A_{ij}] \), \( A_{ij} = \int_{\xi_i} \mathbf{w} \cdot \mathbf{J} \). Using Eq. (15) and a fully-implicit time discretization, the mass balance Eq. (9) becomes:

\[
h_i^{t+1} = K \frac{\alpha}{\beta} \sum_{j=1}^{3} \alpha_j h_j^{t+1} + \frac{\delta}{\beta} h_i^t + \frac{Q_{ij}}{\beta}
\]

where \( \alpha_j = \sum_{j=1}^{3} A_{ij}^{m} \), \( \delta = S_i |T_i| / \Delta t \), \( \beta = \delta + \alpha K \), \( \alpha = \sum_{j=1}^{3} \), and \( Q_{ij} = \int Q_{ij} d\Omega \); replacing Eq. (16) in Eq. (15), one gets:

\[
g_{i}^{t+1} = K \left[ \frac{\alpha}{\beta} \sum_{j=1}^{3} \alpha_j h_j^{t+1} + \frac{\delta}{\beta} h_i^t + \frac{Q_{ij}}{\beta} \right]^{m}
\]

The final system to solve is obtained by writing the continuity of the fluxes between two adjacent elements A and B:

\[
g_{i}^{t+1} + g_{j}^{t+1} = 0
\]

For the edge i, the corresponding diagonal term is:

\[
m_{ii} = m_{i}^{A} + m_{i}^{B}, \quad m_{i}^{A} = K \left( A_{i}^{1} - K \alpha_{i} \right)
\]

while the off-diagonal term is:

\[
m_{ij}^{A} = K \left( A_{i}^{1} - K \alpha_{i} \right)
\]

The matrix of the system is symmetric and positive-definite and for the solution of the system a preconditioned conjugate gradient can be used.

3.2 A 2nd order Finite Volume formulation

It can be proven that [Younes et al., 2006] the mixed hybrid finite element method is algebraically equivalent to the following second order finite volume approach.

Call \( H^A \) the potential in the circumsentrum \( C^A \) of the element A and \( TP_j^A \) the potential head in the centre \( P_{jm}^A \) of the \( f^o \) edge of the same element. Assume a linear variation of the potential inside the element and a potential continuity along the edges; according to these hypotheses the change of the potential between the \( C^A \) and the point \( P_{jm}^A \) is given by (see figure 2):

\[
H^A - TP_j^A = Q_j^A \frac{h_j^A}{K_R b_j^A}
\]

where \( Q_j^A \) is the edge flux, \( b_j^A \) in the length of edge \( j \) and \( h_j^A \) is the minimum distance of \( C^A \) from the edge \( j \). If the distance \( h_j^A \) is taken positive or negative according to the position of \( C^A \) with respect to edge \( j \), oriented in counterclockwise direction, relationship expressed by Eq. (21) still holds if the angle in vertex \( j \) is greater than 90 degrees and the point \( C^A \) falls outside the triangle. In the case of negative distance, \( H^A \) is no more a good approximation of the potential at point \( C^A \), that has to be found with a post-processing analysis. Assuming \( m \) to be the local index of the same edge in the element B next to A, it is possible to cancel the unknown \( TP_j^B \) by summing Eq. (21) with the similar equation holding for the edge \( m \) in element B, that is:

\[
TP_m^B - H^B = -Q_m^B \frac{h_m^B}{K_b b_m^B}
\]

Because \( TP_m^B = TP_j^A \) and \( Q_m^B = -Q_j^A \), you get:

\[
H^A - H^B = \frac{Q_j^A \left( K_R h_j^A + K_A h_m^B \right)}{K_A \left( K_B \right)}
\]

Summing the fluxes through the three edges of element A, you get the finite volume equation:

\[
[T_A^A]^\delta \frac{\partial H^A}{\partial t} = \sum_{j=1}^{3} \varepsilon_j^A \left( H^E - H^A \right)
\]

where \( S^A \) is the storage coefficient, \( [T_A] \) is the area of element A, \( E \) is the index of the element next to the \( j^o \) edge of element A and:

\[
\varepsilon_j^A = \frac{b_j^A K_A K_B}{K_R b_j^A}
\]

(25.a),

\[
K_{h_{AB}} = K_+ h^+ + K_- h^- \]

(25.b).

After time discretization, Eq. (24)-(25) form a linear system of order NE, if NE is the element number. If the quantity \( K_{h_{AB}} \) always satisfies the condition:

\[
K_{h_{AB}} > 0
\]

the linear system, coupled with the proper boundary conditions, has always the so called M-property (a non-singular matrix where the diagonal terms are always positive and the off-diagonal terms are negative or null); the M-property guarantees that local maxima or minima not appear in the solution in a domain without sinks or sources [Younes et al., 2006]) and the eigenvalue positive definite condition. In this hypothesis, the FV formulation is easier.
than the MHFE formulation, because it requires the solution of a system smaller than in the MHFE formulation, with a matrix that satisfy always the M-property.

Unfortunately, if the maximum angle is greater than 90 degrees in one or more elements the sign of $K_{h_{AB}} > 0$ cannot be guaranteed. To overcome this difficulty, the following procedure is proposed:

Call $A$ and $B$ two elements sharing the same edge, numbered with index $j$ in element $A$ and index $m$ in element $B$ (see figure 2). If condition expressed in Eq. (26) is originally not satisfied, it can be met by changing the location of points $C^A$ and $C^B$. Compute first the quantity

$$K_{h_{AB}}^{\text{max}} = K_A h_0^A + K_B h_0^B$$

(27),

that is the value of $k h_{AB}$ attained when the points $C^A$ and $C^B$ have a distance from the edge $j$ of element $A$ equal to the heights $h_0^A$ and $h_0^B$ of the elements $A$ and $B$ with respect to the common edge. This distance from edge $j$ is the maximum possible for points $C^A$ and $C^B$ to avoid negative distances from the other two sides of each element. In a second step compute the distances of the new points $C^A$ and $C^B$. A possible choice is to find the root $\beta$ of the following equation:

$$\left( h_0^A \beta + h_0^A \left( 1 - \beta \right) \right) K_B + \left( h_0^B \beta + h_0^B \left( 1 - \beta \right) \right) K_A = \varepsilon K_{h_{AB}}^{\text{max}}$$

(28),

where $\varepsilon$ is a small quantity (say 0.001). The new distances are obtained by setting:

$$h_0^A = h_0^A \beta + h_0^A \left( 1 - \beta \right)$$

(29,a),

$$h_0^B = h_0^B \beta + h_0^B \left( 1 - \beta \right)$$

(29,b).

The increment of the distances $h_0^A$ and $h_0^B$ from the common edge can be obtained by reducing the distance of $C^A$ and $C^B$ from the nodes $P_j$ and $P_m$, opposite (respectively in element $A$ and $B$) to the common edge. To guarantee that distances from the other edges of the same elements $A$ and $B$ remain positive, for any possible angle values, $C^A$ and $C^B$ must be kept respectively on the lines $P_f C^A$ and $P_m C^B$.

The proposed strategy is equivalent to assume the equality $TP_m = TP_j$ even if the projections of $C^A$ and $C^B$ on the same edge are not the same, as happens for the circumcentres. For this reason, the use of a very small $\varepsilon$ in Eq. (8) implies very small extradiagonal coefficient and a slower convergence, but the use of a much larger value could affect the quality of the solution.

The adopted approximation is similar to the one adopted in the first order finite volume formulation to relate the fluxes with the variation of the average potentials from one element to the next. Its effect on the final solution is to shift locally the accuracy of the solution from the second order to the first order size.

4 Numerical tests

A computational domain $1 \times 1$ m has been assumed for the numerical simulations. Two meshes have been used; the first is structured with 153 equilateral triangular elements (see figure 3,a), the other one is unstructured with 128 elements (see figure 3,b). On the right vertical boundary of the domain the potential head is assumed equal to 1 m (Dirichlet condition); the bottom boundary is assumed impervious to fluxes, while on the other two boundaries an incoming flux is assumed (Neumann condition), equal respectively to 0.12 and 0.125 m$^2$/s for the two meshes.

Two series of simulations have been carried out, assuming homogeneous and heterogeneous hydraulic conductivity; in the second case the value of $K$ inside element $i$ is given by:

$$K = e^{\frac{\sin\left(\frac{\pi x_i}{3\Delta x}\right) + \sin\left(\frac{\pi y_i}{3\Delta y}\right)}{10}}$$

(30),

where $x_i$ and $y_i$ are the coordinates of the centers of mass of the element and parameters $\Delta x = \Delta y$ are assumed equal to 1/8; in figure 4 the contour lines of the points with the same value of $K$ are shown for the unstructured mesh.

Numerical results of the two schemes for the two meshes and both homogeneous and heterogeneous conductivity cases have been compared. The time step $\Delta t$ used for the simulations is 25 s.

In figures 5-6 the equipotential contours are shown for the cases of structured and unstructured meshes and heterogeneous $K$ at the simulation time $T = 500$ s: observe that for both meshes, the contour lines are undistinguishable. A similar behavior occurs in the test-
cases for homogeneous medium, even if for brevity results are not reported here.

The effect of the mesh size has been also investigated. Starting from the meshes used before, at each refinement level \(i+1\), each triangle of the previous \(i^{th}\) mesh is subdivided in 4 equal triangles connecting the midpoints of the three sides (see figure 7) and \(\Delta t\) is halved.

In figures 8-11 results of both the proposed numerical procedures are shown for the case of homogeneous and heterogeneous conductivity for the 2\(^{nd}\) mesh refinement level (2448 elements for the structured mesh and 2048 elements for the unstructured one). Observe that numerical results are undistinguishable, at the graph scale, in the case of structured meshes; for the unstructured meshes results differ near the boundaries and in the zones with higher mesh irregularity (see for example the contour lines values 2.6 or 2.4).

In tables 1,a and 1,b the RMS errors with respect to the solutions of the MHFE method are shown for the proposed FV procedure, as well as for the standard Galerkin FE formulation. In the Galerkin approach, the potential head in the element has been computed as the mean value of the potentials at the three nodes.

The RMS errors are computed as:

\[
RMS = \frac{1}{NE} \sum_{i=1,NE} \sqrt{(H_{FV(GA)}^i - H_{MHFE}^i)^2}
\] (31).

Generally, the scatter of the Galerkin approach is greater than the one of the FV scheme. This is due a) to some difference in the discretization of the boundary conditions, that are assigned to the edges in both the FV and MHFE schemes and to the nodes in the Galerkin scheme, and b) to the smaller degree of freedom (the number of nodes) of this last one with respect to the other two (respectively the number of edges and the number of elements for the MHFE and the FV schemes). We can observe a much smaller error of the FV scheme, with respect to the Galerkin scheme, in the case of heterogeneous domain, due to the strong effect of the flux continuity at the line where the parameter change occurs. We can also observe that the error increases in the FV method for the more refined unstructured meshes. This is because the adopted refinement provides a proportional increment of the irregular elements, that could be avoided using more sophisticated techniques.

The mean values per element of the CPU times are reported in tables 2,a-2,d for the all three numerical procedures.

The computational cost required by the FV2 method is approximately twice the one required by the FE Galerkin scheme, but it is always smaller than the one of the MHFE method. Observe anyway that, increasing the element number refining the mesh, the growth of the computational cost is less than linear.
Fig. 5. Equipotential contours – heterogeneous medium, structured mesh (dashed lines: MHFE method, dotted lines FV method)

Fig. 6. Equipotential contours – heterogeneous medium, unstructured mesh (dashed lines: MHFE method, dotted lines FV method)

Fig. 7. Mesh refinement

Fig. 8. Equipotential contours – homogeneous medium, structured mesh 2nd refinement level (dashed lines: MHFE method, dotted lines FV method)

Fig. 9. Equipotential contours – heterogeneous medium, structured mesh 2nd refinement level (dashed lines: MHFE method, dotted lines FV method)
4 Conclusion

The second order FV formulation, for the solution of the linear groundwater flow problem, has been modified for the case of irregular unstructured triangular meshes. By modifying the position of the circumcentrum for elements with angles higher than 90 degrees, the matrix of the final system is kept always positive-definite and satisfying the M-property.

Numerical results have been compared with the results obtained using the standard MHFE formulation. The performance of method has been investigated using both structured and unstructured meshes, as well as homogeneous and heterogeneous hydraulic conductivity.

For all the studied test cases, the computational costs of the proposed FV procedure are smaller than the ones required by the MHFE formulation to solve the problem.

For structured meshes, results of the new procedure are very close to the ones provided by the MHFE scheme in both homogeneous and heterogeneous medium. For unstructured meshes, results of the new procedure differ from the ones of the MHFE methods, specially for very fine mesh, in the zone characterized by higher mesh irregularity.

It is important to underline that results provided by the proposed FV procedure in cases of unstructured mesh, are much closer to the results of the MHFE methods than the ones computed by a standard FV 1st spatial approximation.

Table 1.a. RMS errors; homogeneous medium

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<th>FV2</th>
<th>FE Galerkin</th>
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Table 1.b. RMS errors; heterogeneous medium

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Table 2.a CPU mean values, homogeneous medium

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Table 2.b CPU mean values, heterogeneous medium

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</tbody>
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order, as shown in figures 12 and 13 (respectively for homogeneous and heterogeneous medium), in the case of the 2nd mesh refinement level.

Fig. 12. Equipotential contours – homogeneous medium, unstructured mesh 2nd refinement level (dashed lines: MHFE method, small dotted lines FV method, large dotted lines 1st order FV method)

Fig. 13. Equipotential contours – heterogeneous medium, unstructured mesh 2nd refinement level (dashed lines: MHFE method, small dotted lines FV method, large dotted lines 1st order FV method)

References:


