About Fourier Series in study of Periodic Signals

Mioara BONCUȚ
University Lucian Blaga of Sibiu, ROMANIA
mioara.boncut @ulbsibiu.ro

Amelia BUCUR
University Lucian Blaga of Sibiu, ROMANIA
amelia.bucur @ulbsibiu

Abstract. This work is about Fourier Series in study of Periodic Signals.A periodic function can be represented by means of Fourier series, which contains all the information about the harmonic structure. The Fourier series is an infinite sum of sinusoidal and cosenoidal terms, however, when the series is truncated, a high frequency phenomenon appears superimposed to the resulting finite Fourier expansion, referred as Gibbs oscillations. In this paper, such a phenomenon is analyzed for a function which is m times differentiable. The case $m=0$ is studied in a discontinuous functions.

Key-Words. Gibbs phenomenon, high frequency oscillations, remainder term, Fourier series, ideal low-pass filter.

I. INTRODUCTION

Periodic functions are signals commonly found in communication systems, which are formed upon an infinite number of harmonic terms whose frequencies are multiples of the fundamental frequency ω_0 [1]. In communication systems there are devices known as filters, whose duty is to let passing a band of frequencies to the next stage, while the others result rejected. When a filter allows passing the low frequencies, it is named a low-pass filter. If such a device is excited by a periodic function, the response will be determined by the finite Fourier expansion, composed only by low frequency terms.

The Gibbs phenomenon is a high frequency oscillation, modulated in amplitude as well as in phase, which appears after getting rid of the upper frequencies in a periodic signal [2]. In this paper, the Gibbs phenomenon is studied on the basis of the remainder term of Fourier series, where explicit formulae are gotten for several cases. In special, the case of a discontinuous function let us validate the general analysis shown here. The validation is reached by treating the finite Fourier expansion as a slow varying-type series, whose integral expression is immediate.

II. FOURIER SERIES FOR FUNCTIONS OF HIGH DIFFERENTIABILITY

Fourier series is a mathematical tool for analyzing the frequency content of a periodic signal $f(x)$. If the function is analytic, it can be differentiated indefinitely., however, the Fourier series is frequently applied to non-periodic functions, which are made artificially periodic by extending its original domain beyond the $[-\pi,\pi]$ interval. Let us suppose that $f(x)$ satisfies the boundary conditions:

$$\begin{gathered} f(\pi)=f(-\pi), \qquad f'(\pi)=f'(-\pi), \\ \vdots \\ f^{(m-1)}(\pi)=f^{(m-1)}(-\pi). \end{gathered} \tag{1}$$

However, due the periodic extension, the boundary conditions cannot be preserved until the mth derivative. Let us write the Fourier series in the trigonometric form [3]:

$$\begin{gathered} f(x)-\frac{a_0}{2}=\sum_{k=1}^{\infty}a_k\cos kx+\sum_{k=1}^{\infty}b_k\sin kx, \\ a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\xi)\cos k\xi d\xi, \qquad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\xi)\sin k\xi d\xi. \end{gathered} \tag{2}$$

where $\omega_0=1$, since $T=2\pi$. The boundary conditions permit us express the coefficients a_k and b_k in terms of the mth derivative, after integrating them by parts m times. When gathering together, they form a complex number:

$$a_k+jb_k=\frac{j^m}{\pi k^m}\int_{-\pi}^{\pi}f^{(m)}(\xi)e^{jk\xi}d\xi, \tag{3}$$

and the Fourier series results in the real part of:

$$f(x)-\frac{a_0}{2}=\sum_{k=1}^{\infty}(a_k+jb_k)e^{-jkx}=\int_{-\pi}^{\pi}f^{(m)}(\xi)G_m(\xi-x)d\xi,$$
$$G_m(\xi-x)=\frac{j^m}{\pi}\sum_{k=1}^{\infty}\frac{e^{jk(\xi-x)}}{k^m},$$
(4)

where $G_m(\xi-x)$ is a kernel-type series. By changing the integrating variable as $\theta=\xi-x$, we have:

$$f(x)-\frac{a_0}{2}=\int_{-\pi}^{\pi}f^{(m)}(\theta+x)G_m(\theta)d\theta. \quad (5)$$

Since the summation is performed over all the terms, it is named a series. On the other hand, if the summation is carried until the nth term, it should be referred as a finite expansion [4], which is the real part of:

$$f_n(x)-\frac{a_0}{2}=\int_{-\pi}^{\pi}f^{(m)}(\theta+x)\left[\frac{j^m}{\pi}\sum_{k=1}^{n}\frac{e^{jk\theta}}{k^m}\right]d\theta. \quad (6)$$

The difference between $f(x)$ and $f_n(x)$ is the so called remainder term of the Fourier series, which indicates how near we are to $f(x)$ from $f_n(x)$, after summing n terms. If the Fourier series converges uniformly to $f(x)$, the remainder term tends to zero when n tends to infinite:

$$\eta_n(x)=f(x)-f_n(x)=\int_{-\pi}^{\pi}f^{(m)}(\theta+x)g_n{}^m(\theta)d\theta,$$
$$g_n{}^m(\theta)=\frac{j^m}{\pi}\sum_{k=n+1}^{\infty}\frac{e^{jk\theta}}{k^m}, \quad (7)$$

where $g_n{}^m(\theta)$ is another kernel-type series. The remainder term thus defined, is built upon a periodic function which is m times differentiable, where the mth derivative becomes discontinuous at the point $x=x_1$. That the function shows a discontinuity should not be viewed as an unfavorable fact, but as an advantage which let us estimate reliable the remainder term of Fourier series while finite expansion approaches to the original function.

III. THE GIBBS PHENOMENON FOR FUNCTIONS OF HIGH DIFFERENTIABILITY

Let us assume the existence of all the derivatives of $f(x)$, of order bigger than m, where $f^{(m)}(x)$ becomes discontinuous at certain point $x=x_1$ of its domain. Formula (7) shows that the remainder term is determined only by the integral operator, instead of the functional form of $f^{(m)}(x)$. Therefore, before stopping at the mth derivative, we can continue until the next one, and treat the discontinuity as a jump in the integral of the $m+1^{th}$ derivative. It implies that the greater part of the integral (7) is located around the immediate vicinity of $x=x_1$. The same will occur when the $m+1^{th}$ derivative becomes not infinite, but much bigger than the other values of the range. Since the function has been made periodic artificially by extending the original domain, we can change the origin at the point $x=x_1$, and integrate around the vicinity of it:

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}f^{(m+1)}(\xi)g_n{}^{m+1}(\xi-x)d\xi=g_n{}^{m+1}(-x)\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}f^{(m+1)}(\xi)d\xi,$$
(8)

after evaluating the definite integral, we get:

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}f^{(m+1)}(\xi)d\xi=f^{(m)}(0^+)-f^{(m)}(0^-)=A, \quad (9)$$

where A is the magnitude of the jump performed by $f^{(m)}(x)$ at the origin. Since $f^{(m+1)}(x)$ is generally regular and grows big enough at the vicinity of $x=0$, we can ignore the rest of the integral since these contributions are negligible, therefore [5]:

$$\eta_n(x)=\int_{-\pi}^{\pi}f^{(m+1)}(\xi)g_n{}^{m+1}(\xi-x)d\xi\approx Ag_n{}^{m+1}(-x). \quad (10)$$

Before getting an explicit formula for the kernel-type series $g_n{}^{m+1}(-x)$, let us analyze its mathematical properties. As can be seen, the series is formed by a periodic term $e^{jk\theta}$ and a slow-varying type function $1/k^{m+1}$, whose variations are negligible in successive values of k. In general, the series can be written like:

$$\sum_{k=1}^{\infty}e^{jk\theta}\varphi(k), \quad (11)$$

where $\varphi(k)$ is the slow-varying type function. When integrating by parts the kth term around $\xi=k$, we have:

$$\int_{k-1/2}^{k+1/2}\varphi(\xi)e^{j\xi\theta}d\xi=\int_{-1/2}^{1/2}\varphi(k+t)e^{j(k+t)\theta}dt$$
$$=\frac{1}{j\theta}\left[\varphi(k+t)e^{j(k+t)\theta}\Big|_{-1/2}^{1/2}-\int_{-1/2}^{1/2}\varphi'(k+t)e^{j(k+t)\theta}dt\right]. \quad (12)$$

Because $\varphi(k)$ is a slow-varying function, its derivative is negligible, which makes possible the approximation:

$$\int_{-1/2}^{1/2}\varphi(k+t)e^{j(k+t)\theta}dt\approx\frac{e^{jk\theta}}{j\theta}\left[\varphi(k+1/2)e^{j\theta/2}-\varphi(k-1/2)e^{-j\theta/2}\right],$$
(13)

but accordingly with the hypothesis, it is valid to consider that $\varphi(k+1/2)\approx\varphi(k-1/2)\approx\varphi(k)$, therefore:

$$\int_{-1/2}^{1/2}\varphi(k+t)e^{j(k+t)\theta}dt\approx e^{jk\theta}\varphi(k)\frac{\operatorname{sen}\theta/2}{\theta/2}. \quad (14)$$

After working out the kth term from this formula, we can substitute it in the original series:

$$\sum_{k=1}^{\infty} e^{jk\theta}\varphi(k) \approx \frac{\theta}{2\operatorname{sen}\theta/2}\sum_{k=1}^{\infty}\int_{k-1/2}^{k+1/2}\varphi(\xi)e^{j\xi\theta}d\xi . \qquad (15)$$

We can see that the limits of the integrals are contiguous, such that can link them together for making one integral:

$$\sum_{k=1}^{\infty} e^{jk\theta}\varphi(k) \approx \frac{\theta}{2\operatorname{sen}\theta/2}\int_{1/2}^{\infty}\varphi(\xi)e^{j\xi\theta}d\xi . \qquad (16)$$

On the basis of this formula, the kernel-type series is approximated by the integral:

$$g_n^{m+1}(x) \approx \frac{x}{2\pi\operatorname{sen}x/2}\, j^{m+1}\int_{n+1/2}^{\infty}\frac{e^{j\xi x}}{\xi^{m+1}}d\xi , \qquad (17)$$

where the inferior limit is set as $n+1/2$, since the series begins from $n+1$. When shifting the integral limits toward the origin, we have:

$$g_n^{m+1}(x) \approx \frac{x}{2\pi\operatorname{sen}x/2}\, j^{m+1}e^{jxn'}\int_{0}^{\infty}\frac{e^{jx\xi}}{(\xi+n')^{m+1}}d\xi , \qquad (18)$$

where $n' = n+1/2$. The integration of (18) is difficult to perform, however, we can get a good approximation in terms of fundamental functions. Let us consider the following one:

$$y(\xi) = \frac{e^{jx\xi}}{(n'+\xi)^m\left[jx(n'+\xi)-m-1\right]} . \qquad (19)$$

Its derivative is formed by three terms:

$$\frac{dy(\xi)}{d\xi} = y(\xi)\left[jx - \frac{m}{n'+\xi} - \frac{jx}{jx(n'+\xi)-m-1}\right], \qquad (20)$$

where $jx/\left[jx(n'+\xi)-m-1\right] \approx 1/(n'+\xi)$, since n is large enough. Hence:

$$\frac{dy(\xi)}{d\xi} \approx y(\xi)\left[jx - \frac{m}{n'+\xi} - \frac{1}{n'+\xi}\right] = \frac{e^{jx\xi}}{(n'+\xi)^m} . \qquad (21)$$

The above approximation fails in the domain of small x, but even there the approximation is very good, excluding the case $m=0$. Hence, (18) becomes:

$$g_n^{m+1}(x) \approx \frac{j^{m+1}e^{jxn'}x}{2\pi\operatorname{sen}x/2}\left.\frac{e^{jx\xi}}{(n'+\xi)^m\left[jx(n'+\xi)-m-1\right]}\right|_0^{\infty} . \qquad (22)$$

The evaluation of the limits brings us the equation:

$$g_n^{m+1}(x) \approx \frac{x}{2\pi\operatorname{sen}x/2}\frac{j^{m+1}e^{jx(n+1/2)}}{(n+1/2)^m\left[m+1-jx(n+1/2)\right]} . \qquad (23)$$

In the calculation of the remainder term, the argument for the kernel-type function is negative. However, before trying to change the sign of x, it is necessary to consider the parity of m. Let m be an odd number, expressed as $m=2s+1$, where $s=0,1,2,\cdots$, then:

$$j^{m+1} = j^{2s+2} = -j^{2s} = (-1)^{s+1} . \qquad (24)$$

On the other hand, if m is an even number, it can be written as $m=2s$, therefore:

$$j^{m+1} = j^{2s+1} = j\cdot j^{2s} = j(-1)^s . \qquad (25)$$

Given that $g_n^{m+1}(x)$ is a complex quantity, it can be written in polar form as:

$$g_n^{m+1}(x) \approx \frac{x}{2\pi\operatorname{sen}x/2}\frac{j^{m+1}}{(n+1/2)^m}\frac{e^{j\left[x(n+1/2)+\alpha\right]}}{\sqrt{(m+1)^2+x^2(n+1/2)^2}} , \qquad (26)$$

where α is a phase angle defined as:

$$\alpha = \arctan\frac{x(n+1/2)}{m+1} \qquad (27)$$

For calculating g_n^{m+1}, we must use the real part of (26), from which the remainder term is expressed like:

If $m=2s+1$ we have:

$$\eta_n(x) \approx \frac{Ax}{2\pi\operatorname{sen}x/2}\frac{(-1)^{s+1}}{(n+1/2)^m}\frac{\cos\left[x(n+1/2)+\alpha\right]}{\sqrt{(m+1)^2+x^2(n+1/2)^2}} . \qquad (28)$$

If $m=2s$ we get:

$$\eta_n(x) \approx \frac{Ax}{2\pi\operatorname{sen}x/2}\frac{(-1)^{s}}{(n+1/2)^m}\frac{\sin\left[x(n+1/2)+\alpha\right]}{\sqrt{(m+1)^2+x^2(n+1/2)^2}} . \qquad (29)$$

The above equations show explicitly the appearance of high frequency oscillations around the discontinuity, due the finite Fourier expansion. Such a phenomenon is known as Gibbs oscillations, after J. Willard Gibbs.

Gibbs phenomenon is an oscillation of frequency $n+1/2$, superimposed to $f(x)$. The phase angle begins with $\alpha=0$ for $x=0$, and grows slowly to a value near of $\pi/2$ for higher values of n. The nodal points for the sinusoidal oscillations and the maximum-minimum points for the cosenoidal oscillations are close to the zeroes given by:

$$x = \frac{k\pi-\alpha}{n+1/2} = \frac{k-1/2}{n+1/2}\pi , \quad k\in\mathbb{Z} . \qquad (30)$$

The maximum amplitude for the oscillations is:

$$\frac{A}{2\pi(n+1/2)^m} . \qquad (31)$$

In general, the phenomenon is quiet independent from the mth order of the derivative which becomes discontinuous. Only the magnitude of the oscillations is influenced inversely by it.

The formulae for the remainder term should be view as sinusoidal or cosenoidal signal modulated in both amplitude and phase. That the Gibbs phenomenon depends on the parity of m can be analyzed in the following way: Let $f(x)$ be written in terms of an even function $g(x)$ and an odd function $h(x)$:

$$g(x) = \frac{1}{2}\left[f(x)+f(-x)\right], \quad h(x) = \frac{1}{2}\left[f(x)-f(-x)\right]. \qquad (32)$$

If $f(x)$ results in an even function, the Fourier series has only cosenoidal terms, while if $f(x)$ is an odd function, the Fourier series is composed just by sinusoidal terms. Therefore, the remainder term should share the parity of $f(x)$ in this way: If an even

derivative of the function becomes discontinuous, the discontinuity must belong to $h(x)$. On the other hand, if an odd derivative of the function becomes discontinuous, the discontinuity must belong to $g(x)$. In the first case $g(x)$ is smoother than $h(x)$, while in the second one, the inverse is true. Hence, in the first case the cosenoidal oscillations for the remainder term are neglected in regard to the sinusoidal oscillations, while in the second case occurs the inverse.

Although the discussion was devoted for the case of a single jump, the Gibbs oscillations behave similarly for a wider class of functions. If a function has a finite number of discontinuities, the Gibbs oscillations for each one should be superimposed for getting the total phenomenon. The resulting pattern will show the high frequency oscillation, modulated in amplitude as well as in phase.

IV. GIBBS OSCILLATIONS FOR DISCONTINUOUS FUNCTIONS: THE CASE $m=0$

When the discontinuity occurs in the function itself, we have the case $m=0$. In this situation, the formula (29) lacks of meaning for small values of x. In such a case, we must treat the remainder term in the following way:

$$\eta_n(x)=\int_{-\pi}^{\pi} f'(\theta+x)\,g_n(\theta)\,d\theta\,,\quad g_n(\theta)=-\frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{\operatorname{sen} k\theta}{k}\,. \tag{33}$$

Let us suppose that the discontinuity occurs at $x=0$. Then, after integrating in the whole period, we found that just the values of $f'(x)$ around the vicinity of $x=0$ should be considered in the integration process:

$$\eta_n(x)=\int_{-\pi}^{\pi} f'(\xi)\,g_n(\xi-x)\,d\xi\approx A g_n(-x)\,, \tag{34}$$

where A is the magnitude of the jump at the discontinuity. For a reliable estimation of $g_n(\theta)$, we can consider it as a slow-varying type series like (11), where $\varphi(k)=1/k$:

$$g_n(\theta)\approx-\frac{\theta}{2\pi\operatorname{sen}\theta/2}\int_{n+1/2}^{\infty}\frac{\operatorname{sen}\theta\xi}{\xi}\,d\xi\,. \tag{35}$$

By changing the integrating variable as $\zeta=\theta\xi$, we have:

$$g_n(\theta)\approx-\frac{\theta}{2\pi\operatorname{sen}\theta/2}\left[\int_0^{\infty}\frac{\operatorname{sen}\zeta}{\zeta}\,d\zeta-\int_0^{(n+1/2)\theta}\frac{\operatorname{sen}\zeta}{\zeta}\,d\zeta\right]. \tag{36}$$

The first integral can be solved by means the residuum theorem, giving the exact value of $\pi/2$. The second one corresponds with the sine integral function, **Fig. 1**:

$$g_n(\theta)\approx-\frac{\theta}{2\pi\operatorname{sen}\theta/2}\left[\frac{\pi}{2}-\operatorname{Si}(n+1/2)\theta\right]. \tag{37}$$

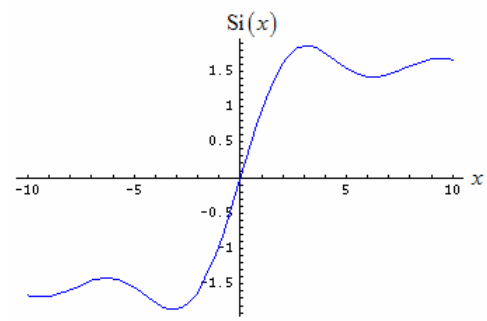

Figure 1. Plot of the sine integral function.

Finally, when changing the sign of θ we have:

$$\eta_n(x)\approx\frac{A}{\pi}\frac{x}{2\operatorname{sen}x/2}\left[\frac{\pi}{2}-\operatorname{Si}(n+1/2)x\right]. \tag{38}$$

In this very case, the remainder term follows a pattern which apparently contradicts the formula (29) for $s=0$, where a sinusoidal behavior is expected. However, a detailed analysis reveals that such contradiction is only apparent. In the region where $f_n(x)$ begins to raise, we cannot talk about Gibbs oscillations, since they start after the first nodal point has been reached, at $nx=1.9264$.

V. GIBBS PHENOMENON IN FILTERED PULSE SIGNALS

In communication systems, the harmonic analysis provides mathematical tools for studying signals and systems. For periodic signals we have the Fourier series, while for non-periodic ones we use the Fourier transform. For analyzing systems we can choose between Fourier transform and Laplace transform. The harmonic analysis lets us work in the frequency domain, where we can design filters which select specific frequency content of the communication signals.

Filters are classified as active or passive, if they are constructed on active or passive constituents, respectively. Filters are also classified as low-pass filters, if they let passing low frequencies; band-pass filters, if they select a band of frequencies; and high-pass filters, if they let passing high frequencies [6], **Fig. 2**. In practice, we don't find such an idealized behavior, since their slopes are big but finite.

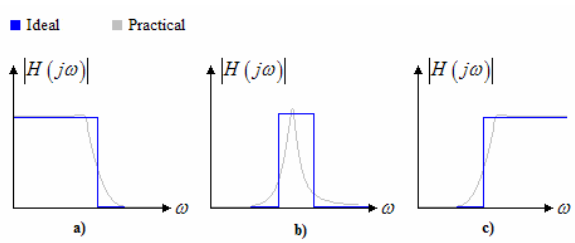

Figure 2. Electrical filters: a) Low-pass filter. b) Band-pass filter. c) High-pass filter.

When a periodic signal passes through a low-pass filter, the upper frequency content is clipped. At the filters' output, the resulting signal seems to be modified: if the signal has corners, after the filtering process they appear to be smoother, i.e., the fine signal details are generated by the high frequency components. Let us suppose that a square pulse passes through an ideal low-pass filter which has a bandwidth $B = n\omega_0$, **Fig. 3**. It is evident that such signal satisfies the boundary condition $f(\pi) = f(-\pi)$.

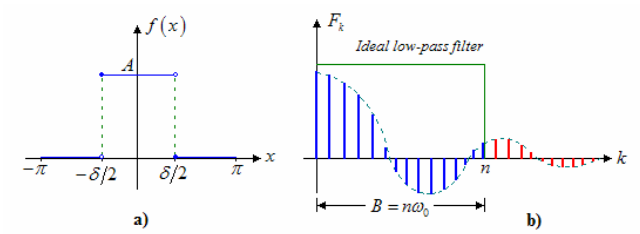

Figure 3. a) Pulse function. b) Spectrum plot and the action of the filter rejecting the high frequencies.

Since the pulse is an even function, it is built from cosenoidal components. At the filters' output, the frequencies upper than B will be rejected, therefore we get the finite expansion for the pulse function:

$$f_n(x) - \frac{A\delta}{2\pi} = \sum_{k=1}^{n} \frac{2A}{k\pi} \sin\left(\frac{k\delta}{2}\right) \cos kx . \tag{39}$$

Again, we get a slow varying-type series, which can be evaluated reliable by means eq. (15):

$$f_n(x) - \frac{A\delta}{2\pi} \approx \frac{Ax}{\pi \sin(x/2)} \int_{1/2}^{n-1/2} \frac{\sin(\xi\delta/2)\cos(\xi x)}{\xi} d\xi . \tag{40}$$

For this integral we can use the trigonometric identity $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta)$, which leads to two integrals:

$$\frac{1}{2}\int_{1/2}^{n-1/2} \frac{\sin\xi(\delta/2 - x)}{\xi} d\xi + \frac{1}{2}\int_{1/2}^{n-1/2} \frac{\sin\xi(\delta/2 + x)}{\xi} d\xi , \tag{41}$$

by changing the integrating variable in both of them we get:

$$\frac{1}{2}\int_{\frac{1}{2}(\delta/2-x)}^{(n-\frac{1}{2})(\delta/2-x)} \frac{\sin u}{u} du + \frac{1}{2}\int_{\frac{1}{2}(\delta/2+x)}^{(n-\frac{1}{2})(\delta/2+x)} \frac{\sin u}{u} du , \tag{42}$$

However, each integral can be divided in two parts:

$$\int_a^b \frac{\sin u}{u} du = \int_0^b \frac{\sin u}{u} du - \int_0^a \frac{\sin u}{u} du , \tag{43}$$

thus, when splitting both integrals in (42) we have:

$$\begin{aligned} &\frac{1}{2}\int_0^{(n-\frac{1}{2})(\delta/2-x)} \frac{\sin u}{u} du - \frac{1}{2}\int_0^{\frac{1}{2}(\delta/2-x)} \frac{\sin u}{u} du \\ &+ \frac{1}{2}\int_0^{(n-\frac{1}{2})(\delta/2+x)} \frac{\sin u}{u} du - \frac{1}{2}\int_0^{\frac{1}{2}(\delta/2+x)} \frac{\sin u}{u} du . \end{aligned} \tag{44}$$

That each integral has the lower limit as zero lets us to use the sine integral function. Therefore, the finite expansion for the pulse function is:

$$\begin{aligned} \frac{A\delta}{2\pi} - f_n(x) \approx &\frac{Ax}{2\pi\sin(x/2)}\left[\mathrm{Si}\frac{1}{2}\left(x+\frac{\delta}{2}\right) - \mathrm{Si}\left(n-\frac{1}{2}\right)\left(x+\frac{\delta}{2}\right)\right] \\ &- \frac{Ax}{2\pi\sin(x/2)}\left[\mathrm{Si}\frac{1}{2}\left(x-\frac{\delta}{2}\right) - \mathrm{Si}\left(n-\frac{1}{2}\right)\left(x-\frac{\delta}{2}\right)\right]. \end{aligned} \tag{45}$$

On the right side we have two interfering oscillations as a result of the discontinuities of the pulse function; the first bracket represents the oscillation of the discontinuity at $x = -\delta/2$, while the second one represents the oscillation of the discontinuity at $x = \delta/2$. Thus, the left side shows that the oscillatory action of Gibbs phenomenon is superimposed to the constant term $A\delta/2\pi$, which is the average of the function in the whole period, i.e., the DC component. The asymptotic behavior of the first summand in each bracket is, accordingly to **Fig. 1**, [7]:

$$\lim_{x\to\infty} \mathrm{Si}\frac{1}{2}\left(x \pm \frac{\delta}{2}\right) = \frac{\pi}{2} . \tag{46}$$

Accordingly to the theory developed previously for the case $m = 0$, we can superimpose two oscillations given by (38), at the discontinuity points of the pulse function. The oscillation at $x = -\delta/2$ is specified by:

$$\eta_{n,1}(x) \approx \frac{A(x+\delta/2)}{2\pi\,\mathrm{sen}\,\frac{1}{2}(x+\delta/2)}\left[\frac{\pi}{2} - \mathrm{Si}\left(x+\frac{\delta}{2}\right)\left(n+\frac{1}{2}\right)\right], \tag{47}$$

while the oscillation at $x = \delta/2$ is given by:

$$\eta_{n,2}(x) \approx -\frac{A(x-\delta/2)}{2\pi\,\mathrm{sen}\,\frac{1}{2}(x-\delta/2)}\left[\frac{\pi}{2} - \mathrm{Si}\left(x-\frac{\delta}{2}\right)\left(n+\frac{1}{2}\right)\right], \tag{48}$$

where the negative sign is due the direction of the jump at $x = \delta/2$. Hence, the total behavior is the superposition of both oscillations:

$$\eta_n(x) = \eta_{n,1}(x) + \eta_{n,2}(x) \tag{49}$$

The validity of equation (38) is demonstrated by considering the self evident similarities between the equations (49) and (45). The main difference lies in that the modulating function of (47) and (48) is shifted toward $x = \pm\delta/2$, while in formula (45) there is not such shifting. Furthermore, the asymptotic behavior (46) makes more evident the validity of the results. We must keep in mind that the equations which describe the Gibbs phenomenon are reliable approximations, and not exact formulae.

REFERENCES

[1] B. Lathi, **Modern Digital and Analog Communication Systems**. Oxford University Press, 3rd Ed. 1998.

[2] J. Foster and F.B. Richards, "*The Gibbs Phenomenon for Piecewise-Linear Approximatio*n", Amer. Math. Monthly **98** (1991) 47-49.

[3] N. Piskunov. **Differential and Integral Calculus**. Routledge 1st Ed. 1965, p. 323-363.

[4] P. Linz, **Theoretical Numerical Analysis**. Dover, New York, 2001, p. 29-68.

[5] C. Lanczos, **Linear Differential Operators**. Dover, New York 1997. Chap. II, p. 49-99.

[6] R. L. Boylestad and Louis Nashelsky, **Electronic Devices and Circuit Theory**. Pearson Prentice Hall 6th Ed. 1996, p. 683-700.

[7] N. G. de Bruijn, **Asymptotic Methods in Analysis**. Dover , New York, 1958, p.p.1.