THE FOX’S H-FUNCTION AND UNSTEADY HYDROMAGNETIC BOUNDARY-LAYER IN A ROTATING MEDIUM

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Abstract. Motivated essentially by the work of P. Puri and P. K. Kulshrestha [7] the author investigate and derive exact analytical solutions for the three-dimensional flow of a viscous fluid in the presence of the transverse magnetic field past an infinite porous flat plate with uniform suction or injection, moving with a time-dependent velocity in a rotating medium. Assuming the normal velocity \( w \) to be a constant (uniform suction/injection), several different cases of the velocity \( -U_0 f(t) \) along the plate are treated here in a unified manner.

Key words. Viscous fluid, rotating flow, boundary-layer phenomenon, Laplace transform, error functions, Fox’s H-function.

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1 Introduction

On the occasion of the IUTAM Symposium on “Mechanics of Passive and Active Flow Control” held in Gottingen, Germany, 7-11 September 1998 [3] Professor Tomomasa Tatsumi from International Institute for Advanced Studies, Kyoto, Japan, said: “It should be noted that the Century of Fluid Mechanics by no means implies that the study of Fluid Mechanics is completed in this century. Instead, the Century of Fluid Mechanics will be succeeded by Century of Complex Fluid Mechanics.

The realm of Fluid Mechanics always expanding with increasing “complexity”. Even in this century, turbulent and chaotic flows have a “complex” property that they are deterministic and random. Besides turbulence, there are several non-classical fluids such as quantum fluids, relativistic fluids and biological fluids, and various flows associated with thermal convection combustion, phase-change, chemical reactions and other fluid phenomena in bio-science, space science and environmental and energy engineering, all of which have some kind of “complexity” beyond the conventional and simple fluids. Some of the above subjects already constitute the applied fields of the contemporary Fluid Mechanics, but I believe that they will become the hard core of the Complex Fluid Mechanics in the coming century.”

In this context the “flow control” is a Complex Fluid Mechanics problem. The flow control is a useful measure to influence a given flow configuration in such a manner that it is changed or improved into a desired flow performance. The problem of boundary-layer control is becoming increasingly important in the field of Complex Fluid Mechanics (see, for details, [3] and [4]).

The last decade has witnessed significant advances and research activity in the area of flow control/management, triggered by technological applications as all as scientific curiosity in understanding the structure of complex flows. There have been various methods, such as suction, blowing, vortex generators, turbulence manipulators that have been used for flow control with varying degrees of success.

Theoretically this problem of flow control has received very little attention. Needless to say, much more mathematical research is needed. So much is abundantly clear.

2 Mathematical Formulation of the Boundary-value Problem and Its Solutions

Let an electrically conducting viscous incompressible fluid filling the semi-infinite space \( z > 0 \) be in contact with an infinite non-conducting porous flat plate \( z = 0 \). Both the fluid and the plate are in a state of solid body rotation with uniform angular velocity \( \Omega \) about the \( z \)-axis in the presence of a constant magnetic field \( H_0 \) normal to the plate. The plate initially at rest is suddenly moved with a velocity \( -U_0 f(t) \) (\( U_0 = \text{const.} \)) in its
own plane along the negative $x$-axis fixed in the plate; $y$-axis is also fixed in the plate normal to its motion.

In a coordinate system rotating with the fluid, the basic equations of motion are given by (cf. [7, pp. 205-206]):

1) continuity equation

$$div \ ⃗{v} = 0$$  \hspace{1cm} (2.1)

2) momentum equation

$$\frac{∂⃗{v}}{∂t} + (⃗{v}.∇)⃗{v} + 2Ω × ⃗{v} + Ω × (Ω × ⃗{r}) =$$

$$= -\frac{1}{ρ} ∇P + ν∇^2 ⃗{v} + \frac{μ}{ρ} j × ⃗{H}$$  \hspace{1cm} (2.2)

3) Maxwell’s equations

$$curl \ ⃗{H} = 4π j, \ div \ j = 0;$$

$$curl \ ⃗{E} = -μ\frac{∂⃗{H}}{∂t}, \ div \ ⃗{H} = 0 \ 9$$  \hspace{1cm} (2.3)

4) Ohm’s law

$$j = σ(⃗{E} + μ⃗{v} × ⃗{H})$$  \hspace{1cm} (2.4)

where $⃗{H}$ denotes the magnetic field vector, $j$ the electric current density vector, $⃗{E}$ the electrical intensity vector, $⃗{v} = (u,v,w)$ the velocity vector, $Ω$ the angular velocity vector, $⃗{r}$ the radius vector, $σ$ the electrical conductivity, $μ$ the permeability, $t$ the time, $ρ$ the density of the fluid, $ν$ the kinematical viscosity of the fluid and $P$ the pressure.

The vector $⃗{H}$ is assumed to be constant with only one component $H_0$ in the $z$-direction. The induced magnetic field produced by the motion of the fluid is ignored. Since no external electric field is applied and the effect of polarization of the ionized fluid is negligible, we can take $⃗{E} = 0$ [5]. The physical variables are functions of $z$ and $t$ only. Under the foregoing conditions, equations (2.1) become $\frac{∂w}{∂z} = 0$, which implies that $w$ is a constant. The equation (2.2), in component form, becomes:

$$\frac{∂u}{∂t} + w\frac{∂u}{∂z} - 2Ωv = -\frac{∂}{∂x}\left(\frac{P}{ρ} - \frac{1}{2}r^2Ω^2\right) +$$

$$+\frac{ν}{ρ}\frac{∂^2u}{∂z^2} - \frac{σμ^2H_0^2}{ρ} u$$  \hspace{1cm} (2.5)

$$\frac{∂v}{∂t} + w\frac{∂v}{∂z} + 2Ωu = -\frac{∂}{∂y}\left(\frac{P}{ρ} - \frac{1}{2}r^2Ω^2\right) +$$

$$+\frac{ν}{ρ}\frac{∂^2v}{∂z^2} - \frac{σμ^2H_0^2}{ρ} v$$  \hspace{1cm} (2.6)

where $r^2 = x^2 + y^2$. Equation (2.7) shows that $\left(\frac{∂}{∂x},y\right)\left(\frac{P}{ρ} - \frac{1}{2}r^2Ω^2\right)$ have the same value as in the free stream, which implies that, both quantities are zero.

It is convenient to introduce the following non-dimensional quantities:

$$z' = \frac{U_0}{v}z, \ v' = \frac{1}{U_0}v, \ t' = \frac{U_0^2}{v}t,$$

$$Ω' = \frac{v}{U_0^2}Ω, \ M^2 = \frac{μ^2H_0^2σv}{U_0^2},$$

$$V_0 = \frac{w}{U_0}$$ (suction/injection), $q = u' + iv'$.

In the sequel primes will be suppressed.

In order to solve the boundary-value problem, we introduce the Laplace transform of $q(z,t)$ by

$$\bar{q}(z,s) = \int_0^\infty e^{-st}q(z,t)dt .$$

Equations (2.5) and (2.6), after applying Laplace transform, reduce to

$$\frac{d^2\bar{q}}{dz^2} - V_0\frac{d\bar{q}}{dz} - (M^2 + V_0^2 + 2iΩ)\bar{q} = 0 \hspace{1cm} (2.8)$$

Equation (2.8) is subject to the following boundary conditions:

$$\bar{q}(0,s) = -U_0\bar{f}(s),$$

$$\bar{q}(∞,s) = 0.$$  \hspace{1cm} (2.9)

$$\bar{q}(x,s) = 0.$$  \hspace{1cm} (2.10)
Solution of (2.8) under the conditions (2.9), (2.10) is given by [7, Eq. (9)]:
\[
\bar{q}(z, s) = -U_0 \tilde{f}(s) \exp \left[ z \left( \frac{V_0}{2} - \sqrt{s + b} \right) \right] \quad (2.11)
\]
where
\[
b = M^2 + \frac{V_0^2}{4} + 2i\Omega . \quad (2.12)
\]

3. Particular Analytical Solutions of Boundary-value Problem

We will now consider particular cases for various value of the function \( f(t) \).

Case 1. Let
\[
f(t) = t^{\lambda - 1} e^{-\beta t} \frac{\Gamma(\lambda)}{\Gamma(\lambda - 1)}, \quad \lambda > 1, \quad \beta \in \mathbb{R}, \quad (3.1)
\]
where \( \Gamma \) is the familiar Gamma function. In this case we have (cf. [6, p. 237, formula 4.2])
\[
\tilde{f}(s) = L\{f(t)\} = \frac{1}{(s - \beta)^{\lambda}}, \quad s > \beta ,
\]
and (2.11) leads to the following solution
\[
\bar{q}(z, s) = -U_0 e^{\frac{z}{2}} \frac{e^{-\frac{z}{2} \sqrt{s + b}}}{(s - \beta)^{\lambda}} . \quad (3.2)
\]

Making use of the known inversion formula (cf. [2, p. 249, Equation (2)])
\[
L^{-1}\left\{ \frac{e^{-\frac{z}{2} \sqrt{s + b}}}{s - \beta} \right\} = \frac{e^{\beta t}}{2} (A + B)
\]
where
\[
A = e^{-z\sqrt{b + \beta}} \text{erfc}\left( \frac{z}{2\sqrt{t}} - \sqrt{(b + \beta)t} \right),
\]
\[
B = e^{z\sqrt{b + \beta}} \text{erfc}\left( \frac{z}{2\sqrt{t}} + \sqrt{(b + \beta)t} \right),
\]
and (cf. [6, p. 237, Equation (4.2)])
\[
L^{-1}\left\{ \frac{1}{(s - \beta)^{\lambda - 1}} \right\} = t^{\lambda - 2} e^{-\beta t} \frac{\Gamma(\lambda - 1)}{\Gamma(\lambda - 1)},
\]
where \( \text{erfc}(\alpha) \) denotes the complementary error function, defined by \( \text{erfc}(\alpha) = 1 - \text{erf}(\alpha) \), \( \text{erf}(\alpha) \) being the error function, so that (obviously)(cf. [9, p. 16]):
\[
\text{erf}(0) = \text{erfc}(\infty) = 0, \quad \text{erf}(\infty) = \text{erfc}(0) = 1
\]
and by taking the convolution theorem for Laplace transform (see, for example, [1, p. 131, Entry 4.1 (20)] and [6, p. 209, Equation (1.18)]) leads us in this case from (3.2) to the following analytical solution:
\[
q(z, t) = L^{-1}\{\bar{q}(z, s)\} = \frac{V_0}{2} e^{\frac{z}{2} - \beta t} \frac{\Gamma(\lambda - 1)}{2\Gamma(\lambda - 1)} \int_0^t (t - \xi)^{\lambda - 2} e^{2\beta \xi} (A + B) d\xi \quad (3.3)
\]

When \( M = 0, \quad \Omega = 0 \) and \( \nu = 1 \), the solution (3.3) reduces to the solution (29) obtained by S. Turbatu [11].

Case 2. Let
\[
f(t) = t^{\lambda - 1} H_{\rho, r}^{m, n}\left[ a_{j, \alpha_j}^{(\lambda)}\right] \quad (3.4)
\]

Therefore, in this case, the function \( f(t) \) is given in terms pf Fox’s \( H \) - function (see, for details [8, Chapter 2]; see also [9, p. 40, et seq.]). Indeed, the \( H \) - function involved in \( f(t) \) is known to contain, as its special cases, a remarkably large number of potentially useful functions such as the Bessel functions \( J_\nu \) and \( I_\nu \), the Macdonald function \( K_\nu \), the Whittaker functions \( M_{k, \mu} \) and \( W_{k, \mu} \), the parabolic cylinder function \( D_\nu \), the Gauss hypergeometric function \( \frac{1}{2} F_1 \) and its generalization \( \rho F_q \), the Fox-Wright generalized hypergeometric function \( \Psi \) and its such further special cases as the Wright-Bessel function \( J_\nu^\mu \), the generalized Mittag-Leffler function \( E_{\alpha, \beta} \), etc. , MacRobert’s \( E \) - function, Meijer’s \( G \) - function, and so on (see, for example, [8, pp. 18-19] and [9, Section II.5]).

In this case, we have (cf. [8, p. 15, Eq. (2.4.2)]):
\[ \tilde{f}(s) = L_0 \{ f(t) \} = \frac{1}{s^\lambda} H_{m+n+1}^{\alpha+1, r} \left[ \frac{\omega}{s} \begin{pmatrix} 1 - \lambda, 1, (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, r} \end{pmatrix} \right] (3.5) \]

where

\[
\begin{aligned}
& s > 0; \quad \lambda > - \min_{1 \leq j \leq m} \left( \frac{b_j}{\beta_j} \right); \quad \sum_{j=1}^{p} \beta_j - \sum_{j=1}^{p} \alpha_j \geq 0
\end{aligned}
\]

We thus obtain from (2.11)

\[
\tilde{q}(z, s) = -U_0 s^{-\lambda} \exp \left[ \frac{1}{2} V_0 \left( \frac{1}{2} - \sqrt{z + b} \right) \right] \cdot H_{m+n+1}^{\alpha+1, r} \left[ \frac{\omega}{s} \begin{pmatrix} 1 - \lambda, 1, (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, r} \end{pmatrix} \right] (3.6)
\]

which, in view of the convolution theorem for the Laplace transform [1, p. 131, Entry 4.1 (20)] and another known result (cf. e.g. [2]), leads us finally to the following exact analytical solution of the boundary-value problem (3.7)

\[
q(z, t) = -U_0 \frac{V_0}{2} e^{\frac{1}{2} t} \int_0^t (t - \xi)^2 (A^* + B^*) d\xi
\]

where

\[
A^* = e^{-z}\sqrt{b} \text{erfc} \left( \frac{z}{2\sqrt{\xi}} - \sqrt{b} \xi \right),
\]

\[
B^* = e^{-z}\sqrt{b} \text{erfc} \left( \frac{z}{2\sqrt{\xi}} + \sqrt{b} \xi \right)
\]

For \( M = 0 \), \( \Omega = 0 \) and \( \nu = 1 \) \( b = \frac{V_0^2}{4} \) we find from (3.7) the Turbatu and Srivastava’s solution [10, Equation (3.9)].

Numerous applications of our exact analytical solution (3.7) associated with simpler choices in (3.4), can be derived fairly easily, for example the solution (3.3).

**REFERENCES**