Fuzzy Measure Based Approximate Reasoning with Distance-based Operators

MÁRTA TAKÁCS

John von Neumann Faculty of Informatics,
Budapest Tech
1034 Budapest, Bécsi út 96/b.
HUNGARY

Abstract: - In the paper a distance based operators based approximate reasoning method is represented, applying fuzzy measure. Distance based operators satisfy most of properties as generally defined parametrical operators. Based on this the possible relationship with the integrals based on fuzzy measures is introduced. It can be shown, that the reason used for fuzzy integral introduction is similar to the reasons in approximate reasoning by fuzzy control.

Key-Words: - distance based operators, approximate reasoning, fuzzy measures

1 Introduction

In practical representation of the fuzzy logic theory it is very important to be visually recognized. If we examine an FLC model with triangular or trapezoid membership functions, we can recognize the relationships between classical integral calculus and the decision making in FLC. It is very important in the cases of non-continuous fuzzy operators, where the use of classical degree of firing in approximate reasoning (AR) is not correct in all cases. So the new groups of operators must find their place in theoretical system of AR and parametrical norms, and based on theoretical background step should be made for further applications.

The uninorms were introduced, as a generalization of t-norms and conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in unit interval. The recent results on uninorms we can find in [1]. New trends in information aggregation, starting from the classical Zadheian operators through the group of entropy-based and evolutionary operators, to distance-based operators which can found in [2] and [4], with the proposition that this non-classic group of norms and conorms are guided by uninorm theory. Paper [4] described the main idea of using these norms in fuzzy logic control (FLC), where the fuzzy rule output is nothing else, but a fuzzy set weighted with the degree of coincidence of the rule premise and system input. You can read the similar reason at introducing a new type of fuzzy integral in [3]. Fuzzy measure theory needed in fuzzy integral calculus can be found in [5], [6] and [3].

2 Known fuzzy preliminaries

The fuzzy logic control has been carried out searching for different mathematical models in order to supply rules:

IF $x$ is $A_i$ THEN $y$ is $B_i$

where $x$ is $A_i$ is the rule premise, and $y$ is $B_i$ is the rule consequence, in the i-th rule of the rule system of $p$ rules. $A_i$ and $B_i$ are fuzzy sets on the universes $X$ and $Y$. $x$ is the rule input variable from the universe $X$, and $y$ is the rule output variable from universe $Y$.

The important part of a rule-based system is the inference mechanism. One of the widely used methods is a generalised modus ponens (GMP) in which the main point is, that the inference $y$ is $B'$ is obtained from the rule outputs $B'_i$ when the proposition is: the rule IF $x$ is $A_i$ THEN $y$ is $B_i$, and the system input $x$ is $A'$.

The connection $\text{Imp}(A, B)$ is generally defined, and it can be some type of t-norm. GMP sees the real influences of the implication choice on
the inference mechanisms in fuzzy systems, of course, where the general rule consequence is obtained by

\[ B'(y) = \sup_{x \in X} T(A'(x), \text{Imp}(A(x), B(y))) \] (1)

On a general level, \text{Imp} is the relationship between rule base premise and rule base consequence, satisfying the following conditions:

(out1) If the input coincides with one of the premises, then the resulting output coincides with the corresponding consequence, i.e., \((\exists i \in \{1, 2, \ldots, n\})(A' = A_i) \text{ then } B' = B_i\).

(out2) For each normal input \(A'\) the output is not contained in all consequences, i.e., \((\exists i \in \{1, 2, \ldots, n\})(B'_i < B_i)\).

(out3) The rule output belongs to the convex hull of \(B_i, (i \in I)\), where \(I = \{\| \leq i \leq n, \text{Supp}(A') \cap \text{Supp}(A_i) \neq 0\}\).

In [7] we can find an axiom system on the same principle. One of the conditions for those conditions is the correct partition of the output universe \(Y\), and covering over this universe with the \(B_i(y)\) sets.

2.1. Fuzzy measures

Fuzzy measure is defined as a function \(m : \Sigma \rightarrow [0, 1]\), where \(\Sigma\) is a \(\sigma\) algebra of fuzzy subsets of \(X\) (\(X\) is a non empty set). The interval \([0, 1]\) can be modified in interval \([0, 1]\), as usually in FLC. Function \(m\) must have properties, sometimes generalized properties, described as:

M1. boundary condition, \(m(\emptyset) = 0\),

M2. monotonicity, for every \(A\) and \(B\) from set of fuzzy subsets, where \(A \subseteq B\), then \(m(A) \leq m(B)\).

M3. continuity, that for either \(A_1 \subseteq A_2 \subseteq \ldots\) it is \(\lim_{j \to \infty} m(A_j) = m(\lim_{j \to \infty} A_j)\).

In order to generalize fuzzy measure theory we can find the so called \(S\) measure types, with properties

MP1. \(m(A \cup B) = S(m(A), m(B))\), for \(A \cap B = \emptyset\).

MP2. \(m(A \cap B) = T(m(A), m(B))\), for separable \(A\) and \(B\) too.

\((T, S)\) is a pair of t-norm and conorm, or pair of conjunctive and disjunctive type of fuzzy operators [3]. It is very important, that the parametrical \((T, S)\) pair, with parameter \(q\) has further conditions:

MP3. If \(q = 0\), we have a probability measure,

MP4. If \(q = 1\), we have a possibility measure.

MP5. For parameter \(q \in [0, 1]\), and for every \(A\) and \(B\) from set of fuzzy subsets

\[
m[A \cup B] = \begin{cases} \max(m(A), m(B)) & \text{if } m(A) > q \\ m(A) + m(B) - q & \text{otherwise} \end{cases}
\]

2.2. Fuzzy integrals

In [3] the next utility function is examined

\[
U(u_1, u_2; \mu_1, \mu_2) = S(T(u_1, \mu_1), T(u_2, \mu_2))
\] (2)

where \(u_1, u_2\) are utility values in the interval \([0, 1]\), and \(\mu_1, \mu_2\) are two degrees of plausibility from the set

\[
\Phi_{k,p} = \{(\alpha, \beta) | (\alpha, \beta) \cup (p, 1)^2, \alpha + \beta = 1 + q \} \cup \{ (\alpha, \beta) | \min(\alpha, \beta) \leq p, \max(\alpha, \beta) = 1 \}
\] (3)

Based on this \((S, U)\) integral was introduced [3].

Def.1. Let \(S\) be a \(t\)-conorm, \(\Sigma\) is a \(\sigma\) algebra of fuzzy subsets of \(X\) and \(m : \Sigma \rightarrow [0, 1]\) an \(S\) measure.

A set \(A \in \Sigma\) is \(S\) measure faithful if each \(B \in \Sigma\), where \(B \subseteq A\) we have \(S(u, v) < 1\), wherever \(u < m(B)\), and \(v < m(A \backslash B)\).

A partition \(\beta = \{B_k | B_k \in \Sigma, k \in N^1\}\) is called \(S\)m partition if the sets \(B_k\) are faithful.

The \(S\) measure is faithful, if there is a faithful partition of \(X\).

A measurable step function \(\rho : X \rightarrow [0, 1]\) with conditions described in [3], has a canonical representation

\[
\rho = \sum_{i=1}^{n} U(a_i, q), \text{ for } x \in A, \text{ and } \rho = \sum_{i=1}^{n} U(a_i, 0)
\] otherwise.

Def.2. Let \(m : \Sigma \rightarrow [0, 1]\) be a \(S\)m faithful \(S\) measure.
Given an $Sm$ faithful partition $\beta = \{B_i, \mu_i \in \Sigma, i \in N\}$, the $(S,U)$ integral of a measurable function $\rho : X \rightarrow [0,1]$ is defined by:

$$
\int \rho dm = \int_{k \epsilon K} \left( \sum_{i=1}^{n} S(U(a_i, m(A_i \cap B_k))) \right)
$$

(4)

3 $(S,U)$ integral and the approximate reasoning model in FLC with distance based reasoning operators

The maximum distance minimum operator with respect to parameter $e \in [0,1]$ is defined as

$$T_{e}^{\text{max}} = \begin{cases}
\max(x,y) & \text{if } y-e > e-x \\
\min(x,y) & \text{if } y-e < e-x \\
\min(x,y) & \text{if } y=x \text{ or } y-e = e-x
\end{cases}$$

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\end{cases}$$

Several pairs of the distance based operators for the rule outputs obtaining and for the FLC output obtaining have been tried out in FLC systems, with special emphasis on the pairs $(T_{e}^{\text{max}}, S_{e}^{\text{max}})$ and $(T_{e}^{\text{min}}, S_{e}^{\text{min}})$. It was tested in a simulation system [4]. The choosing of pairs $(T_{e}^{\text{max}}, S_{e}^{\text{max}})$ and $(T_{e}^{\text{min}}, S_{e}^{\text{min}})$ by the simulation, using the same $e$ value, gives results with negligible difference. So it was sufficient trying out the pairs $(T_{e}^{\text{max}}, S_{e}^{\text{max}})$ for example. The choosing of the pair $(T_{e}^{\text{max}}, S_{e}^{\text{max}})$, where $e$ is near zero, returns in a short time the desired state of the system, but it is not stable. If $e$ is near 1, the situation is known, because it develops into a (min, max) pair. The desired state is obtained easier, and the systems stay stable. It can be observed, that continual choosing of $e$ from zero till 1 results continual improvement in stability, and continual increasing time of obtaining desired state in system. The choosing of pair $(T_{e}^{\text{max}}, S_{e}^{\text{max}})$ gives acceptable result by both criteria. It was to be expected again, that the reconciling of the $T$ and $S$ pairs in FLC, where this pair is dual in fuzzy theory sense, gives the best results in a simple simulation system.

In recent years the use of continuous, monotonic $t$-norms in fuzzy control is strongly preferred over other types of $t$-norms. On the other hand, special types of operators have been developed, with several boundary conditions. Using family-tree of the uninorms, we obtain new possibilities to explain small and large degrees of fuzzy memberships, and the “pessimistic” (intersection-type, see MP3), and “optimistic” (union-type, MP4) connections between fuzzy sets. It turned out, that the distance based operators group and the novel inference mechanism, in well-known rule base system and the GMP the coincidence of the rule premise and the system input appears in a new form.

Because of the non-monotonic property of the distance-based operators, it was unreasonable to use the classical degree of firing, to give expression to coincidence of the rule premise (fuzzy set $A_i$), and system input (fuzzy set $A'_i$), therefore a degree of coincidence ($Doc$) for these fuzzy sets has been initiated. It is nothing else, but the proportion of area under membership function of the modified entropy-based intersection of these fuzzy sets, and the area under membership function of the their union (using max as the fuzzy union).

The rule output fuzzy set $(B_i)$ should not achieved as a cut of rule consequence $(B_i)$ with
It is supposed that the rule output can be better obtained in the same area-proportional sense. For the rule consequence, rule premise and system input triangular membership functions are generally used, therefore from the area of rule-consequence triangle an area for rule output is carved out in proportion with Doc.

This reason has two advantages:
- it considered the width of coincident of $A_i$ and $A'$, and not only the height
- the rule output is weighted with a measure of coincident of $A_i$ and $A'$ in each rule.

For a given input fuzzy set $A'(x)$, in a mathematical-logical sense, the output fuzzy set $B_i(y)$ in one rule, will be generated with the expression

$$B_i'(y) = \max \{B_i(y), 1 - \sqrt{1 - Doc_i} \}$$

where $Doc_i$ is the degree of coincidence, and gives expression to coincidence of the rule premise (fuzzy set $A$), and system input (fuzzy set $A'$) in the $i$-th rule of the rule system:

$$Doc_i = \frac{\int_x T_{x}^{\min} (A_i(x), A'(x)) dx}{\int_x \max(A_i(x), A'(x)) dx}$$

where $T_{x}^{\min}$, $T_{x}^{\max}$, $S_{x}^{\min}$ and $S_{x}^{\max}$ are $t$-norm and $t$-conorm, respectively, for calculating measure $m$, we can use pair $(T_{x}^{\min}, S_{x}^{\max})$, because they are parametrical norms, and satisfy all conditions required for fuzzy measures and fuzzy integrals.

It is easy to prove, that $Doc \in [0,1]$ and $Doc=1$ if $A_i$ and $A'$ cover over each other, and $Doc=0$ if $A_i$ and $A'$ have no point of contact, and satisfy axioms of fuzzy reasoning (out1)-(out3).

This system was tested several times. The conclusion was, that using this type of distance based operators and novel Doc as the fuzzy sets coincidence expression, the controlled system obtains the desired state better then by classical approach.

The FLC rule base output is constructed as crisp value calculated from associative using $(S,U)$ integrals for rule outputs, based on (4) as follows:

$$B_{\text{out}}'(y) = S \left( \frac{\sum_{j=1}^{p} U(q_j, m(A_i \cap A'))}{\sum_{j=1}^{p} U(q_j, m(A_i \cap A'))} \right)$$

where $q_j$ is utility value obtained from rule consequence $B_i'$, and it is in accordance with (5).

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Let we introduce

$$m(A_i \cap A') = Doc$$

as the weight measure for computing rule output $B_i'$ in $i$-th rule in rule base.

In $i$-th one rule output is obtained as

$$B_i'(y) = \frac{\sum_{j=1}^{p} U(q_j, m(A_i \cap A'))}{\sum_{j=1}^{p} U(q_j, m(A_i \cap A'))}$$

4 Conclusion

Using special defined measures in [3], we obtain a method, which is close to the visual description of decision-making in FLC, for triangular or trapezoid membership function. The method has theoretical background in fuzzy integrals $(S,U)$.

Acknowledgment

The research was partially supported by the Hungarian Scientific Research Project OTKA T048756.
References


