Robustness Design of Nonlinear Systems: Fuzzy Lyapunov Approach

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Abstract: This study addresses a fuzzy Lyapunov method for the stability analysis of time-delay fuzzy systems subject to external disturbances. The Takagi-Sugeno (T-S) fuzzy model and parallel distributed compensation (PDC) scheme are first employed to design a nonlinear fuzzy controller in order to stabilize the time-delay fuzzy systems. According to the controlled system, the H infinity criterion is derived based on the fuzzy Lyapunov method, which is defined in terms of fuzzy blending quadratic Lyapunov functions. Based on the stability criterion, the time-delay fuzzy systems are guaranteed to be stable.

Key-Words: - Fuzzy Lyapunov method, delay-dependent, T-S fuzzy systems.

1. Introduction

The Takagi-Sugeno (T-S) model was first described by (Takagi and Sugeno, 1985). This model can effectively represent complex nonlinear systems using fuzzy sets, and then by applying fuzzy reasoning to a set of linear input-output submodels (Cao and Frank, 2001). These fuzzy models represent local dynamics in different state space regions with linear models. The overall model of the system is attained by the fuzzy “blending” of these fuzzy models. Since they utilize linear models in the consequent parts, traditional system theory becomes straightforward for analysis. Various controllers based on T-S fuzzy models have since been presented (for example, see Cao et al., 1997; Chen et al., 2004; K. Tanaka and Kosaki, 1997; Tanaka and Sano, 1994 and the references therein).

Linear matrix inequality (LMI) theory is a new and rapidly growing field that provides a helpful alternative to the analytical method (for more details, see Boyd et al., 1994; Nesterov and Nemirovsky, 1994). Many problems arising in system and control theory can be reduced to a few standard convex or quasiconvex optimization problems involving the LMI. These resulting optimization problems can be easily solved with numerical computation, making LMI techniques into highly efficient and practical tools for solving complex control problems (Park et al., 2002). Moreover, stability is an essential property of control systems, and has been widely investigated in literature on fuzzy dynamic systems (see Cao et al., 1996; Chen, 2006; Chen et al., 1993; Cuesta et al., 1999; Feng et al., 1997; Tanaka and Sugeno, 1992; Wang et al., 1996; Margaliot and Langholz, 2003 and the references therein).

The problem of the stabilization of time-delay systems has been explored for several years. Furthermore, engineering processes commonly involve time delays, which frequently arise in chemical processes and long transmission lines in pneumatic, hydraulic and rolling mill systems. The problem of stability analysis of time-delay systems has been one of the main concerns of research into the attributes of such systems. Many monographs on this subject have been published, such as Cao and Frank, 2000; Cao and Frank, 2001; Hsiao et al.,
construct a global fuzzy logic controller. To compensation (PDC) scheme is utilized to briefly reviewed, and the parallel distributed time-delay system, the T-S fuzzy modeling is follows. First, to analyze the stability of a time-delay T-S fuzzy systems.

However, this work focuses on two issues. One issue is that most of the above reported results indicate that the stability analysis is based on T-S fuzzy systems, but not nonlinear systems. Restated, the T-S fuzzy model is adopted to represent practical nonlinear systems and controllers are designed only for fuzzy systems. However, the effect of the modeling error between nonlinear systems and T-S fuzzy models is not addressed. The modeling error might influence the performance of the controlled systems. The other issue is the extension the Lyapunov functions for stability issues of T-S type systems. In order to avoid conservatism of stability and stabilization problems, multiple Lyapunov functions have been paid increasing attention e.g., (Chen et al., 2001; Tanaka et al., 2003; El-Farra et al., 2005). Therefore, this investigation develops fuzzy Lyapunov approaches for the stability analysis of nonlinear systems subjected to external disturbances. To derive a relaxed stability condition, this work extends the fuzzy Lyapunov approaches on stability analysis of time-delay T-S fuzzy systems by applying the linear matrix inequality (LMI) theory. A novel robustness delay-dependent $H^\infty$ LMI-based stability criterion is proposed. Namely, this study considers the T-S based fuzzy control for external disturbances. To derive a relaxed stability concept is thereby proposed, and LMI concept is employed to derive the stability conditions. Furthermore, the $H^\infty$ robust control is also used to achieve the control performance and derive the delay-dependent stability conditions by single Lyapunov functions. To avoid conservatism, fuzzy Lyapunov functions are proposed to extend the stability analysis of time-delay T-S fuzzy systems, while the controller design problem is reformulated into the linear matrix inequality (LMI) problem. Finally, a numerical example of nonlinear structural system with time delays is given to demonstrate the results, and conclusions are drawn.

2. System Description and Problem Preliminary

Consider nonlinear systems represented as follows:

$$\dot{x}(t) = f(x(t), u(t)) + g(x(t - \tau)) + \phi(t)$$ (1)

where $f$ and $g$ are nonlinear vector-valued functions; $t$ denotes time; $\tau$ is the time delay, and is a positive real number; $x(t) \in R^n$ is the state vector; $\phi(t)$ denotes the external disturbance; $\dot{x}(t)$ is derivative of $x(t)$, and $u(t) \in R^m$ is the input vector.

**Definition 1** (Khalil, 1992): The solution of a dynamic system are said to be uniformly ultimately bounded (UUB) if there exist positive constants $\beta$ and $\kappa$, and for every $\delta \in (0, \kappa)$ there is a positive constant $T = T(\delta)$, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \leq \beta, \forall t \geq t_0 + T.$$ 

A nonlinear system is described using a set of fuzzy IF-THEN rules developed primarily from Takagi and Sugeno (1985). The T-S model consists of a set of If-Then rules. Each rule represents the local linear input-output relation of the nonlinear system, and has the following form:

**IF** $z_1(t)$ is $M_{i_1}$ and $\cdots$ and $z_g(t)$ is $M_{i_g}$ **THEN**

$$\dot{x}(t) = A_i x(t) + \bar{A}_i x(t - \tau) + B_i u(t) + \phi(t),$$

$$i = 1, 2, \cdots, r$$ (2)

where state vector

$$x^T(t) = [x_1(t), x_2(t), \cdots, x_n(t)] \in R^{in}.$$
control input

\[ u^T(t) = [u_1(t), u_2(t), \ldots, u_m(t)] \in \mathbb{R}^{1 \times m}, \]

and unknown disturbance

\[ \phi^T(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_p(t)] \in \mathbb{R}^{1 \times p} \]

where \( H \) is the fuzzy system, and \( B_i \) are constant matrices with appropriate dimensions; \( M_{i,j}, (p = 1, 2, \ldots, g) \) is the fuzzy set; \( r \) is the rule number; \( z_i(t) \) are the premise variables.

The parallel distributed compensation (PDC) scheme has the same premise components as the T-S model. The linear control rule \( i \) is derived based on Eq. (2) in the consequent component of the \( i \)th model rule.

\[ u(t) = -K_i x(t), \quad i = 1, 2, \ldots, r \]

(3)

where \( K_i \) is the local feedback gain matrix. The final control \( u \) is inferred using the Sum-Product reasoning method:

\[ u(t) = -\sum_{i=1}^{r} w_i(t) K_i x(t) \sum_{i=1}^{r} w_i(t) \]

(4)

where \( w_i(t) \) is the activation degree of the \( i \)th rule, calculated as:

\[ w_i(t) = \prod_{p=1}^{g} M_{i,p}(z_p). \]

According to the above T-S fuzzy model and PDC scheme, the model of nonlinear system (1) is represented by following closed-loop controlled system.

\[ \dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(t) h_j(t) \{(A_i - B_i K_i) x(t) + \bar{A}_i x(t - \tau)\} + e(t) + \phi(t) \]

(5)

where

\[ e(t) = f(x(t), u(t)) + g(x(t - \tau)) - \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(t) h_j(t) \{(A_i - B_i F_i) x(t) + \bar{A}_i x(t - \tau)\} . \]

Suppose that a bounding matrix \( \Delta H \) exists

\[ e(t) \leq \left\| \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(t) h_j(t) \Delta H x(t) \right\| \]

and \( \Delta H = \delta H \), and \( \left\| \delta \right\| \leq 1 \). Then the following can be readily obtained:

\[ e^T(t) e(t) \leq [H x(t)]^T [H x(t)] \]

(6)

This means that the modeling error is bounded by the matrix \( H \). The proof of Eq. (6) and the procedures for determining \( \delta \) and \( H \) is provided by (Chen et al., 1999).

3. LMI Conditions

This section derives linear matrix inequality (LMI) conditions using Lyapunov theory to verify the stability of time-delay T-S fuzzy systems. Considering an LMI \( F(x) > 0 \), the LMI problem is to find \( x^{feas} \) such that \( F(x^{feas}) > 0 \), or determine that the LMI is infeasible. This is a convex feasibility problem and an LMI definition is clear described in (Lu et al., 1998; Boyd et al., 1994).

Before a typical stability condition for time-delay T-S fuzzy system (5) is proposed, some useful concepts are given below:

**Lemma 1** (Li and Souza, 1997): For any \( A, B \in \mathbb{R}^n \) and for any symmetric positive definite matrix \( G \in \mathbb{R}^{n \times n} \) or \( R \), we have

\[ -2 A^T B \leq A^T G A + B^T G^{-1} B. \]

**Lemma 2** (Wang et al., 1996): The equilibrium point of closed-loop fuzzy system \( \dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(t) h_j(t) \{(A_i - B_i K_i) x(t)\} \)

is asymptotically stable in the large if there exists a common positive definite matrix \( P \) such that

\[ (A_i - B_i K_i)^T P + P(A_i - B_i K_i) < 0, \quad i = 1, 2, \ldots, r. \]

**Lemma 3** (Hsiao et al., 2005): Consider the following \( H^\infty \) control performance to attenuate the influence of the excitation \( \phi(t) \) on the state variable \( x(t) \), if the initial condition is considered.

\[ \int_{0}^{t_f} x(t)^T Q x(t) dt \leq x(0)^T P x(0) + \eta^2 \int_{0}^{t_f} \phi(t)^T \phi(t) dt \]

(9)

where \( P \) are some positive definite matrices, \( t_f \) denotes the terminal time of the control, \( \eta \) is a prescribed value which denotes the effect of \( \phi(t) \) on \( x(t) \), and \( Q \) is a positive definite weighting matrix.

**Lemma 2** gives a sufficient condition for ensuring asymptotic stability of closed-loop fuzzy system

\[ \dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(t) h_j(t) \{(A_i - B_i K_i) x(t)\} \]

Based on the above inequalities, a stability
condition can be derived as follows:

**Theorem 1:** There exists fuzzy $H^\infty$ controller so that the closed-loop time-delay T-S fuzzy system (5) is stable in the large if there exist a symmetric and common positive definite matrix $P \in R_+^{n \times n}$, positive constants $\sigma$, $\eta$ and the feedback gains $K_i$ such that the following inequality is satisfied:

$$
\Delta + R + P \sum_{i=1}^{r} R_i^{-1} A_i^T P + \frac{1}{\eta} PP + \sigma H_\tau P + \sigma^{-1} P P + Q < 0
$$

(10)

where

$$
\Delta = (A_i - B_i K_i)^T P + P (A_i - B_i K_i)
$$

(11)

with $P = P^T > 0$ for $i, l = 1, 2, \cdots, r$

**Remark 1:** Since condition (10) implies that condition (11) can be negative, i.e. $\Delta < 0$, then $V(t) < 0$ when $\phi(t) = 0$ and $\tau = 0$. This means that the closed-loop T-S fuzzy system $\dot{x}(t) = \sum_{i=1}^{r} \sum_{l=1}^{r} h_i(t) h_l(t) \{(A_i - B_i K_i) x(t)\}$ is asymptotically stable if disturbances and modeling errors are not considered. This stability condition of Theorem 1 can be reduced to that of Lemma 2.

**Remark 2:** Theorem 1 can be recast as the linear matrix inequality (LMI) problem by the following procedure.

New variables $H_{il} = A_i - B_i K_i$, $W = P^{-1}$, $\bar{R} = R^{-1}$, $\bar{Q} = Q^{-1}$, $\bar{\sigma} = \sigma^{-1}$, $\bar{\eta} = \eta^{-2}$ and $Y_{il} = H_{il} W$ are introduced and Eq. (10) is rewritten as follows:

$$
\begin{bmatrix}
Y_{il} & HW & W & W \\
(HW)^T & -\bar{\sigma} & 0 & 0 \\
W & 0 & -\bar{Q} & 0 \\
W & 0 & 0 & -\bar{R}
\end{bmatrix} < 0,
$$

(12)

where

$$
Y_{il} = Y_{il} + Y_{il}^T + \bar{\eta} + \bar{A}_i \bar{R} A_i^T + \bar{\sigma}
$$

and $i, l = 1, 2, \cdots, r$.

To avoid conservatism of stability and stabilization problems for the time-delay T-S fuzzy systems from being subjected to disturbances, the fuzzy Lyapunov function approach is employed to derive more generalized criteria, as described in the next section.

4. Fuzzy Lyapunov Function Approach

**Definition 2** (Tanaka et al., 2001, 2003): Equation (13) is said to be a fuzzy Lyapunov function for the T-S fuzzy system if the time derivative of $V(x(t))$ is always negative at $x(t) \neq 0$.

$$
V(x(t)) = \sum_{i=1}^{r} h_i(t) x^T(t) P_i x(t)
$$

(13)

where $P_i$ is a positive definite matrix.

An upper bound of the time derivative, i.e. $|\dot{h}_i(t)| \leq \phi_i$, is adopted to ensure that the term of the time derivative $\dot{h}_i(t)$ can be solved numerically (Chen et al., 2006). The following stability condition is obtained by taking the time derivative of Eq. (5).

**Theorem 2:** The fuzzy system (5) is stable in the large if there exist common positive definite matrices $P_1, P_2, \cdots, P_r$ such that inequality

$$
\begin{align*}
\sum_{\rho=1}^{r} \phi_i P_i \\
+ (A_j - B_j K_j) P_i + P_i (A_j - B_j K_j) + R + P_i A_i R^{-1} A_i^T P_i \\
+ \bar{\eta} P_i + \bar{\sigma} H_\tau (t) H(t) + \sigma P_i P_i + Q < 0
\end{align*}
$$

(14)

where

$$
\bar{\psi}_{il} = Y_{il} + Y_{il}^T + \bar{\eta} + \bar{A}_i \bar{R} A_i^T + \bar{\sigma}
$$

and $i, j, l = 1, 2, \cdots, r$, and the symbol $*$ denotes the transposed elements in the symmetric positions.

**Remark 4:** The conditions (14-15) imply that conditions (10-12) are satisfied when the single Lyapunov function is considered. This means the stability criterion of Theorem 2 can be reduced to that of Theorem 1.

5. Conclusions

This study considers a fuzzy Lyapunov method to derive the robustness stability condition for the time-delay fuzzy system with external disturbances. The fuzzy Lyapunov function is defined by fuzzy-blending quadratic Lyapunov functions. First, the nonlinear systems are described by the T-S fuzzy model, and the PDC scheme is used to construct the T-S fuzzy control. To ensure that the controlled system can be
stabilized, the robustness design is proposed to overcome the modeling error. Delay-dependent stability conditions of closed-loop controlled systems are then derived based on fuzzy Lyapunov functions to avoid conservatism.

Acknowledgment
The authors would like to thank the National Science Council of the Republic of China, Taiwan, for financial support of this research under Contract No. NSC 96-2628-E-366-004-MY2, NSC 96-2415-H-156-003-MY2, NSC 96-2628-E-132-001-MY2 and NSC 95-2415-H-156-007. The authors are also most grateful for the constructive suggestions from anonymous reviewers all of which has led to the making of several corrections and suggestions that have greatly aided us in the presentation of this paper.

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