Joint Effect Analysis of Phase Noise, Carrier Frequency Offset, Doppler Spread and Nonlinear Amplifier on the Performance of OFDM System

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Abstract: In this paper, we provide an exact closed-form expression of the effective signal-to-noise ratio (SNR) for orthogonal frequency division multiplexing (OFDM) performance under the combined effect of phase noise, frequency offset, and amplifier nonlinearities of the soft envelope limiter (SEL). This formula describes explicitly the various parameters with the system performance. After deriving the effective SNR, we evaluate the bit-error-rate (BER) for slowly fading Rayleigh channel for QPSK and 16QAM modulation formats. We also obtain the minimum total degradation (TD) versus phase noise rate for different values of frequency offset rate as a criterion of performance evaluation. We show that the degradation is increased as the phase noise levels and frequency offset levels are increased. Even worse results are obtained for 16QAM modulation format compared with QPSK modulation format.

Key-Words: OFDM, CFO, Doppler spread, and amplifier nonlinearity.

1 Introduction
Orthogonal frequency division multiplexing (OFDM) is considered a practical scheme to combat multipath channel fading (eg.[1]). However, OFDM has some disadvantages. One of the main disadvantages is the distortion generated by the high power amplifier (HPA) at the transmitter end. Other sources of impairments are the effect of the phase noise and the frequency offset which is due to the deviation between the transmitter and the receiver, or by Doppler shifts.

Several studies have been proposed to analyze the effect of amplifier nonlinearities [2]-[4], the effect of phase 1 noise [5]-[8], the effect of CFO [9],[10], and the effect of the Doppler spread [11]-[13] separately. The joint effect of both amplifier nonlinearity and the phase noise in M-QAM OFDM system was analyzed in [14], however, do 6 not provide, even for additive white Gaussian noise(AWGN) channel, a closed-form analytical expression that shows the exact quantitative relations between various parameters and system performance. In this paper, we describe the combined effect analysis of phase noise, CFO, Doppler spread, and amplifier nonlinearity on OFDM system performance.

The paper is organized as follows. In section 2, the system model is presented including nonlinear amplifier model, phase noise model, Doppler spectrum model, channel model, and OFDM system model. In section 3 , performance analysis is provided by evaluating the effective SNR, the BER and the TD of the system. The numerical results of the system performance are presented in section 4. Finally, the paper is concluded in section 5.

2 System Model

2.1 Nonlinear amplifier model
The nonlinear distortion at the transmitter causes some interference both inside and outside the signal bandwidth [3]. In this paper, we concentrate our attention on the in-band interference of the SEL only. The AM/AM (amplitude modulation / amplitude modulation) function is given by

$$A(r) = \begin{cases} r, & r \leq A_s \\ A_s, & r \geq A_s \end{cases}$$

(1)

Where $A_s$ is the input amplitude of the maximum amplifier output power.

2.2 Phase noise model
Phase noise, $\phi(t)$ generated at both transmitter and receiver oscillators, can be described (eg. [6], [15]) as a g, continuous-path Brownian motion (or
Wiener process \( ) \) with zero mean and \( \text{variance} \sigma^2_\phi \) which is given by
\[
\sigma^2_\phi = 2\pi \beta \vert f \vert \quad (2)
\]
Where the parameter \( \beta \) represents the two-sided 3-dB linewidth of Lorentzian power spectral density of the local oscillator.

### 2.3 Doppler spectrum model

In this work, we use the worst extreme case of the Doppler spectrum. It is the two-path model [12], where its autocorrelation function is given by Doppler spectrum:
\[
R_d(\tau) = \cos(2\pi f_d \tau) \quad (3)
\]
Where \( f_d \) is the maximum Doppler frequency.

It should be noted that the two-path model corresponds to an OFDM with a fixed frequency offset of \( f_d \). So, this model can be used to analyze the effect of both CFO and Doppler spread.

### 2.4 Channel model

We assume a multipath fading channel which is modeled in the time domain by \( M \) delayed impulses
\[
h(t, \tau) = \sum_{r=0}^{N_p-1} \alpha_r(t) \delta(\tau - \frac{\tau_r(t)T}{N}) \quad (4)
\]
Where \( N_p \) is the number of paths and \( \delta(\tau) \) is a Dirac delta function. \( \alpha_r(t) \) and \( \tau_r(t) \) are the attenuation coefficients and delay times of the \( r \)-th path, respectively. \( N \) is the number of the OFDM subcarriers and \( T = T_s - T_g \) is the duration of the useful OFDM symbol where \( T_s \) is the transmitter OFDM symbol and \( T_g \) is the duration of the guard interval, respectively. \( \{\alpha_r(t)\}_{r=0}^{N_p-1} \) are modeled as zero-mean complex Gaussian random variables. Also, \( \{\tau_r(t)\}_{r=0}^{N_p-1} \) are modeled as independent uniform random variables in \((0, T_g)\) where the parameter \( t \) can be omitted in a slow fading channel. For arbitrary symbol of an \( N \)-subcarrier OFDM system, the corresponding channel gains in the frequency domain are expressed by \( \{H_k\}_{k=0}^{N-1} \) with autocorrelation function \( E[H_k^2] = 1 \), as given in [6].

### 2.5 OFDM system model

Due to the presence of the amplifier nonlinearities, phase noise, and the frequency offset the OFDM transmitted signal is given by (eg. [6], [10], [12], and [14])
\[
s(t) = a e^{j2\pi f t} \sum_{k=0}^{N-1} a_k e^{j\phi(t)} e^{j2\pi k t} + \\
\sum_{k=0}^{N-1} D_k e^{j2\pi f t} e^{j2\pi k t} \quad (5)
\]
Where \( a \) is the complex gain factor due to the amplifier nonlinearity and \( f \) is the frequency offset which is equal to \( f_d + f_0 \) where \( f_d \) is the Doppler spread and \( f_0 \) is the CFO. \( a_k \) is the data symbol which is assumed to be zero-mean random variable with variance \( E_s = E(\vert a_k \vert^2) \) where \( E_s \) is the transmitted energy per data symbol and \( E(\cdot) \) denotes the expectation of the argument. \( D_k \) is the NLD due to HPA with variance \( \sigma^2_D \).

The detected values at the input of a decision element is given by
\[
Z_k = aI_0 a_k H_k + I_0 D_k H_k + \\
\sum_{r=0, r \neq k}^{N-1} I_{k-r}(aa_r + D_r)H_r + N_k \quad (6)
\]
Where \( I_k \) is given by
\[
I_k = \frac{1}{T} \int_0^T e^{j2\pi f t} e^{j\phi(t)} e^{-j2\pi k t} dt \quad (7)
\]
Equation (6) contains four components. The first component is the desired signal where \( I_0 \) is the common error (CE) due to the presence of phase noise and frequency offset. The second component is the nonlinear interference (NL) signal. The third component is the intercarrier interference (ICI) component and the last component is an AWGN component.

### 3 Performance analysis

#### 3.1 Effective SNR derivation

In order to derive the effective SNR, it is convenient to separate the CE in its mean value and in its varying part [5], [10]. So, (6) will be
\[
Z_k = aE[I_0]H_k + aE[I_0 - E[I_0]] + \\
I_0 D_k H_k + \sum_{r=0, r \neq k}^{N-1} I_{k-r}(aa_r + D_r)H_r + N_k \quad (8)
\]
Where the second term is termed now, common error interference (CEI). Hence, the powers of the desired signal, CEI, NLI, ICI and AWGN is given by (see the Appendix)

\[ P_{\text{des}} = E_x \left| \mathbf{a} \right|^2 E^2 \left( |I_0| \right) \]

\[ = E_x \left| \mathbf{a} \right|^2 \exp \left( -\sigma_\alpha^2 \sin^2 (\Delta f) \right) \]

(9)

Where \( \sin c = \sin (\pi x) / \pi x \).

\[ P_{\text{CI}} = E_x |\mathbf{a}|^2 \text{Var}(I_0) = E_x |\mathbf{a}|^2 \left( \frac{\sigma_\alpha^2}{4} + 4\pi^2 (\Delta f)^2 T^2 \right) \]

\[ \frac{2\sigma_\alpha^2}{4} - 4\pi^2 (\Delta f)^2 T^2 \]

\[ = \frac{\sigma_\alpha^2}{4} + 4\pi^2 (\Delta f)^2 T^2 \]

\[ \frac{2\sigma_\alpha^2}{4} - 4\pi^2 (\Delta f)^2 T^2 \]

\[ = \frac{2\sigma_\alpha^2}{4} + 4\pi^2 (\Delta f)^2 T^2 \]

\[ \frac{2\sigma_\alpha^2}{4} - 4\pi^2 (\Delta f)^2 T^2 \]

\[ = \frac{\sigma_\alpha^2}{4} + 4\pi^2 (\Delta f)^2 T^2 \]

\[ e^{-\sigma_\alpha^2 \sin^2 (\Delta f)} \]

\[ P_{\text{NI}} = E_x |\mathbf{D}|^2 = E_x \left( \frac{\sigma_\alpha^2}{4} + 4\pi^2 (\Delta f)^2 T^2 \right) \]

\[ \frac{2\sigma_\alpha^2}{4} - 4\pi^2 (\Delta f)^2 T^2 \]

\[ = \frac{2\sigma_\alpha^2}{4} + 4\pi^2 (\Delta f)^2 T^2 \]

\[ = \frac{\sigma_\alpha^2}{4} + 4\pi^2 (\Delta f)^2 T^2 \]

\[ e^{-\sigma_\alpha^2 \sin^2 (\Delta f)} \]

\[ P_{\text{AWGN}} = \sigma^2 \]

(11)

Thus, the effective SNR is given by

\[ \text{SNR}_{\text{eff}} = \frac{P_{\text{des}}}{P_{\text{CI}} + P_{\text{NI}} + P_{\text{des}}} \]

and after some algebra, we get

\[ \text{SNR}_{\text{eff}} = \frac{\sigma^2 e^{-2\pi^2 \sin^2 (\Delta f)}}{1 + \sigma_\alpha^2 + \sigma_\alpha^2 (1 - e^{-2\pi^2 \sin^2 (\Delta f)})} \]

(15)

Where SNR is the signal-to-noise ratio which is equal to \( E_s / \sigma^2 \) and \( \sigma_\alpha^2 \) is the NLD variance \( \sigma_\alpha^2 \) normalized to AWGN variance, \( \sigma^2 \).

In the absence phase noise and frequency offset, we obtain the effective SNR exactly as was obtained in [3] for the effect of the amplifier nonlinearity only. In the absence of amplifier nonlinearity and frequency offset, we get the effective SNR exactly as was obtained in [5] for the effect of phase noise only using discrete time signal model analysis. Similarly, in the absence of amplifier nonlinearity and phase noise, the effective SNR is exactly as was given for CFO only [10].

The normalized NLD variance, \( \sigma_\alpha^2 \), for the SEL is given by

\[ \sigma_\alpha^2 = \text{SNR}(1 - e^{-\text{IBO}} - \alpha^2) \]

(16)

\[ \alpha = 1 - e^{-\text{IBO}} + \frac{\pi \sigma (\text{IBO})}{2} \text{erfc} (\sqrt{\text{IBO}}) \]

(17)

Which is a real value. IBO is the input backoff which represents the ratio between the input saturation power, \( A_s \) and the input mean power.

In the previous analysis, we used Gaussian approximation method to get \( \text{SNR}_{\text{eff}} \). Although Gaussian approximation is not accurate, it is acceptable for small values of frequency offset which are the practical values in OFDM system [9]. Also, although the ICI due to phase noise is non-Gaussian [7], [8] and tends to Gaussian at high values of normalized linewidth \( (\beta T \approx 1) \) [5], [7]. Gaussian approximation can be used and gets acceptable results for moderate SNR even in small values of linewidth.

### 3.2 BER calculations

Under slowly Rayleigh fading channel, the BER for QPSK and 16QAM modulation formats with coherent detection technique and Gray encoding are given by [16]

\[ BER_{\text{QPSK, Ray}} = \frac{1}{2} \left[ 1 - \frac{1}{\sqrt{1 + 1/\text{SNR}_{\text{eff}}}} \right] \]

(18)
\[ BER_{\text{RayQAM}} = \frac{3}{8} \left( 1 - \frac{1}{\sqrt{1 + 5/(2\text{SNR}_{\text{eff}})}} \right) \] \tag{19}

3.3 Total degradation measure

A useful system performance measure is the total degradation (TD) as a function of the HPA IBO. It is given by

\[ TD_{\text{dB}} = \text{SNR}_{\text{dB}} - \text{SNR}_{\text{dB}}' + \text{IBO}_{\text{dB}} \] \tag{20}

Where \( \text{SNR}_{\text{dB}} \) is the required SNR in dB at the input of the threshold detector to obtain a fixed BER for a given value of the IBO, phase noise, and frequency offset. \( \text{SNR}_{\text{dB}}' \) is the required SNR to obtain the same BER in the absence of nonlinearity and \( \text{IBO}_{\text{dB}} \) is the IBO in dB.

The optimum operating point of the amplifier is obtained by minimizing (20).

4 Numerical results

Fig. 1 and Fig. 2 show the BER against SNR for different values of normalized linewidth, \( \beta T \), normalized frequency offset, \( \Delta f T \), and IBO under slowly Rayleigh fading channel for the two modulation formats QPSK and 16QAM respectively. It is clear that the loss will be increased as one, some or all the previous parameters are increased.

Total degradation curves are obtained in Fig. 3 versus \( \text{IBO}_{\text{dB}} \) for 16QAM modulation format under slowly Rayleigh fading channel when \( BER = 0.02 \) for different values of \( \beta T \) and \( \Delta f T \). It shows that \( TD_{\text{dB}} \) will be 10.21 dB for \( \beta T = 0, \Delta f T = 0.025 \) at \( IBO = 10 dB \), and will take the value of 5.68 dB for the same values of normalized linewidth and normalized frequency offset at \( IBO = 4 dB \). Also, \( TD_{\text{dB}} \) will be 15.91 dB for \( \beta T = 0.005, \Delta f T = 0.025 \) at \( IBO = 10 dB \), and will take the value of 17.52 dB for the same values of normalized linewidth and normalized frequency offset at \( IBO = 4 dB \).

Fig. 4 illustrates minimum \( TD_{\text{dB}} \) versus normalized linewidth for different values of normalized frequency offset, \( \Delta f T = 0.025 \), \( \Delta f T = 0.05 \) under slowly Rayleigh fading channel for \( BER = 0.02 \) for QPSK and 16QAM modulation formats. For QPSK, \( TD_{\text{dB}} \) are about 6.19 dB and 6.7 dB at \( \beta T = 0.0025 \), and are increased to about 7.7 dB and 8.4 dB at \( \beta T = 0.005 \). For 16QAM modulation format, the degradations are increased which are about 7.65 dB.
and 9.3 dB at $\beta T = 0.0025$ and will be about 12.69 dB and 15.6 dB at $\beta T = 0.005$.

Fig. 4. Min. TD versus normalized linewidth for different values of normalized frequency offset for QPSK and 16QAM at BER=0.02.

5 Conclusion
We have obtained simple closed-form expression for the effective SNR of OFDM under the combined effect of phase noise, CFO, Doppler spread, and amplifier nonlinearity. The formula has obtained without linearization of phase noise levels. This formula can be applied to both AWGN and slowly Rayleigh fading channels. BER curves have investigated versus SNR for QPSK and 16QAM under slowly Rayleigh fading channel. TD has also evaluated against normalized linewidths for different values of normalized frequency offset. Although this paper focuses on SEL, the analysis can be applied to other kinds of power amplifiers such as traveling wave tubes (TWTA) and solid-state power amplifier (SSPA).

APPENDIX

1 Computation of the desired power, $P_{dc}$:
From (7), $E\left(\left|I_0\right|^2\right)$ is given by

$$E\left(\left|I_0\right|^2\right) = \frac{1}{T} \int_0^T E\left(e^{j\phi(t)}\right) e^{-j2\pi n t} dt$$

(21)

where $E\left(e^{j\phi(t)}\right)$ is given by [15] as follows

$$E\left(e^{j\phi(t)}\right) = e^{-\frac{\sigma_0^2}{2}}$$

(22)

From (22) into (21), we get

$$E^2\left(\left|I_0\right|^2\right) = e^{-\sigma_0^2} \sin c^2 (\Delta T)$$

(23)

Thus, from (23), we get (9).

2 Computation of the interference powers; $P_{C_{EI}}$, $P_{NL_{I}}$ and $P_{IC_{I}}$

$$E\left(\sum_{r=0}^{\infty} \left|I_r\right|^2\right) = E\left(\sum_{r=0}^{\infty} \left|I_0\right|^2\right) = \sum_{r=0}^{\infty} E\left(\left|I_r\right|^2\right)$$

(24)

where $E\left(\left|I_0\right|^2\right)$ is given by [17], (10.27)

$$E\left(\left|I_r\right|^2\right) = \int \left[1 - \left|\alpha\right|^2\right] e^{-j2\pi x} R_{\phi}(Tx) R_\Delta(Tx) dx$$

(25)

where $R_\phi (T x)$ is the autocorrelation function of the phase noise, which is given by [15]

$$R_\phi (T x) = e^{-\frac{\sigma_0^2}{2}}$$

(26)

and $R_\Delta (T x)$ is the autocorrelation function of the frequency offset which is given by (3).

Using the formula [18], (4-4-16), (4-4-17)

$$\sum_{r=0}^{\infty} e^{-j2\pi x} = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

(27)

Thus, from (3), (25), (26), (27) into (24), and after some algebra, we obtain

$$E\left(\sum_{r=0}^{\infty} \left|I_r\right|^2\right) = \frac{\sigma_0^2}{4 + 4\pi^2 (\Delta T)^2}$$

(28)

$$= \frac{2\sigma_0^2}{4} - 4\pi^2 (\Delta T)^2 - 2e^{-\sigma_0^2/2}$$

(29)

From (25), (27), we get

$$E\left(\left|I_0\right|^2\right) = 1 - \sum_{r=0}^{\infty} E\left(\left|I_r\right|^2\right)$$

Thus, from (23), (24), (28), and (29), we get the interference powers (10), (11), and (12).

References:


