FUZZY MULTISUBMEASURES AND APPLICATIONS

ALINA GAVRILUȚ
"Al.I. Cuza" University,
Faculty of Mathematics,
Bd. Carol I, No. 11, Iași, 700506,
ROMANIA

ANCA CROITORU
"Al.I. Cuza" University,
Faculty of Mathematics,
Bd. Carol I, No. 11, Iași, 700506,
ROMANIA

Abstract: Properties of fuzzy multisubmeasures are discussed. Also, some applications are presented: decomposition theorems concerning pseudo-atoms, an extension result by preserving non-pseudo-atomicity or fuzzy character and properties of regularity.

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1. Introduction.

The importance of fuzzy measures is pointed out by their applications in economics, statistics, theory of games, social sciences, engineering domains. The fuzzy measures were introduced by Sugeno [15] in 1974. Since then, the non-additive case generated much research over the past 30 years (see Dinculeanu [1], Klimkin and Svistula [11], Pap [13], Rao and Rao [14], Suzuki [16], for example). Recently, new applications of fuzzy measures were found in utility theory (see Liginlal and Ow [12]).

In Gavriluţ [9] and Gavriluţ and Croitoru [10] we introduced and studied the notions of atom, pseudo-atom, Darboux property and non-atomicity for set multifunctions, developing there a theory for the multivalued case and also pointing out the differences from the single valued case.

In this paper we introduce and study the notion of a fuzzy multivalued set function and present some applications of fuzzy multisubmeasures such as: decomposition theorems concerning pseudo-atoms, an extension result by preserving non-pseudo-atomicity or fuzzy character and properties of regularity.

We now introduce the notations and several definitions used throughout the paper. Let $T$ be an abstract nonvoid set and $C$ a ring of subsets of $T$.

Definition 1.1. A set function $\nu : C \to [0, \infty]$ is said to be:

(i) monotone if $\nu(A) \leq \nu(B)$, for every $A, B \in C$, with $A \subseteq B$;

(ii) a submeasure (in the sense of Drewnowski [3]) if $\nu$ is monotone and $\nu(A \cup B) \leq \nu(A) + \nu(B)$, for every $A, B \in C$, with $A \cap B = \emptyset$;

(iii) $\sigma$-continuous if $\lim_{n \to \infty} \nu(A_n) = 0$, for every $(A_n)_{n \in \mathbb{N}} \subseteq C$, with $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$;

(iv) a Dobrakov submeasure ([2]) if $\nu$ is a submeasure and it is also $\sigma$-continuous.

(v) fuzzy if $C$ is a $\sigma$-algebra and $\nu$ satisfies the conditions:

1) $\nu$ is monotone,

2) $\nu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \nu(A_n)$, for every $(A_n)_{n \in \mathbb{N}} \subseteq C$, with $A_n \subseteq A_{n+1}$, for every $n \in \mathbb{N}$,

3) $\nu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \nu(A_n)$, for every $(A_n)_{n \in \mathbb{N}} \subseteq C$, with $A_{n+1} \subseteq A_n$, for every $n \in \mathbb{N}$ and there is $n_0 \in \mathbb{N}$ such that $\nu(A_{n_0}) < +\infty$.

Definition 1.2. Let $\nu : C \to [0, \infty]$ be a set function.

(i) We say that $\nu$ has the Darboux property (DP) if for every $A \in C$ and every $\nu \in (0, 1)$, there exists a set $B \in C$ such that $B \subseteq A$ and $\nu(B) = \nu(A)$.

(ii) A set $A \in C$ is said to be an atom of $\nu$ if $\nu(A) > 0$ and for every $B \in C$, with $B \subseteq A$, we have $\nu(B) = 0$ or $\nu(A \setminus B) = 0$.

(iii) $\nu$ is said to be non-atomic (NA) if it has no atoms (that is, for every $A \in C$ with $\nu(A) > 0$, there exists $B \in C$, $B \subseteq A$, such that $\nu(B) > 0$ and $\nu(A \setminus B) > 0$).

(iv) A set $A \in C$ is called a pseudo-atom of $\nu$ if $\nu(A) > 0$ and $B \in C$, $B \subseteq A$ implies $\nu(B) = 0$ or $\nu(B) = \nu(A)$.

Remark 1.3. If $\nu$ is finitely additive, then $A \in C$ is an atom of $\nu$ if and only if $A$ is a pseudo-atom of $\nu$. 

In what concerns non-atomicity and the Darboux property, we remind now the following relationships established in literature for set functions. Suppose \( \nu : C \rightarrow \mathbb{R}_+ \).

**Remark 1.4.** Let \( \nu : C \rightarrow \mathbb{R}_+ \).

I) If \( C \) is a \( \delta \)-ring and \( \nu \) is a measure, then \( NA \Leftrightarrow DP \) (Dinculeanu [11]).

II) If \( C \) is a \( \sigma \)-algebra and \( \nu \) is a Dobrakov submeasure, then \( NA \Leftrightarrow DP \) (Klimkin and Svistula [11]).

III) If \( C \) is an algebra and \( \nu \) is bounded finitely additive, then \( DP \Rightarrow NA \) (Rao and Rao [14]).

In the sequel, \( X \) will be a real normed space, with the distance \( d \) induced by its norm, \( P_0(X) \) the family of non-empty closed subsets of \( X \), \( P_f(X) \) the family of non-empty closed bounded subsets of \( X \), \( P_{b_f}(X) \) the family of non-empty closed bounded convex subsets of \( X \).

Let \( h \) be the Hausdorff pseudometric on \( P_f(X) \):

\[
h(M, N) = \max\{e(M, N), e(N, M)\},
\]

where \( e(M, N) = \sup_{x \in M} d(x, N) \), for every \( M, N \in P_f(X) \).

As we know, \( h \) is a metric on \( P_{b_f}(X) \) and if \( X \) is a Banach space, then \( (P_{b_f}(X), h) \) becomes a complete metric space.

We define \(|A| = h(A, \{0\})\), for every \( A \in P_f(X) \), where \( 0 \) is the origin of \( X \).

On \( P_0(X) \) we introduce the Minkowski addition \( + \), defined by:

\[
A + B = \overline{A + B}, \text{ for every } A, B \in P_0(X).
\]

2. Properties of fuzzy multivalued set functions

Further we introduce the notion of a fuzzy multivalued set function and we point out some of its properties.

**Definition 2.1.** Let \( \mu : C \rightarrow P_f(X) \) be a multivalued set function, with \( \mu(\emptyset) = \{0\} \). \( \mu \) is said to be:

(i) **monotone** if \( \mu(A) \subseteq \mu(B) \), for every \( A, B \in C \), with \( A \subseteq B \);

(ii) a **multisubmeasure** (Gavrilut [4]) if it is monotone and \( \mu(A \cup B) \subseteq \mu(A) + \mu(B) \), for every \( A, B \in C \), with \( A \cap B = \emptyset \);

(or, equivalently, for every \( A, B \in C \);

(iii) a **multimeasure** if \( \mu(A \cup B) = \mu(A) \cdot \mu(B) \), for every \( A, B \in C \), with \( A \cap B = \emptyset \);

(iv) **increasing convergent** with respect to \( h \) if \( \lim_{n \to \infty} h(\mu(A_n), \mu(A)) = 0 \), for every \( (A_n)_{n \in \mathbb{N}} \subseteq C \), with \( A_n \subseteq A_{n+1} \), for every \( n \in \mathbb{N} \) and \( \bigcap_{n=1}^{\infty} A_n = \emptyset \) (that will be denoted \( A_n \not\subset A \));

(v) **decreasing convergent** with respect to \( h \) if \( \lim_{n \to \infty} h(\mu(A_n), \mu(A)) = 0 \), for every \( (A_n)_{n \in \mathbb{N}} \subseteq C \), with \( A_n \supseteq A_{n+1} \), for every \( n \in \mathbb{N} \) and \( \bigcap_{n=1}^{\infty} A_n = \emptyset \) (that will be denoted \( A_n \not\supset A \));

(vi) **order-continuous** (briefly, o-continuous) with respect to \( h \) if \( \lim_{n \to \infty} |\mu(A_n)| = 0 \), for every \( (A_n)_{n \in \mathbb{N}} \subseteq C \), with \( A_n \not\subset \emptyset \);

(vii) **exhaustive** with respect to \( h \) if \( \lim_{n \to \infty} |\mu(A_n)| = 0 \), for every mutual disjoint sequence of sets \( (A_n)_{n \in \mathbb{N}} \subseteq C \);

(viii) **fuzzy** if \( \mu \) is monotone, increasing convergent and decreasing convergent.

**Examples 2.2.**

I) If \( \nu : C \rightarrow \mathbb{R}_+ \) is an o-continuous submeasure (finitely additive measure, respectively), then the multivalued set function \( \mu : C \rightarrow P_f(\mathbb{R}) \), defined by

\[
\mu(A) = [0, \nu(A)], \text{ for every } A \in C,
\]

is a fuzzy multisubmeasure (multimeasure, respectively), called the multisubmeasure (multimeasure, respectively) induced by \( \nu \).

II) If \( m_1, m_2 : C \rightarrow \mathbb{R}_+ \), \( m_1 \) is an o-continuous finitely additive measure and \( m_2 \) is an o-continuous submeasure (finitely additive measure, respectively), then the multivalued set function \( \mu : C \rightarrow P_f(\mathbb{R}) \), defined by

\[
\mu(A) = [-m_1(A), m_2(A)], \text{ for every } A \in C,
\]

is a fuzzy multisubmeasure (multimeasure, respectively).

III) Let \( C = \{\emptyset, \{1\}, \{2\}, \{1, 2\} \} \) and \( \nu : C \rightarrow \mathbb{R}_+ \) be the submeasure defined for every \( A \in C \), by:

\[
\nu(A) = \begin{cases} 
0, & \text{if } A = \emptyset \\
1, & \text{if } A = \{1\} \text{ or } A = \{2\} \\
\frac{3}{2}, & \text{if } A = \{1, 2\}
\end{cases}
\]

Then the multisubmeasure induced by \( \nu \) is fuzzy.

In fact, the multisubmeasure induced by a submeasure is fuzzy if and only if the submeasure is fuzzy.

**Remark 2.3.**

I) Any monotone multisubmeasure is a multisubmeasure.
II) If $\mu$ is decreasing convergent, then $\mu$ is o-continuous.

III) If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is an exhaustive multisubmeasure, then $\mu$ takes his values in $\mathcal{P}_{bf}(X)$ (according to Gavriluţ [5]).

**Theorem 2.4.** Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure. Then:

I) $\mu$ is o-continuous if and only if $\mu$ is fuzzy.

II) If $\mathcal{C}$ is a $\sigma$-ring and $\mu$ is fuzzy, then $\mu$ is exhaustive.

**Proof.** I) Suppose $\mu$ is o-continuous. Since $\mu$ is a multisubmeasure, it results that $\mu$ is monotone. We prove that $\mu$ is increasing convergent. Let $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, $A_n \not\subseteq A$, with $A \in \mathcal{C}$. If we denote $B_n = A \setminus A_n$, for every $n \in \mathbb{N}^*$, then $B_n \searrow \emptyset$. Since $\mu$ is o-continuous, it results $\lim_{n \to \infty} |\mu(B_n)| = 0$. From the inequality: $e(\mu(A), \mu(A_n)) \leq |\mu(A \setminus A_n)| = |\mu(B_n)|$, for every $n \in \mathbb{N}^*$, it follows $\lim_{n \to \infty} e(\mu(A), \mu(A_n)) = 0$ and so, $\lim_{n \to \infty} h(\mu(A), \mu(A_n)) = 0$. Thus $\mu$ is increasing convergent. The fact that $\mu$ is decreasing convergent analogously follows. Consequently, $\mu$ is fuzzy. If $\mu$ is fuzzy, then $\mu$ is decreasing convergent, hence it is o-continuous.

II) Let $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$ be a mutual disjoint sequence of sets and let $B_n = \bigcup_{k=n}^{\infty} A_k$, for every $n \in \mathbb{N}^*$. Then $B_n \searrow \emptyset$ and, since $\mu$ is decreasing convergent, we have $\lim_{n \to \infty} |\mu(B_n)| = \lim_{n \to \infty} h(\mu(B_n), \{0\}) = 0$. Since $\mu$ is monotone and $A_n \subseteq B_n$, it results $|\mu(A_n)| \leq |\mu(B_n)|$, for every $n \geq 1$. Consequently, $\lim_{n \to \infty} |\mu(A_n)| = 0$ and, therefore, $\mu$ is exhaustive. \[\square\]

**Definition 2.5.** For a multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, we consider $|\mu| : \mathcal{C} \to \mathbb{R}_+$, defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$.

**Remark 2.6.** If $\mu$ is a multisubmeasure, then $|\mu|$ is a submeasure.

**Definition 2.7.** Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a multivalued set function, with $\mu(\emptyset) = \{0\}$.

(i) We say that $\mu$ has the Darboux property if for every $A \in \mathcal{C}$, with $\mu(A) \supseteq \{0\}$ and every $p \in (0, 1)$, there exists a set $B \in \mathcal{C}$ such that $B \subseteq A$ and $\mu(B) = p\mu(A)$.

(ii) A set $A \in \mathcal{C}$ is said to be an atom of $\mu$ if $\mu(A) \supseteq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(B \setminus A) = \{0\}$.

(iii) If $\mu$ is monotone, then $\mu$ is said to be non-atomic if it has no atoms; that is, for every $A \in \mathcal{C}$, with $\mu(A) \supseteq \{0\}$, there exists $B \in \mathcal{C}$, with $B \subset A$, $\mu(B) \supseteq \{0\}$ and $\mu(A \setminus B) \supseteq \{0\}$.

(iv) A set $A \in \mathcal{C}$ is called a pseudo-atom of $\mu$ if $\mu(A) \supseteq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(B) = \mu(A)$.

(v) If $\mu$ is monotone, then $\mu$ is said to be non-pseudo-atomic if it has no pseudo-atoms; that is, for every $A \in \mathcal{C}$, with $\mu(A) \supseteq \{0\}$, there exists $B \in \mathcal{C}$, with $B \subset A$, $\mu(B) \supseteq \{0\}$ and $\mu(B) \not\subseteq \mu(A)$.

**Examples 2.8.**

I) Suppose $T$ is a countable set. Let $\mathcal{C} = \{A : A \subseteq T, A$ is finitely or c$A$ is finitely and the multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined for every $A \in \mathcal{C}$ by

$\mu(A) = \begin{cases} \{0\}, & \text{if } A \text{ is finitely} \\ \{0, 1\}, & \text{if } cA \text{ is finitely} \end{cases}$

Then every $A \in \mathcal{C}$, such that $cA$ is finite, is an atom of $\mu$.

II) Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure and $A, B \in \mathcal{C}$, with $B \subseteq A$. Then $\mu(A \setminus B) = \{0\}$ implies $\mu(A) = \mu(B)$. It follows that every atom of $\mu$ is a pseudo-atom of $\mu$. The converse is not valid:

Let $T = \{x, y, z\}$, $\mathcal{C} = \mathcal{P}(T)$ and let $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ be the multisubmeasure defined for every $A \in \mathcal{C}$ by

$\mu(A) = \begin{cases} [0, 3], & \text{if } A \neq \emptyset \\ \{0\}, & \text{if } A = \emptyset. \end{cases}$

Let $A = \{x, y\}$. Then $A$ is not an atom of $\mu$, but $A$ is a pseudo-atom of $\mu$. So, there are pseudo-atoms of a multisubmeasure, which are not atoms.

**Remark 2.9.** Let $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ be a multisubmeasure and let $A, B \in \mathcal{C}$, with $B \subseteq A$. Then $\mu(A) = \mu(B)$ implies $\mu(A \setminus B) = \{0\}$. It follows that every pseudo-atom of $\mu$ is an atom of $\mu$. Consequently, if $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ is a monotone multisubmeasure, then $A \in \mathcal{C}$ is an atom of $\mu$ if and only if $A$ is a pseudo-atom of $\mu$.

In Gavriluţ and Croitoru [10], we obtained the following result:

**Theorem 2.10.** Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$ and $|\mu| : \mathcal{C} \to \mathbb{R}_+$.

I) Suppose $\mu$ is a multisubmeasure. If $\mu$ has the Darboux property, then $\mu$ is non-atomic.

II) Suppose $\mathcal{C}$ is a $\sigma$-algebra and $\mu$ is the multisubmeasure induced by a Dobrakov submeasure $\nu : \mathcal{C} \to \mathbb{R}_+$ (i.e. $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$).
Then \( \mu \) has the Darboux property if and only if \( \mu \) is non-atomic.

**Remark 2.11.** The converse of Theorem 2.10-I) is not true. For that, see Example 3.16 of a non-atomic multisubmeasure which has not the Darboux property.

### 3. Applications

First, we give several decomposition results concerning pseudo-atoms.

**Theorem 3.1.** Suppose \( \mu : C \to \mathcal{P}_0(X) \) is monotone, with \( \mu(\emptyset) = \{0\} \) and let \( A, B \in C \) be pseudo-atoms of \( \mu \).

I) If \( \mu(A \cap B) \supseteq \{0\} \), then \( A \cap B \) is a pseudo-atom of \( \mu \) and \( \mu(A \cap B) = \mu(A) \cap \mu(B) \).

II) Suppose, moreover, \( \mu : C \to \mathcal{P}_f(X) \) is a multisubmeasure. If \( \mu(A \cap B) = \{0\} \), then \( A \cap B \) and \( B \setminus A \) are pseudo-atoms of \( \mu \) and \( \mu(A \cap B) = \mu(A), \mu(B \setminus A) = \mu(B) \).

**Proof.** 1) Let \( E \subseteq C, E \subseteq A \cap B \), with \( \mu(E) \supseteq \{0\} \). Since \( E \subseteq A \cap B \subseteq A \) and \( A \) is a pseudo-atom, it results \( \mu(E) = \mu(A) \). By the monotonicity of \( \mu \), it follows \( \mu(E) = \mu(A \cap B) \subseteq \mu(A) \). Thus \( \mu(E) = \mu(A \cap B) \), which shows that \( A \cap B \) is a pseudo-atom of \( \mu \). Also we have \( \mu(A \cap B) = \mu(A) \). Analogously, we prove that \( \mu(A \cap B) = \mu(B) \).

II) We prove that \( \mu(A \cap B) \supseteq \{0\} \). Suppose, on the contrary, that \( \mu(A \cap B) = \{0\} \). Since \( \mu \) is a multisubmeasure, we obtain \( \mu(A) = \mu(A \cap B) \cup (A \cap B) \subseteq \mu(A \cap B) \cup (A \cap B) = \{0\} \), which is false because \( A \) is a pseudo-atom of \( \mu \). So, \( \mu(A \cap B) \supseteq \{0\} \).

Then \( A \cap B \) is a pseudo-atom of \( \mu \) and \( \mu(A \cap B) = \mu(A) \). Analogously, \( B \setminus A \) is a pseudo-atom of \( \mu \) and \( \mu(B \setminus A) = \mu(B) \).

**Corollary 3.2.** Let \( \mu : C \to \mathcal{P}_f(X) \) be a multisubmeasure and let \( A, B \in C \) be pseudo-atoms of \( \mu \). Then there exist mutual disjoint sets \( C_1, C_2, C_3 \in C \), with \( A \cup B = C_1 \cup C_2 \cup C_3 \), such that, for every \( i \in \{1, 2, 3\} \), either \( C_i \) is a pseudo-atom of \( \mu \), or \( \mu(C_i) = \{0\} \).

**Proof.** We consider \( C_1 = A \cap B, C_2 = A \setminus B, C_3 = B \setminus A \) and use Theorem 3.1.

**Proposition 3.3.** Let \( \mu : C \to \mathcal{P}_f(X) \) be a multisubmeasure and let \( A, B \in C \) be pseudo-atoms of \( \mu \). If \( \mu(A \cap B) \supseteq \{0\} \), then \( A \cap B \) is a pseudo-atom of \( \mu \) and \( \mu(A \cap B) = \mu(A) \).

**Proof.** According to Theorem 3.1-I), it follows that \( A \cap B \subseteq \mu(A) \) is a pseudo-atom of \( \mu \). We now consider \( C = A \cap B = (A \setminus B) \cup (B \setminus A) \in C \). Then

\[
\mu(C) \subseteq \mu(A \setminus B) + \mu(B \setminus A) = \{0\},
\]

which implies \( \mu(C) = \{0\} \). Moreover, \( A \cap B \cap C = \emptyset \) and \( A \cup B = (A \cap B) \cup C \).

The next decomposition theorem (see Gavriluţ and Croitoru [10]) presents a property of compacity type.

**Theorem 3.4.** Let \( C \) be a \( \sigma \)-ring, \( \mu : C \to \mathcal{P}_f(X) \) a fuzzy multisubmeasure and denote \( S = \{A \in C \mid A \) is a pseudo-atom of \( \mu \) and \( |\mu(A)| \geq \alpha \}, \) where \( \alpha > 0 \).

Then the set \( E = \bigcup \mathcal{S} \) is represented as the union of at most finite pairwise disjoint sets of \( S \) and of a set \( F \) with \( \mu(F) = \{0\} \).

(In the representation of \( E \), the set \( F \) and the other elements from \( S \) may not be pairwise disjoint).

Analogously, we can prove another theorem:

**Theorem 3.5.** Suppose \( C \) is a \( \sigma \)-ring and \( \mu : C \to \mathcal{P}_f(X) \) is a fuzzy multisubmeasure. Let \( (\alpha_n)_{n \in \mathbb{N}^*} \subseteq (0, +\infty), \) with \( \alpha_{n+1} < \alpha_n \), for every \( n \in \mathbb{N}^* \), such that \( \lim \alpha_n = 0 \) and let \( S_n = \{A \in \mathcal{C} \setminus A \) is a pseudo-atom of \( \mu \) and \( \alpha_n + 1 \leq |\mu(A)| < \alpha_n \}, \) for every \( n \in \mathbb{N}^* \). Then the set \( E_{n+1} = \bigcup \mathcal{S}_{n+1} \) is represented as the union of at most finite pairwise disjoint sets of \( S_{n+1} \) and a set \( F \) with \( \mu(F) = \{0\} \).

Using Theorem 3.5 for \( \alpha_n = \frac{1}{n}, n \in \mathbb{N}^* \), we have obtained the following result:

**Theorem 3.6.** Let \( \mu : C \to \mathcal{P}_f(X) \) be a fuzzy multisubmeasure, with \( |\mu| : C \to \mathbb{R}_+ \). Then there exists a sequence of pairwise disjoint pseudo-atoms \( (A_n)_{n \in \mathbb{N}^*} \) of \( \mu \), having the property that for every \( E \in C \) and every \( \varepsilon > 0 \), there are a subsequence \( (A_n)_{n \in \mathbb{N}^*} \subseteq (A_n) \), \( p_1, p_2 \in \mathbb{N} \) and \( B \in C \) with \( B \cap A_{n+1} = \emptyset \), for every \( n \in \mathbb{N}^* \), satisfying the conditions:

(i) \( E = (\bigcup_{n=1}^{p_1} A_n) \cup (\bigcup_{n=p_1+1}^{p_2} A_n) \cup \bigcup_{n=p_2+1}^{\infty} A_n \cup B \),

(ii) \( |\mu(A_n)| > \varepsilon, \forall n \in \{1, 2, \ldots, p_1\} \),

(iii) \( |\mu(A_n)| \leq \varepsilon, \forall n \in \{p_1 + 1, p_1 + 2, \ldots, p_2\} \),

(iv) \( |\mu(\bigcup_{n=1}^{p_2} A_n)| \leq \varepsilon \),

(v) \( B \) contains no pseudo-atoms of \( \mu \).
We now prove that any fuzzy multisubmeasure on a $\sigma$-algebra can be uniquely extended to a wide family of sets.

**Lemma 3.7.** Let $\mu : C \rightarrow \mathcal{P}_f(X)$ be exhaustive, with $\mu(\emptyset) = \{0\}$. Then for every $\varepsilon > 0$ and every $A \subseteq T$, there exists a set $K \subset C$ such that $K \subset A$ and $|\mu(B \setminus K)| < \varepsilon$, for every $B \in C$, with $K \subset B \subset A$.

From now on, we suppose $X$ is a Banach space.

**Theorem 3.8.** (i) If $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is an exhaustive multisubmeasure, then for every $A \subseteq T$, there exists $\lim_{B \in \mathcal{C}, B \subset A} \mu(B)$ in $\mathcal{P}_f(X)$ (where $(\mu(B))_{B \in \mathcal{C}, B \subset A}$ is a net with the sets of indices directed by inclusion).

(ii) If we denote $\mu^*(A) = \lim_{B \in \mathcal{C}, B \subset A} \mu(B)$, for every $A \subseteq T$, then $\mu^* : \mathcal{P}(T) \rightarrow \mathcal{P}_f(X)$ is monotone and exhaustive. Moreover, $\mu^* / \mu = \mu^*$.

(iii) If $\mu$ is non-pseudo-atomic, then the same is $\mu^*$.

**Proof.** (i) One proves that $(\mu(B))_{B \in \mathcal{C}, B \subset A}$ is a Cauchy net in $\mathcal{P}_f(X)$. Since $\mathcal{P}_f(X)$ is complete with respect to $h$, the net $(\mu(B))_{B \in \mathcal{C}, B \subset A}$ is convergent in $\mathcal{P}_f(X)$. Its limit exists in $\mathcal{P}_f(X)$ and is unique.

(ii) It results from the definition of $\mu^*$.

(iii) Suppose that, on the contrary, there exists a pseudo-atom $A_0 \in \mathcal{P}(T)$ for $\mu^*$. Then $\mu^*(A_0) > \{0\}$ and for every $B \subset A_0$, we have that $\mu^*(B) = \{0\}$ or $\mu^*(B) = \mu^*(A_0)$. Because $\mu^*(A_0) > \{0\}$, there exists $C_0 \in \mathcal{C}$ so that $C_0 \subset A_0$ and $\mu^*(C_0) > \{0\}$. Since $\mu$ is non-pseudo-atomic, there is $D_0 \in \mathcal{C}$ so that $D_0 \subset C_0$, $\mu(D_0) > \{0\}$ and $\mu^*(D_0) = \mu^*(A_0)$.

If $\mu^*(D_0) = \{0\}$, then $\mu(D_0) = \mu^*(D_0) = \{0\}$, which is false.

If $\mu^*(D_0) = \mu^*(A_0)$, then $\mu(D_0) = \mu^*(C_0)$.

By $\mu^*(C_0) \leq \mu^*(A_0) = \mu^*(D_0)$, a contradiction. So, $\mu^*$ is non-pseudo-atomic, as claimed.

In the sequel, $C$ will be an algebra of subsets of $T$.

**Lemma 3.9.** Let $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ be exhaustive, with $\mu(\emptyset) = \{0\}$. Then for every $\varepsilon > 0$ and every $A \subseteq T$, there exists $D \in \mathcal{C}$ such that $A \subset D$ and $|\mu(D \setminus B)| < \varepsilon$, for every $B \in C$, with $A \subset B \subset D$.

Now, let $C_\mu = \{A \subset T\}$ for every $\varepsilon > 0$, there exist $K, D \in C$ such that $K \subset A \subset D$ and $|\mu(B)| < \varepsilon$, for every $B \in C$, with $B \subset D \setminus K$.

Obviously $C \subset C_\mu$ and it is easy to check that $C_\mu$ is an algebra.

**Theorem 3.10.** Let $\mathcal{C}$ be a $\sigma$-algebra and $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ a fuzzy multisubmeasure. Then $\mu$ uniquely extends to a fuzzy multisubmeasure $\mu^* : C_\mu \rightarrow \mathcal{P}_f(X)$.

**Proof.** I) Since $\mathcal{C}$ is, particularly, a $\sigma$-ring and $\mu$ is fuzzy, then, according to Theorem 2.4-II), $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is exhaustive. By Theorem 3.8, $\mu$ extends to an exhaustive monotone multivalued set function $\mu^* : C_\mu \rightarrow \mathcal{P}_f(X)$.

II) We prove that $\mu^* / C_\mu$ is a multisubmeasure.

Now, let be $A_1, A_2 \subset C_\mu$, with $A_1 \cap A_2 = \emptyset$. We prove that $\mu^*(A_1 \cup A_2) \leq \mu^*(A_1) + \mu^*(A_2)$. Indeed, since $A_1 \cup A_2 \subset C_\mu$, then there are $K, D \in C$ such that $K \subset A_1 \cup A_2 \subset D$ and $h(\mu^*(A_1 \cup A_2), \mu^*(B)) < \frac{\varepsilon}{\delta}$, for every $B \in C$, with $B \subset D$.

Analogously, for $A_1 \subset C_\mu$ there are $K_i, D_i \subset C$ such that $K_i \subset A_i \subset D_i$ and $h(\mu^*(A_i), \mu^*(B)) < \frac{\varepsilon}{\delta}$, for every $B \in C$, with $K_i \subset B \subset D_i$, $i = 1, 2$.

Let $\tilde{K}_1 = K_1 \cup (\mathcal{K} \setminus K_2) \cap D_1 \subset C$ and $\tilde{K}_2 = (D_2 \cap D) \setminus \tilde{K} \subset C$. One can easily check that $\tilde{K}_1 \cap \tilde{K}_2 = \emptyset$, $\tilde{K}_1 \subset \tilde{K}_1 \subset D_1, K_2 \subset \tilde{K}_2 \subset D_2$ and $K \subset \tilde{K}_1 \cup \tilde{K}_2 \subset D$. Therefore, $h(\mu^*(A_1), \mu^*(K_1)) < \frac{\varepsilon}{\delta}$, $h(\mu^*(A_2), \mu^*(K_2)) < \frac{\varepsilon}{\delta}$ and $h(\mu^*(A_1 \cup A_2), \mu^*(\tilde{K}_1 \cup \tilde{K}_2)) < \frac{\varepsilon}{\delta}$.

Because $\mu(K_1 \cap \tilde{K}_2) \subseteq \mu(\tilde{K}_1) + \mu(\tilde{K}_2)$, then $e(\mu(\tilde{K}_1 \cap \tilde{K}_2), \mu(\tilde{K}_1) + \mu(\tilde{K}_2)) = 0$. We immediately get that $e(\mu(A_1 \cup A_2), \mu(A_1) + \mu(A_2)) < \varepsilon$, for every $\varepsilon > 0$, hence $\mu^*(A_1 \cup A_2) \leq \mu^*(A_1) + \mu^*(A_2)$, as claimed.

III) We prove that $\mu^*$ is fuzzy, or, equivalently, by Theorem 2.4-I), that $\mu^*$ is $\sigma$-continuous. Let $(A_n)_{n \in \mathbb{N}} \subset C_\mu$, with $A_n \cap \emptyset = \emptyset$. For every $n \in \mathbb{N}$, there are $K_n, D_n \subset C$ such that $K_n \subset A_n \subset D_n$ and $|\mu(B)| < \frac{\varepsilon}{\delta}$, for every $B \in C$, with $B \subset D_n \setminus K_n$. Then, particularly, $|\mu(D_n \setminus K_n)| < \frac{\varepsilon}{\delta}$.

Let $\tilde{K}_n = \bigcap_{i=1}^{n} K_i$, for every $n \in \mathbb{N}$. Then $\tilde{K}_n \setminus \emptyset$ and, since $\mu$ is fuzzy (equivalently, $\sigma$-continuous) on $C$, we get that $\lim_{n \to \infty} |\mu(\tilde{K}_n)| = 0$.

Consequently, because $D_n \setminus \tilde{K}_n \subseteq$
Theorem 3.12. Let \( \mu \) be an arbitrary fuzzy measure on a ring \( C \), then \( \mu \) is fuzzy.

**Proof.** Indeed, if \( \mu \) is a \( R'_c \)-regular multisubmeasure, then, according to Gavriluț [5], it is also \( o \)-continuous, hence \( \mu \) is fuzzy. \(\)

**Remark 3.13.** I) If \( C = B_0 \), then a multisubmeasure \( \mu \) is \( R'_c \)-regular if and only if it is fuzzy. Indeed, according to Gavriluț [5], a multisubmeasure \( \mu : B_0 \rightarrow \mathcal{P}_f(X) \) is \( R'_c \)-regular if and only if it is \( o \)-continuous.

II) For a multisubmeasure defined on \( B_0 \) or \( B \), \( R'_c \)-regularity is equivalent to \( R'_c \)-regularity (see Gavriluț [4]). So, if \( \mu : B_0 \rightarrow \mathcal{P}_f(X) \) is a multisubmeasure, then \( \mu \) is \( R'_c \)-regular if and only if it is fuzzy.

In Gavriluț [9] it is proved that, on \( B \), every atom of a multisubmeasure \( \mu \) is \( R'_c \)-regular with respect to \( \mu \). Also, a condition which characterizes the atoms of \( \mu \) is given:

**Theorem 3.14.** Let \( \mu : B \rightarrow \mathcal{P}_f(X) \) be a multisubmeasure and \( A \in B \) with \( \mu(A) \geq \{0\} \).

(i) If \( A \) is an atom of \( \mu \), then \( A \) is \( R'_c \)-regular with respect to \( \mu \).

(ii) \( A \) is an atom if and only if there is an unique \( a \in A \) such that \( \mu(A \setminus \{a\}) = \{0\} \).

(iii) If for every \( t \in T \), there exists \( A_t \in B \) so that \( t \in A_t \) and \( \mu(A_t) = \{0\} \), then \( \mu \) is non-atomic.

**Theorem 3.15.** Let \( \mu : B \rightarrow \mathcal{P}_b(X) \) be a multisubmeasure. Then:

(i) \( \mu \) is non-atomic if and only if for every \( t \in T \), \( \mu(\{t\}) = \{0\} \).

(ii) \( \mu \) has the Darboux property, then for every \( t \in T \), \( \mu(\{t\}) = \{0\} \).

The converse of Theorem 3.15-(ii) is not valid in general, as we can see from the following example:

**Example 3.16.** Let \( T \) be a compact space and \( \mu : B \rightarrow \mathcal{P}_{bc}(\mathbb{R}) \) be the multisubmeasure defined by:

\[
\mu(A) = \left\{ \begin{array}{ll} [-m(A), m(A)], & \text{if } m(A) \leq 1 \\ [-m(A), 1], & \text{if } m(A) > 1, \end{array} \right.
\]

for every \( A \in B \), where \( m : B \rightarrow \mathbb{R}_+ \) is a bounded finitely additive set function having the Darboux property. Then \( \mu \) is non-atomic, hence \( \mu(\{t\}) = \{0\} \), for every \( t \in T \), but \( \mu \) does not have the Darboux property.

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References: