A New Mathematical Model Based on Laplace and Modified Z Transform For Investigation of Three-Phase Switched Circuits

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Abstract : - A new mathematical method for time-domain analysis of power converters with periodic switching circuits is proposed. The method is based on mixed using of the Laplace and modified Z-Transform in linear periodically time-varying systems. The model was used for the analysis of three-phase voltage source inverters with Space Vector PWM feeding a three-phase static load, but it is applicable for all types of converters with an explicitly determined output voltage (converters with forced commutation) and periodical modulation. Instead of solution of algebraic equations for the initial current conditions the change of switching instants is reflected in the solution only by a change in two values $m_k$ and $n_k$. From the modulated waveforms we can easily obtained equations for the six step waveforms. The derived equations are validated using a 3 kW three-phase inverter.

Key-Words:- Mathematical model, Laplace transform, Modified Z-Transform

1 Introduction

Several methods have been presented for the time analysis of linear circuits containing periodically operated switches in electronic opened-loop systems [1],[2],[3]. However, the approach used in these methods depends heavily on matrix manipulations as they require matrix inversion as well as exponentiation. Besides, it requires solution of many algebraic equations.

Many electronic systems such as the inverters with Pulse Width Modulation (PWM) can be modeled with periodically varying parameters. In these inverters Space Vector PWM (SVPWM) has attracted great interest in recent years [4] since the harmonic characteristics are better than those of the other methods. At present, most of AC drives use some type of SVPWM.

Recent developments in high switching frequency power devices, such as IGBT, offer the possibility of developing high frequency PWM control techniques. Voltage waveforms of such modulated inverters contain many pulses and gaps. It is important to known current response for such complicated voltage waveforms in a drive design. In order to satisfy the required conditions for differential state equations describing the circuit behavior the continuity conditions due to the steady-state current at the transitions of states are used, i.e.:

$$i(t^+_s) = i(t^-_s) \quad (1a)$$

where $t_s$ are switching instants is a period.

Using (1a) and (1b) for the whole period of PWM we get algebraic equations that must be solved to obtain steady-state solution. In case of transient solution when current is not periodic during $T$ we must use (1a) and (1b) for the whole transient duration. The solution of the current response is intricate, as the number of the pulses is increasing.

This paper brings a new mathematical model that uses the Laplace and modified Z-transform (mixed p-z approach). The model enables one to determine both steady state and transient state in a relatively simple and lucid formula. Method for finding the Laplace transform of the voltage vector is also presented. The solution is not dependent on the number of the pulses of the PWM pattern. The change of the switching instants is reflected in the solution by a change in only two values.

2 Mathematical Model

We are going to investigate the three-phase half-bridge voltage source inverter fed from a DC voltage source and feeding a balanced three-phase Y-connected load. Generally, voltages and currents of three-phase circuits are explained by three variables, respectively. In case of three-phase load fed from a voltage source inverter shown in Fig.1, the phase voltages with respect to neutral point are:
The required voltage vector $V_{AV}$ is within the sampling period $\Delta T$ modulated as follows:

$$e^{j\rho}V_{AV} = V_{1} \frac{\Delta T_{1}}{T} + V_{2} \frac{\Delta T_{2}}{T} \quad ,$$

$$\Delta T = \Delta T_{1} + \Delta T_{2} + \Delta T_{0} \quad (6)$$

This type of modulation is called the Space Vector Modulation (SVM).

In (6) $\Delta T_{1}$ is dwell time of vector $V_{1}$, $\Delta T_{2}$ is dwell time of vector $V_{2}$, and $\Delta T_{0}$ is dwell time of zero vector $V_{0}$, or $V_{7}$.

$\Delta T$ is sampling interval.

$$\Delta T = T / N \quad (7)$$

$\rho$ is an angle that defines position of the reference vector $V_{AV}$ with respect to real axis in complex $\alpha\beta$ plane. $V_{1}$ and $V_{2}$ are adjacent to the voltage vector $V_{AV}$ in a given sector $n$, and the conduction per unit times are given from (6) by:

$$\Delta \epsilon_{1} = \Delta T_{1} / T = \epsilon_{1B} - \epsilon_{1A} = g \sin(60^\circ - \rho) / N_{1}$$

$$\Delta \epsilon_{2} = \Delta T_{2} / T = \epsilon_{2B} - \epsilon_{2A} = g \sin \rho / N_{1} \quad (8)$$

$$\Delta \epsilon_{0} = \Delta T_{0} / T = 1 / N_{1} - g \sin(60^\circ + \rho) / N_{1}$$

$\epsilon_{1A}$ and $\epsilon_{1B}$ are respectively, the beginning and end of duration of vector $V_{1}$, $\epsilon_{2A}$ and $\epsilon_{2B}$ are respectively, the beginning and end of duration of vector $V_{2}$, $\Delta \epsilon_{1}, \Delta \epsilon_{2}$ and $\Delta \epsilon_{0}$ are respectively, per unit dwell times (duty ratios) of the applied vectors.

$$g = \frac{V_{AV}}{2\sqrt{3}V_{dc}} \quad (9)$$

$g$ is the transformation (modulation) factor, $V_{dc}$ is the voltage of DC bus.

By substituting phase voltages for each switching state into (4), the following discrete space vectors are obtained:

$$V(n) = \frac{2V_{dc}}{3} e^{j\pi n / 3}, \quad n = 0, 1, 2, \ldots \quad (10)$$

These vectors thus form vertices of hexagon as shown in Fig.2.

As was mentioned, more vectors within sampling period are used. As the SVM is a periodical with $T$, the voltage vector can be expressed in $n$-th sector as,

$$V(n, \epsilon) = \sum_{k=1}^{M} \frac{2V_{dc}}{3} e^{jmn / 3} f(\epsilon, k) e^{jmn(k) / 3} \quad (11)$$

$M$ is number of the vectors, which are used within a sector $T$.

From (11) it can be seen, that all vectors are rotated in the next sector through $\pi / 3$, and in each sector are vectors modulated with time dependency given by $f(\epsilon, k)$, and also with the angle dependency given by $e^{jmn(k) / 3}$.
\( f(\epsilon, k) \) is a switching function which takes values 1 inside of \( \Delta \epsilon_k \) or 0 outside of \( \Delta \epsilon_k \). \( \alpha(k) \) defines the sequence of the phase shift of the used vectors, and for SVM with two adjacent vectors has value 1 or 0.

3 Laplace Transform of Voltage Vectors

To find the Laplace transform of (11) we can use relation between the Laplace and modified Z transform [8]. Using (5), and its derivation
\[ dt = T d\epsilon \]
we can write for the Laplace transform of the periodic voltage vector:
\[ V(p) = \sum_{n=0}^{\infty} \int V(n, \epsilon) e^{-p(n+\epsilon)T} d\epsilon = T \int V(z, \epsilon) e^{-pT \epsilon} d\epsilon \]  \hspace{1cm} (12)
Where is noted: \( z = e^{pT} \), \( z \) is operator of Z-transform.

\[ V(z, \epsilon) \] is the modified Z transform of \( V(n, \epsilon) \) [7,8] defined by equation:
\[ V(z, \epsilon) = \sum_{n=0}^{\infty} V(n, \epsilon) z^{-n} \]  \hspace{1cm} (13)

With regard to SVM strategy mentioned, we get from (12) and (13)
\[ V(p) = \frac{2V_{dc}}{3} \left( e^{pT} - e^{p\pi T/3} \right) \sum_{k=1}^{M} e^{j\pi \alpha(k)/3} (e^{-pT \epsilon_{kA}} - e^{-pT \epsilon_{kB}}) \]  \hspace{1cm} (14)
where \( \epsilon_{kA}T \) and \( \epsilon_{kB}T \) are respectively, the beginning and the end of application of k-th non-zero vector.

4 Current Response

Now, we suppose that voltage with the Laplace transform \( V(p) \) is feeding load with admittance:
\[ Y(p) = \frac{A(p)}{B(p)} = \sum_{s=m}^{l} \frac{A(p_s)}{B'(p_s)} \frac{1}{p - p_s} \]  \hspace{1cm} (15a)
where:
\[ B'(p_s) = \left[ \frac{dB}{dp} \right]_{p = p_s} \]  \hspace{1cm} (15b)
\( p_s \) are roots of the equation:
\[ B(p) = 0 \]
\( L_a \) is a order of the polynomial \( B(p) \). Thus, using (2) and (3) the Laplace transform of the load current can be expressed as:
\[ I(e^{pT}, p) = V(p)Y(p) = R(e^{pT})Q(p) \]  \hspace{1cm} (16)

As can be seen from (16), the Laplace transform of the current vector consists of two multiplicative parts. One \( (R(e^{pT})) \) is a function of \( z \)-operator, the other \( (Q(p)) \) is a function of \( p \)-operator.

\[ R(e^{pT}) = \frac{e^{pT}}{e^{pT} - e^{p\pi T/3}} \]  \hspace{1cm} (17)
\[ Q(p) = \frac{2V_{dc}}{3p} \frac{\Delta(p)}{B(p)} \sum_{k=1}^{M} e^{j\pi \alpha(k)/3} (e^{-pT \epsilon_{kA}} - e^{-pT \epsilon_{kB}}) \]
By transforming (16) into modified z-space we get:
\[ I(z, \epsilon) = R(z)Z_m(Q(p)) \]  \hspace{1cm} (18)

In order to find \( Z_m \) transform of \( Q(p) \) we must use the translation theorem in Z-transform which holds
\[ Z_m \{ e^{-p.a.F(p)} \} = z^{-x}F(z, \epsilon - a + x) \]  \hspace{1cm} (19)
with \( Z_m \{ \} \) denoting the modified Z transform operator.

And where parameter \( x \) is given by
\[ x = \begin{cases} 
1 & \text{for } 0 \leq \epsilon < a \\
0 & \text{for } a \leq \epsilon < 1 
\end{cases} \]  \hspace{1cm} (20)

If we want to express translation for k-th pulse, with the beginning \( \epsilon_{kA} \) and the end \( \epsilon_{kB} \), (pulse-width \( \Delta \epsilon_k = \epsilon_{kB} - \epsilon_{kA} \)) we can use two parameters, namely \( m_k \) and \( n_k \) to determine per unit time for prepulse, inside-pulse and postpulse, respectively.

\( m_k \) is a parameter that defines the beginning of k-the pulse \( \epsilon_{kA} \), \( n_k \) is a parameter that defines the end of k-pulse \( \epsilon_{kB} \), According to (20) we can write:
\[ m_k = \begin{cases} 
1 & \text{for } 0 \leq \epsilon < \epsilon_{kA} \\
0 & \text{for } \epsilon_{kA} \leq \epsilon < 1 
\end{cases} \]
and
\[ n_k = \begin{cases} 
1 & \text{for } 0 \leq \epsilon < \epsilon_{kB} \\
0 & \text{for } \epsilon_{kB} \leq \epsilon < 1 
\end{cases} \]  \hspace{1cm} (21)

Using parameters \( m_k, n_k \), and Heaviside theorem (15a), we can express (18) with help of (21) and (11) in the modified Z-space:
The trajectory is given by

\[ I(z, \varepsilon) = \frac{2V_{dc}}{3} \frac{z}{(z - e^{j\pi/3})} \sum_{k=1}^{M} \frac{A(p_\alpha)}{B(p_\beta)} \frac{z^{-n_k}}{z - 1} \] (22)

Equation (22) has simple poles \( e^{j\pi/3}, 1, e^{j\pi/3} \). The inverse \( Z \) transform of (22) can be found using the residue theorem.

\[ I(n, \varepsilon) = \frac{1}{2\pi j} \int I(z, \varepsilon) z^{-n-1} dz \] (23)

If doing so, we can express the time dependency of the load current by the following formula:

\[ i(n, \varepsilon) = \sum_{k=1}^{M} \frac{2V_{dc} e^{j\pi\alpha(k)/3}}{3} \frac{A(p_\alpha) e^{-j\pi\alpha/3}}{B(0)} \left( e^{j\pi/3} - 1 \right) \sum_{s=1}^{L} \frac{A(p_\beta B(p_\gamma) e^{j\pi/3} - p_\alpha T_e)}{z - e^{j\pi/3} L - e^{-j\pi/3} L} \] (24)

The solution contains two parts. Since \( p_\alpha \) includes a negative real part (we consider stable systems), the second portion of (24) consisting \( e^{j\pi T_{v1}/3} \) attenuates, for \( n \to \infty \), forming the transient component of the current space vector \( i_1(n, \varepsilon) \). The term

\[ e^{j\pi(n+1)/3} = \cos \pi(n+1)/3 + j \sin \pi(n+1)/3 \]

therefore, the first part of (24) is the steady-state component of the current space vector \( i_0(n, \varepsilon) \).

As an example, let us consider three-phase R-L series load. Equation (24) has only one simple root:

\[ p_1 = \frac{-R}{L} \] (25)

By substituting \( p_1 \) into (14) we can write for the load current components:

a) steady-state component:

\[ i_0(n, \varepsilon) = \sum_{k=1}^{M} \frac{2V_{dc} e^{j\pi\alpha(k)/3}}{3} \frac{A(p_\alpha) e^{-j\pi\alpha/3}}{B(0)} \left( e^{j\pi/3} - 1 \right) \sum_{s=1}^{L} \frac{A(p_\beta B(p_\gamma) e^{j\pi/3} - p_\alpha T_e)}{z - e^{j\pi/3} L - e^{-j\pi/3} L} \] (26)

b) transient component:

\[ i_1(n, \varepsilon) = \sum_{k=1}^{M} \frac{2V_{dc} e^{j\pi\alpha(k)/3}}{3} \frac{A(p_\alpha) e^{-j\pi\alpha/3}}{B(0)} \left( e^{j\pi/3} - 1 \right) \sum_{s=1}^{L} \frac{A(p_\beta B(p_\gamma) e^{j\pi/3} - p_\alpha T_e)}{z - e^{j\pi/3} L - e^{-j\pi/3} L} \] (27)

Fig. 3 shows trajectory of the steady-state current vector in complex \( \alpha \beta \) plane. This trajectory is given by (26). The parameters of the modulation are: \( N_s = 7, g = 0.8 \).
From (24) we can derive easily the solution for six-step waveform (without modulation).

Analytical expressions for the steady-state currents of the system with a three-phase VSI with six-step waveform feeding a three-phase static inductive load were presented in [9] (Equations (1)-(8)). These expressions were derived by existing methods using (1a) and (1b), which necessitates solving algebraic equations to express the initial value of the load phase current \( i_0 \). From the proposed mathematical model we can determine the solution in a very simple form.

In Eq (26), which is valid for the steady-state, we substitute:

\[
M=1 \quad \text{(one pulse per sector)}, \quad \varepsilon_{1A}=0, \quad \varepsilon_{1B}=1, m_1=0, n_1=1.
\]

By substituting these values into (26) we obtain for the steady-state vector current of the RL load:

\[
i_S(n, \varepsilon) = \begin{bmatrix}
-\pi & -\pi \\
\pi & e^{j\varepsilon}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3}V_{dc}e^{j\pi/3} \\
\frac{1}{3}V_{dc}e^{-j\pi/3}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3} \varepsilon & -1 \\
-\frac{1}{3} & \frac{R}{L}
\end{bmatrix}
\]

Putting \( n=0 \) and \( 0<\varepsilon\leq1 \), we get solution for the first sixth of the period, for \( n=1 \) and \( 0<\varepsilon\leq1 \), we get solution for the second sixth of the period, etc.

The A-phase current is given by real part of (28)

\[
i_A(n, \varepsilon)=\text{Re}\left\{ i_S(n, \varepsilon) \right\}
\]

For the voltage vector with six-step waveform we can write:

\[
V(n, \varepsilon)=\frac{2}{3}(V_{dc}e^{j\varepsilon/3})=V(n)
\]

And the A phase voltage is given by a real part:

\[
v_A(n, \varepsilon)=v_A(n)=\text{Re}\left\{ V(n, \varepsilon) \right\} = \frac{2}{3}(V_{dc}\cos\pi n/3)
\]

Fig.5 shows phase A current given by (29) and phase A voltage, for six-step waveforms (without modulation).

If we compare Fig.5 with the waveforms in [9] we can see that the results are identical. But the presented mathematical model contains only one equation (28) which is valid for the whole output period \( (n=0,1,2,3,4,5, \ldots, 0<\varepsilon\leq1) \). The model in [9] necessitates solution for every sixth of the period, which means six equations per one period. Besides, it requires solving the initial value of the load phase current.

5 Experimental Results

Validation of the derived analytical equations was also carried out using measurements with a 3 kW three-phase inverter supplying RL load. A three-phase static inductive load has the parameters: \( R=623 \Omega, L=502 \Omega \).

An IGBT inverter utilized Space Vector PWM with sampling intervals \( N=7 \), modulation factor \( g=0.8 \), and with a fundamental frequency of the output voltage of 50 Hz.

Fig.6 shows experimental waveforms of the phase A steady-state load current (upper trace) and the phase A load voltage (lower trace).

The corresponding theoretical phase A steady-state current and phase voltage given are shown in Fig.4. As can be seen, there is very good agreement between measured and theoretical results, with correlation being better than 5% over most of the load range.

Fig.7 shows phase A voltage and current measured in the inverter without modulation-six step waveforms.

If we compare Fig.7 with theoretical waveforms given in Fig.5 we can see very high correlation. The simple form of equations (28) and (31) can be used directly to assess the system performance.
All the dependencies were graphical visualized by the programme MATHCAD [11]

![Graphical visualization](image1)

![Graphical visualization](image2)

Fig.7 Experimental results. Phase A voltage and current-six step waveforms

6 Conclusion

An approach for the analysis of linear system containing periodically operated switches is described. The approach was demonstrated for the inverter with Space Vector PWM, but it is applicable for all types of converters with explicitly determined output voltage. The mathematical model uses the Laplace and modified Z transforms. The steady-state and transient components of the load current are determined in a simple and lucid form that it avoids involved matrix inversion as well as exponentiation. Experimental results prove the feasibility of the proposed mathematical model as compared with the conventional methods. The theory is based on a relatively simple model, but correlation between measurements and calculations is very good.

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References


