Energy Approach for Microscopic Particles Manipulation using Optical Tweezers

CARLOS AGUILAR IBAÑEZ
CIC
Instituto Politécnico Nacional
Av. Juan de Dios Báñez s/n Esq. Manuel Othón, Mexico City 07700
MEXICO

HEBERT SIRA MEDINA
CINVESTAV
Instituto Politécnico Nacional
Av. Politécnico Nacional 2508, Mexico City 07300

ARMANDO BARRAÑON
Dept. de Ciencias Básicas
Universidad Autónoma Metropolitana
Av. San Pablo 180, Mexico City 02200
MEXICO

Dept. of Physics
University of Texas
USA

Abstract: - In this article we use Lyapunov Second Theorem as well as LaSalle Theorem in order to determine the attraction region of an optical trap and to design an steering strategy to move a microscopic particle by means of optical tweezers. A variational approach based on Lagrangian formulation is used to obtain the equations of motion of a particle in the optical trap[1]. System is proved to be locally asymptotically stable under the assumption that Stoke’s drag force acts on the particle. Then, a stable control strategy to move the particle from an initial rest position to a desired final position is derived from Lyapunov second theorem and LaSalle invariance theorem.

Key-Words: - Optical tweezers, Lyapunov theorem, LaSalle invariance theorem, Stability, Microparticles, Lagrangian, Particle steering

1 Introduction

Since 1986, Ashkin et. al. proved experimentally that a single-beam gradient force trap can exercise a negative light pressure on particles with size in the range from 10 micrometers up to 25 nanometers in water [2]. As a matter of fact, when a 1 watt laser beam is concentrated on a spot of about one wavelength, a dielectric sphere one micrometer in diameter is subject to an intensity of one hundred million watts and an acceleration of about one million g, although particles levitation can also be attained and used for visual observation in microscopes and high vacuum devices [3].Svoboda et. al. used optical traps to study the position variance of silica beads driven by single molecules of motor protein kinesin in order to test thermal ratchet models for motor movement, finding lower values of the variance than those expected in these ratchet models of motor movement [4]. Arlt and Padgett have developed an hologram that generates a beam with a dark focus surrounded by regions f higher intensity, namely an optical bottle beam that could trap cold atoms so that low intensity single beam blue-detuned atom traps could be built [5]. Felgner et. al. used optical traps to measure the flexural rigidity of microtubules incubated with taxol after polymerization which turned out to be more flexible than those without taxol added [6]. Optical tweezers have been used to...
transport gaseous Bose-Einstein condensates over distances up to 44 cm, which might lead to producing a continuous atom laser avoiding losses due to scattered light [7]. Also, Helfer et al. have used optical tweezers and single-particle tracking of the thermally excited position fluctuations of probe beads attached to membranes in order to determine the viscoelastic behavior of actin-coated membranes [8].

Fluctuations in the beam can be considered as kinetic heating leading to particle escape, which can be avoided by means of feedback optimal damping [9]. Also, Dalibard et al. proposed a stable three-dimensional optical trap with a small escape probability [10]. Stable magneto-optical traps with a lifetime of atoms in the magnetic trap in the range of 300s have been used to study elastic collision cross sections in fermionic atoms by means of evaporative cooling [11]. In this article we use Lyapunov second theorem as well as LaSalle invariance theorem in order to determine the attraction region of an optical trap and to design an steering strategy. The paper is organized as follows. In Section 2 we present the physical model of the optical tweezers and we establish the problem statement. In Section 3 we discuss the stability properties with its respective domain of attraction. The transfer maneuver strategy is also introduced in this section. In Section 4 we present some numerical simulations. Finally, Section 5 is devoted to conclusions.

2 Problem Formulation

Consider a particle of mass $m$, which is trapped in a Potential field with a Gaussian distribution ($[1, 15]$). Let $x$ and $y$ be the horizontal position and vertical position of the particle, with respect to a fixed reference frame. The nonlinear model of this system, which can be obtained from the Euler-Lagrange equations is given by

$$\dot{x} = \frac{\gamma}{m}x'$$

$$-\frac{2 \ln(2) p_0(x-x_0)}{ma^2} e^{\ln(2)} \left( \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} \right)$$

$$\dot{y} = \frac{\gamma}{m}y'$$

$$-\frac{2 \ln(2) p_0(y-y_0)}{mb^2} e^{\ln(2)} \left( \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} \right)$$

(1)

(2)

Remark 1: The variables $x_0$ and $y_0$ can be seen as the control variables of the system (1). That is, changing adequately the positions of $x_0$ and $y_0$, we can steer the particle of mass $m$, from an initial position to another desired position. It is easy to show by simple linearization that the above system has one stable equilibrium point, defined by $(x_0, y_0, 0, 0)$, and a set of unstable equilibrium points given by $(x \to \pm \infty, y \to \pm \infty, 0, 0)$. It means that if the particle is located far enough from the centroid $(x_0, y_0)$, then it cannot be trapped by the Gaussian potential, unless we move the centroid to a neighborhood near the particle.

To simplify the algebraic manipulation in the forthcoming developments, we normalize the earlier equations by introducing the following scaling transformations

$$x_1 = \frac{x}{a}; y_1 = \frac{y}{b}; x_2 = \frac{x'}{ak}; y_2 = \frac{y'}{bk};$$

(2)

$$\lambda = \frac{b^2}{a^2}; \gamma = \frac{\gamma}{km}; \tau = kt; k = \sqrt{\frac{2 \ln(2) p_0}{a^2 m}};$$

(3)

with the new controllers defined by $x_a = x_0 / a$ and $y_b = y_0 / b$. So, system (1) can be rewritten as:

$$\dot{x}_2 = -\lambda x_2$$

$$-(x_1 - x_a) \exp[-\ln(2)(x_1 - x_a)^2 + (y_1 - y_b)^2]$$

$$\dot{y}_2 = -\lambda y_2$$

$$-\frac{(y_1 - y_b)}{\lambda} \exp[-\ln(2)(x_1 - x_a)^2 + (y_1 - y_b)^2]$$

(4)

We show that the above normalized system is locally asymptotically stable at the origin $(x_a, y_b)$, with an estimable domain of attraction. That is, the original optical-trap-device proposed in [13] allows us to capture one particle if it lies in an estimable domain of attraction. Based on this, we propose an energy based steering strategy for moving the particle from one initial position to another final position. We must emphasize that the stability properties system (4) were not considered in [13].

We must recall that the domain of attraction is formed by the set of all initial conditions that converge to the equilibrium point. That is, the region of attraction of system (4), denoted by $R_A$, is defined by

$$R_A = \{(q_1, q_2) : \phi(t, q_1, q_2) \to (\pi, 0) \text{ as } t \to \infty\}$$

(5)
where \( \overline{q} = (x_a, y_b) \) and \( \phi(t, q_1, q_2) \) denotes the solution of (4) which starts in the initial state \( q_1 = (x_1, y_1) \) and \( q_2 = (x_2, y_2) \), at time \( t = 0 \).

Finally, we introduce the following notation. We use \( \omega, \sigma \) and \( \omega(0) \) to denote the vector states \( \omega^T = (x_1, y_1, x_2, y_2) = (q_1, q_2), \sigma^T = (x_a, y_b, 0, 0) \), and \( \omega(0) = (x_1(0), y_1(0), x_2(0), y_2(0)) \).

### 3 Problem Solution

In this section we first show that the normalized model is stable in the Lyapunov sense, for some domain of stability which can be estimated with high precision. For that purpose, let us introduce the following energy function:

\[
V(\omega) = \frac{1}{2} (x_2^2 + \lambda y_2^2) + \frac{1}{2 \ln(2)} \left( 1 - e^{-\ln 2 (e_1^2 + e_2^2)} \right)
\]

(6)

where \( e_1 = x_1 - x_a \) and \( e_2 = y_1 - y_b \). Notice that \( V \) is locally positive definite in some neighborhood of \( \omega_0 \), as a matter of fact, \( \omega_0 \) is an isolated local minimum of \( V \). On the other hand, as the time derivative of \( V \) along of the trajectories of the system (4), leads to:

\[
V'(\omega) = -\gamma \frac{1}{2} (x_2^2 + \lambda y_2^2)
\]

(7)

then the equilibrium \( \omega_0 \) of model (4) is locally stable in the sense of Lyapunov. To assure that system (4) is locally asymptotically stable we need to invoke LaSalle’s invariance theorem [17]. Recalling that the invariance principle requires that the trajectories are bounded and remain inside the set where \( V' \leq 0 \). In order to apply this theorem, we define the following set

\[
B_\delta = \{ \omega = (q_1, q_2) : V(\omega) \leq \delta \}
\]

(8)

where \( \delta \) is the largest value such that \( B_\delta \) is a compact set. As \( V \) is formed by a strictly positive definite kinetic energy, plus a locally positive function around to the origin \( q_0 \), then one way to estimate \( B_\delta \) is to fix \( \delta = \delta_\mu = \mu/(2 \ln 2) \); \( \mu \in (0, 1) \) (See Appendix).

**Remark 2:** The set (8) defines an invariance set, since for all initial conditions \( \omega_0 \) such that \( \omega_0 \in B_\delta \), it follows that \( \omega(t) \in B_\delta \) for \( t = 0 \). That is, every trajectory of system (4) starting in \( B_\delta \) stays for all future in the compact set \( B_\delta \). This a consequence of the fact that \( V \) is a non-increasing function. Indeed, \( V(\omega_0) \leq \delta_\mu \) implies that \( V(\omega(t)) \leq V(\omega_0) \leq \delta_\mu \).

Continuing with LaSalle’s theorem, we define the following invariant set

\[
S = \{ \omega = (q_1, q_2) \in B_\delta : V'(\omega) = 0 \}
\]

(9)

and we proceed to compute the largest invariant set \( M \subset S \). LaSalle’s theorem assures that all the solutions that starts in \( B_\delta \) asymptotically converges to \( M \); where \( M \) is the largest invariant set contained in \( S \) [17]. From (7), it follows that \( x_2 = 0 \) and \( y_2 = 0 \) are in the set \( S \), hence, we have that \( x_1 = \pi \) and \( y_1 = \gamma \), where \( \pi \) and \( \gamma \) are constants. On the other hand, in the invariant set \( S \), we clearly have that \( x_2 = 0 \) and \( y_2 = 0 \), substituting of all these values in (4), we have

\[
|\pi - x_0| = 0 \text{ and } |\gamma - y_0| = 0
\]

(10)

Hence, \( x_1 = x_a \) and \( y_1 = y_b \) are in the set \( S \). Therefore the largest invariant set \( M \) contained in \( S \) is constituted by the single rest equilibrium point. So that from LaSalle’s theorem, we have that all the solutions starting in \( B_\delta \) asymptotically converge towards the largest invariant set \( M \), which is the rest point \( \sigma \). Notice that the set \( B_\delta \) can be taken as one estimate of the region of attraction \( R_A \).

We summarize of the previous discussion as follows.

**Proposition 1:** The nonlinear system (4) is locally asymptotically stable for any strictly positive , with a computable domain of attraction defined by (8). On the other hand if \( \gamma = 0 \) then the mentioned system is stable in Lyapunov sense, with \( B_\delta \), as the domain of attraction.

Physically, the last proposition means that the proposed physical-device allows us to trap the particle of mass \( m \), as long as the initial energy of the particle is less than the estimate of \( \delta_\mu \), which is related to the well depth . That is, if a particle is initialized in \( B_\delta \), then it will converge asymptotically to \( \sigma \). Constant \( \mu \) can be selected almost equal to one (at least theoretically). However, for computational purpose, we take \( \mu = 0.9 \) to avoid numerical mistakes.
3.1 Steering tasks
In this section we propose a simple methodology for steering the particle of mass \( m \), from an initial rest position to a desired final rest position. Physically, we want to move the particle from a given initial position \( q_i = (x_i, y_i) \) to another rest equilibrium position \( q_f = (x_f, y_f) \), by following a convenient trajectory. The tracking errors between the actual and the desired positions are defined as follows:

\[
e^*_x = x_i - x_f(t); \quad e^*_y = y_i - y_f(t)
\]

Assumption 1: Suppose that the particle rests very close to the position \((x_i, y_i)\) where the smooth variables \( x_a(t) \) and \( y_b(t) \) are selected such that:

\[
e^*_x(0) < c_1; \quad e^*_y(0) < c_2
\]

\[
e^*_x(0) < c_3; \quad e^*_y(0) < c_4
\]

with \( c_i \) small enough; for \( i = 1, \ldots, 4 \).

Remark 3: The above set of inequalities means that the variables \( x_a \) and \( y_b \) must be selected close enough to the actual initial conditions of the particle

Then, differentiating twice the above errors with respect to time and after using (4), the following error dynamics are obtained:

\[
e^*_{x x} = -\mathcal{P}^*_x - e^*_x \exp[-\ln(2)[e^*_x + e^*_y]] - \Phi(x_a)
\]

\[
e^*_y = -\mathcal{P}^*_y - \frac{1}{\lambda} e^*_y \exp[-\ln(2)[e^*_x + e^*_y]] - \Phi(y_b)
\]

(13)

where

\[
\Phi(x_0) = \mathcal{P}^{\prime} x_0 + x_0'' , \Phi(y_0) = \mathcal{P}^{\prime} y_0 + y_0''
\]

Selecting \( x_a \) and \( y_b \) such that \( \Phi(x_a) = \Phi(y_b) = 0 \), we have that

\[
x_a(t) = x_f + (x_i - x_f) \exp(-\mathcal{P}^*_x);
\]

\[
y_b(t) = y_f + (y_i - y_f) \exp(-\mathcal{P}^*_y)
\]

(14)

Hence, system (13) is equivalent to system (4), if the control variables \( x_a \) and \( y_b \) are defined as (14).

Therefore, system (13) is locally asymptotically stable with the domain of attraction described in Proposition 1. Physically, it means that the proposed trap allows us to change the initial position of the particle when the trajectory is selected as (14). Notice that, the estimates of \( c_3 \) and \( c_4 \) are given by:

\[
c_3 = \left| x_i(0) - \mathcal{P}(x_i - x_f) \right|
\]

\[
c_4 = \left| y_i(0) - \mathcal{P}(y_i - y_f) \right|
\]

(15)

We summarize all the above by presenting the proposition:

**Proposition 2**: Under the Assumption A1 and provided that:

\[
\left( c_i^2 + \lambda c_4^2 \right) + \frac{1}{\ln 2} \left( 1 - \exp(- \ln(2) (c_i^2 + c_4^2)) \right) \leq \frac{1}{\ln 2}
\]

(16)

then the errors \( e_x \) and \( e_y \) locally asymptotically converge to zero.

The above proposition suggests how to steer the particle from an initial rest position to a final desired position. That is, we can carry out this task if the estimated initial tracking errors satisfy the inequality (16). On the other hand, if the constants \( c_3 \) and \( c_4 \) are very large, then we recommend to make a few smaller movements, by taking smaller values of \( x_f \) and \( y_f \), until the particle reaches the final desired position.

4 Numerical simulations
In this section we carried out some numerical simulations using an implementation of the four order Runge-Kutta algorithm to show the behavior of the proposed Optical Tweezers.

In the first numerical experiment we illustrate the nonlinear behavior of the system (4). The physical parameters, except \( \gamma \), were taken from a real experiment in [13, 1], namely: \( a = 600nm, \quad b = 300nm, \quad x_0 = 0, \quad y_0 = 0, \quad p_0 = 1x10^{-6} nJ, \quad m = 1x10^{-15} Kg \). The parameter \( \gamma \) was estimated using Stokes formula given by \( \gamma = 6\pi r \rho = 1.89x10^{-10} Kg / s , \) where \( \eta = 1x10^{-3} Kg / (ms) \) is the viscosity of water and \( r = 10nm \), is the radius of the particle. (see [14]) [1] for details). For the first experiment the initial conditions were set as \( \omega = (x_0 = 1200nm, y_0 = 900nm, x'_0 = 0, y'_0 = 0) \).

Notice that this initial condition lies in the domain of attraction, since \( V(\omega) \geq 9.27x10^{-6} J < p_0 \). Fig 1 shows the phase space plot of a particle in the actual coordinates system. As we can see, both position system coordinates converge asymptotically to the rest position, in a period of time of \( t \equiv 1x10^{-4} \).
Fig. 1.- Shows the phase space plot of a particle in the actual coordinates system. As we can see, both position system coordinates converge asymptotically to the rest position, in a period of time of \( t \approx 1 \times 10^{-4} \) s.

In the second simulation we carried out the translation task from the initial position \((x_i = 0, y_i = 0)\) to the final position \((x_i = 6 \times 10^{-6} \text{ m}, y_i = 2.45 \times 10^{-6} \text{ m})\). The initial condition was set as \((x_0 = 0, y_0 = 0, x_0' = 0.1 \text{ m/s}, y_0' = -0.11 \text{ m/s})\). That is, it is located is, it is located a little bit away from the equilibrium point. To do it, we used the tracking signals proposed in (14). Figure 2 shows the phase space plot of the system in the actual coordinates. As we can see, the proposed steering strategy works fine even if the particle is not initially located at its rest position.

Fig. 2.- Shows the phase space plot of the system in the actual coordinates. As we can see, the proposed steering strategy works fine even if the particle is not initially located at its rest position.

Fig. 3 depicts how \( x \) and \( y \) follow the positions of \( x_0 \) and \( y_0 \), respectively. Notice that after \( 4 \times 10^{-15} \) s elapsed, both tracking errors converge almost to zero.

Finally, we performed the tracking task without the presence the damping force and using the same initial condition and the same physical parameters as in the second simulation. In figure 4, we can see that the translation task started at \( q_i = (0, 0) \) and finished at \( q_f = (7.8 \times 10^{-7} \text{ m}, 7.8 \times 10^{-7} \text{ m}) \), in a finite time interval \([t_i = 1 \times 10^{-5} \text{ s}, t_f = 7.65 \times 10^{-5} \text{ s}]\). As we can see, the translation task is efficiently carried out. Besides, the positions of the particle remain oscillating around to the tracking signals, due to the lack of the damping force.

Fig. 3.- Depicts how \( x \) and \( y \) follow the positions of \( x_0 \) and \( y_0 \), respectively. Notice that after \( 4 \times 10^{-15} \) s elapsed, both tracking errors converge almost to zero.

Fig. 4.- The positions of the particle remain oscillating around to the tracking signals, due to the lack of the damping force.

5 Conclusions
In this work we present a simple strategy to move a microscopic particle from an initial rest position to a given final position. We show that the system is local
asymptotically stable when a viscous force acts over the particle. We also show that the system is only stable when the viscous force is absent, since in this case all the state variables are confined to move inside of a certain domain of attraction. Roughly speaking, the steering strategy consists of tracking a convenient trajectory, ensuring that the error between the actual particle position and the position of the geometric center of the optical tweezers always lies inside of a proper domain of attraction. The convergence stability analysis was carried out by using the simple Lyapunov method in conjunction with LaSalle theorem. Numerical simulations were carried out to show the performance and effectiveness of the proposed steering strategy.

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7 Appendix
We show that the set \( B_\delta \) is compact set where. From definition (8) where \( \delta = \mu / (2 \ln 2) \) and \( 0 < \mu < 1 \), we have

\[
B_\delta = \{ \omega = (q_1, q_2) : \frac{1}{2} (x^2 + \lambda y^2) + \frac{1}{2 \ln 2} (1 - \exp(-\ln 2 (e^2 + e_y^2))) \leq \frac{\mu}{2 \ln 2} \}
\]

(17)

From the above \( q_1 \) and \( q_2 \) satisfy

\[
\left( x^2 + \lambda y^2 \right) \leq \frac{\mu}{2 \ln 2}
\]

and

\[
1 - \exp(-\ln 2 (e^2 + e_y^2)) < \mu
\]

Evidently the above two inequalities define respectively an ellipse region. Consequently, both sets are a compact set for all \( 0 < \mu < 1 \).

References: