Double temperature-enhanced occupancy of metastable states in fluctuating bistable potentials

ROMI MANKIN
Tallinn University
Department of Natural Sciences
Narva Road 25, 10120 Tallinn
Estonia

ERKKI SOIKA
Tallinn University
Department of Natural Sciences
Narva Road 25, 10120 Tallinn
Estonia

AKO SAUGA
Tallinn University of Technology
Department of Economics
Kopli 101, 11712, Tallinn
Tallinn University
Department of Natural Sciences
Narva Road 25, 10120 Tallinn
Estonia

Abstract: Overdamped motion of Brownian particles in an asymmetric bistable potential driven by an additive nonequilibrium three-level noise and a thermal noise is considered. An exact formula for the mean occupancy of the metastable state is derived and the phenomenon of noise-enhanced stability is investigated. It is established that in a certain region of system parameters variations of temperature cause double resonance peaks of the mean occupancy of a metastable state at low and moderate temperatures. The possibility that the occupancy of a metastable state can be controlled by thermal noise at moderate temperatures is important, because temperature can easily be varied in experiments, e.g. in experimental cell biology.

Key-words: Noise-enhanced stability, stochastic dynamics, trichotomous noise, metastable state, thermal noise, bistable potential.

1 Introduction

The recent years have witnessed an increasing interest in the dependence of the mean exit time of metastable and unstable systems on noise intensity [1], [2]. Noise can modify stability of a system in a counterintuitive way such that the system remains in a metastable state for a longer time than in the deterministic case [1]. Related investigations involving noise-induced stability [3] or noise-enhanced stability [4], [5] belong to a highly topical interdisciplinary realm of studies, ranging from condensed matter physics to molecular biology, or to cancer growth dynamics [1], [6], [7].

Motivated by the investigations of the effect of a periodic electric field on cell membrane proteins [8], [9] the author of [3] has considered the overdamped motion of a Brownian particle in an asymmetric bistable potential fluctuating according to dichotomous noise. This biologically motivated model clearly demonstrates the effect of noise-induced stability as for intermediate fluctuation rates the mean occupancy of minima with an energy above the absolute minimum is enhanced.

In the present paper we consider a model similar to the one presented in [3], except that the dichotomous noise is replaced with a trichotomous noise. Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, including all cases of dichotomous noises [10]. Furthermore, it is remarkable that for trichotomous noises the flatness parameter φ, contrary to the cases of the Gaussian colored noise, $\varphi = 3$, and symmetric dichotomous noise, $\varphi = 1$, can be anything from 1 to ∞. This extra degree of freedom can prove useful in modeling actual fluctuations.

The main contribution of this paper is as follows. We provide an exact formula for the analytic treatment of the dependence of the occupancy probability of a metastable state on various system parameters (viz. temperature, potential asymmetry, correlation time, flatness, and noise amplitude). We establish a new thermal fluctuations-induced phenomenon, namely, for certain values of the system parameters there exist three ranges of temperature values where the occupancy of the metastable state is enhanced. It is remarkable that one of the temperature regimes where the enhancement of stability occurs, is relevant for cell biology. Thus, in the case of living cells, the result may reveal a possibility to control the stability of metastable states by varying the temperature.

The structure of the paper is as follows. Section 2 presents the basic model investigated in this work. A master equation description of the model is given and the formula for the occupancy probability of the metastable state is found. Section 3 analyzes the behavior of the occupancy probability. The phenomenon of double enhanced stability of the metastable state versus temperature is established. Section 4 contains some brief concluding remarks.

2 Model and the exact solution

As an archetypical model for systems with a metastable state and strongly coupled with noisy environment, we consider one-dimensional overdamped Brownian motion in a fluctuating sawtooth-like asymmetric bistable potential

$$U(X,Z) \ = U(X) + X \cdot Z(t), \tag{1}$$

where $X(t)$ is the displacement of a Brownian particle at the time t and the variable $Z(t)$ is a Markovian trichotomous noise [10], which consists of jumps between three values: $z_1 = a$, $z_2 = 0$, $z_3 = -a$, $a > 0$. The jumps follow in time the pattern of a Poisson process, the values occurring with the stationary probabilities $p_s(a) = p_s(-a) = q$ and $p_s(0) = 1 - 2q$, where $0 < q < 1/2$. In a stationary state the fluctuation process satisfies $\langle Z(t) \rangle = 0$ and $\langle Z(t+\tau)Z(t) \rangle = 2qa^2 \exp(-\nu\tau)$, where the switching rate ν is the reciprocal of the noise correlation time $\tau_c = 1/\nu$, i.e., $Z(t)$ is a symmetric zero-mean exponentially correlated noise. The trichotomous process is a particular case of the kangaroo process [11] with the flatness parameter $\varphi = \langle Z^4(t) \rangle / \langle Z^2(t) \rangle^2 = 1/2q$.

We describe the overdamped motion of Brownian particles in dimensionless units, using the Langevin equation

$$\frac{dX}{dt} = h(X) - Z(t) + \xi(t), \quad h(x) = -\frac{dU(x)}{dx}, \tag{2}$$

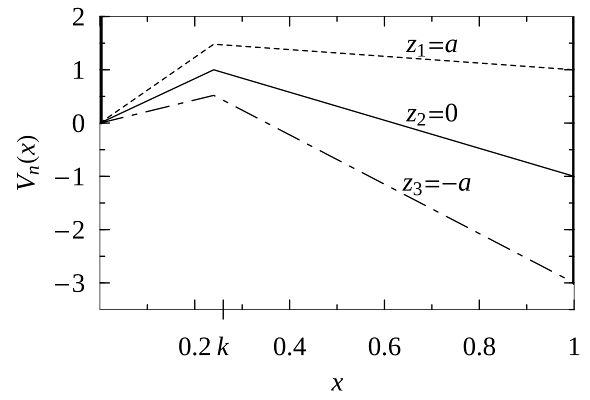

Figure 1: Representation of different states of the net potentials $V_n(x) = U(x) + z_n x$ with $z_1 = a$, $z_2 = 0$, $z_3 = -a$. The potential $U(x)$ is given by Eq. (3) at the parameter values $k = 0.24$, $a = 2$, and $\varepsilon = 1$. All quantities are dimensionless.

where the thermal noise $\xi(t)$ satisfies $\langle \xi(t) \rangle = 0$ and $\langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2)$. D is the thermal noise strength, which for the sake of brevity will be called temperature. The piecewise linear asymmetric bistable potential considered has the profile

$$U(x) \ = \begin{cases} \dfrac{1}{k}x, \quad x \in (0,k); \\ 1 + \dfrac{1+\varepsilon}{1-k}(k-x), \quad x \in (k,1); \\ U(0) = U(1) = \infty. \end{cases} \tag{3}$$

A schematic representation on the three configurations assumed by the "net potentials" $V_n(x) = U(x) + z_n x$, $n = 1,2,3$, associated with the right-hand side of Eq. (2), is shown in Fig. 1. In this work, we restrict ourselves to the system parameters region where the net potentials $V_n(x)$ for all states $n = 1,2,3$ of the non-equilibrium noise Z have two minima. More precisely, we assume that

$$a \ < \frac{1+\varepsilon}{1-k}, \quad a < \frac{1}{k}, \quad 0 < k < \frac{1}{2}, \\ 0 \ < \varepsilon < a(1-2k). \tag{4}$$

The master equation corresponding to Eq. (2) reads

$$\frac{\partial}{\partial t}P_n(x,t) = \\ -\frac{\partial}{\partial x}\left\{ [h(x) - z_n]\, P_n(x,t) - D\frac{\partial}{\partial x}P_n(x,t) \right\} \\ + \sum_{m=1}^{3} U_{nm} P_m(x,t), \tag{5}$$

where $P_n(x,t)$ is the joint probability density for the position variable $x(t)$ and the fluctuation variable

$z(t)$; and $U_{nm} = \nu[q + (1-3q)\delta_{n2} - \delta_{nm}]$. The stationary probability density in the x space $P^s(x)$ is then evaluated via the stationary probability densities $P^s_n(x)$ for the states (x, z_n):

$$P^s(x) = \sum_{n=1}^{3} P^s_n(x). \tag{6}$$

As the "force" $h(x) = -dU(x)/dx$ is piecewisely constant, $h(x) = h_1 = -1/k$ for $x \in (0,k)$ and $h(x) = h_2 = (1+\varepsilon)/(1-k)$ for $x \in (k,1)$, Eq. (5) splits up into two linear differential equations with constant coefficients for the two vector functions $\boldsymbol{P}^s_i(x) = (P^s_{1i}, P^s_{2i}, P^s_{3i})$, $i = 1,2$, defined on the intervals $(0,k)$ and $(k,1)$, respectively. The solution reads

$$P^s_{ni}(x) = p(z_n) \sum_{j=1}^{5} Y_{ij} A_{nij} e^{-\lambda_{ij} x/D}, \tag{7}$$

where

$$p(z_n) = (1-2q)\delta_{n2} + q(\delta_{n1} + \delta_{n3}),$$

$$A_{nij} = \frac{\nu D}{\nu D - \lambda_{ij}(h_i - z_n + \lambda_{ij})},$$

Y_{ij} are constants of integration, and $\{\lambda_{ij}, j = 1, \ldots, 5\}$ is the set of roots of the algebraic equation

$$\begin{aligned} &\lambda_i^5 + 3\lambda_i^4 h_i + \lambda_i^3(3h_i^2 - a^2 - 2\nu D) \\ &+\lambda_i^2 h_i (h_i^2 - a^2 - 4\nu D) \\ &+\lambda_i \nu D\left[\nu D + 2(qa^2 - h_i^2)\right] + h_i \nu^2 D^2 = 0, \\ &i = 1,2. \end{aligned} \tag{8}$$

Nine independent conditions for the ten constants of integration Y_{ij} can be determined at the points of discontinuity, by requiring continuity for the quantities $P^s_{ni}(x)$ and for the stationary current densities $j_{ni}(x) := (h_i - z_n)P^s_{ni}(x) - D\frac{d}{dx}P^s_{ni}(x)$ at the point $x = k$ and the vanishing of the current densities $j_{ni}(x)$ at the boundary points $x = 0, 1$, i.e.,

$$\begin{aligned} P^s_{n1}(k) &= P^s_{n2}(k), \quad j_{n1}(k) = j_{n2}(k), \\ j_{n1}(0) &= j_{n2}(1) = 0, \quad n = 1,2,3. \end{aligned} \tag{9}$$

It follows from Eq. (5) that the system of linear algebraic equations (9) contains only nine linearly independent equations for Y_{ij}. By including the normalization condition

$$\sum_{n=1}^{3} \int_0^1 P^s_n(x)dx = 1 \tag{10}$$

a complete set of conditions is obtained for ten constants of integration Y_{ij}:

$$\begin{aligned} &\sum_{j=1}^{5} \Big(Y_{1j}A_{n1j}e^{-\lambda_{1j}k/D} \\ &\qquad - Y_{2j}A_{n2j}e^{-\lambda_{2j}k/D}\Big) = 0,\ n = 1,2,3; \\ &\sum_{j=1}^{5} \Big[Y_{1j}A_{n1j}\,(h_1 + \lambda_{1j})\,e^{-\lambda_{1j}k/D} \\ &\qquad - Y_{2j}A_{n2j}\,(h_2 + \lambda_{2j})\,e^{-\lambda_{2j}k/D}\Big] = 0, \\ &\qquad\qquad n = 1,3; \\ &\sum_{j=1}^{5} Y_{1j}A_{n1j}\,(h_1 - z_n + \lambda_{1j}) = 0,\ n = 1,3; \\ &\sum_{j=1}^{5} Y_{2j}A_{n2j}\,(h_2 - z_n + \lambda_{2j})\,e^{-\lambda_{2j}/D} = 0, \\ &\qquad\qquad n = 1,3; \\ &\sum_{j=1}^{5} \left[\frac{Y_{1j}}{\lambda_{1j}}\left(1 - e^{-\lambda_{1j}k/D}\right)\right. \\ &\qquad \left. + \frac{Y_{2j}}{\lambda_{2j}}\left(e^{-\lambda_{2j}k/D} - e^{-\lambda_{2j}/D}\right)\right] = \frac{1}{D}. \end{aligned} \tag{11}$$

The remaining task is to solve the Eqs. (11) for Y_{ij}, which is a simple linear algebraic problem.

Now, the occupancy probabilities W_1 and W_2 of the left and right potential wells, respectively, are given by

$$\begin{aligned} W_1 &= \sum_{n=1}^{3} \int_0^k P^s_{n1}(x)dx = \\ &= D \sum_{j=1}^{5} \frac{Y_{1j}}{\lambda_{1j}}\left(1 - e^{-\lambda_{1j}k/D}\right), \\ W_2 &= 1 - W_1. \end{aligned} \tag{12}$$

The behavior of W_1 at different system parameters regimes will be considered in Sec. 3. All numerical calculations are performed by using the software Mathematica 5.0.

3 Enhancement of the stability of the metastable state

Of central interest to us are the occupancy probability W_1 of the left potential well (see Eq. (12)) and its responses to the switching rate ν and to the temperature D. Figure 2 exhibits the ratio W_1/W_2 as a function of the switching rate ν at different values of temperature. It can been seen that the functional dependence of W_1/W_2 on the correlation time $\tau_c = 1/\nu$ is of a bell-shaped form. Notably, at low temperatures for intermediate values of ν the mean occupancy of the metastable state (the left potential well) is much larger

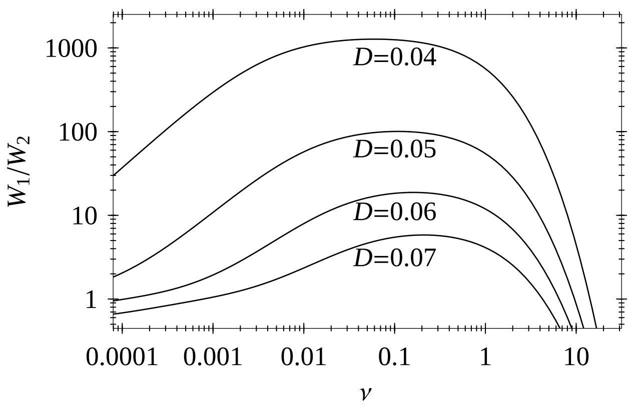

Figure 2: The ratio W_1/W_2 vs the noise switching rate ν at various temperatures D. The occupancy probabilities W_1 and W_2 of left and right potential wells, respectively, are computed by means of Eq. (12). The parameter values: $a = 2$, $\varepsilon = 1$, $k = 0.125$, and $q = 0.35$. The curves correspond to the values of the temperature $D = 0.04$, $D = 0.05$, $D = 0.06$, $D = 0.07$, from top to bottom.

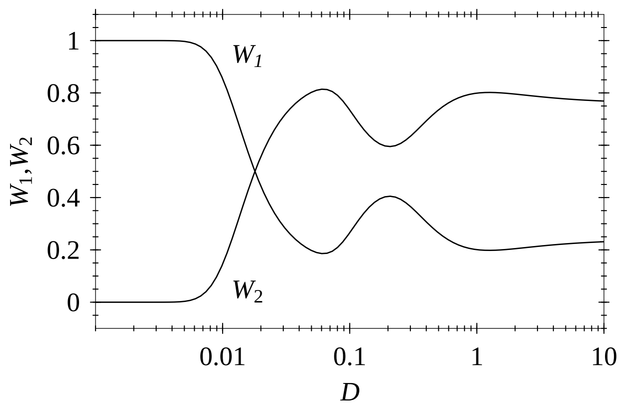

Figure 3: The occupancy probabilities W_1 and W_2 of the left and right potential wells, respectively, versus temperature D (Eq. (12)). The parameter values are $a = 2$, $\varepsilon = 1$, $k = 0.24$, $q = 0.499$, and $\nu = 0.1$. At large values of the temperature, $D > 10$, the probabilities W_1 and W_2 saturate to the values k and $1 - k$, respectively.

than the mean occupancy of the stable state, i.e., such fluctuations enhance the occupancy of the left minimum, although the most of the time it is not the absolute minimum of the potential. Thus, we observe a noise-induced stability for the metastable state.

In the case of a dichotomous noise, the phenomenon of noise-induced stability in models similar to Eq. (2) is examined already in Ref. [3], where analogous results of Fig. 2 are presented and a comprehensive physical interpretation of the effect is given. So our result exposed in Fig. 2 shows that the phenomenon of the noise correlation time induced stability is robust enough to survive a modification of the noise as well as the potential profile.

It is of interest to examine the behavior of the exact expression of W_1 (Eq. (12)) versus temperature. In Figure 3 we have plotted the occupancy probabilities W_1 and W_2 as functions of the dimensionless temperature D for an intermediate value of the correlation time $\tau_c = 10$. For increasing values of D the probability W_1 starts from the value $W_1 \approx 1$ and decreases to the minimum. Next it grows to the local maximum and decreases to the other minimum. Finally, at high temperatures, it grows to the value k.

The interesting peculiarity of Fig. 3 is that there are three temperature regimes where thermal fluctuations cause an enhancement of the occupancy of the metastable state: (i) At high temperatures the effect is trivial. In this case the Brownian particles "do not see" the structure of the potential profile and move as in a simple rectangular potential well. (ii) For low values of the temperature the effect of enhancement is very pronounced, i.e., nearly all particles are concentrated in the left potential well that has higher energy most of the time. This result is in accordance with the phenomenon of noise correlation time induced stability (see Fig. 2 and [3]). (iii) In the case of moderate values of the temperature a new resonance-like behavior is observed — enhancement of stability also occurs in a finite interval of the temperature, where the lowest depth of the potential wells is comparable with the thermal energy of the particle.

Though we are not aware of any simple physical explanation for the last-mentioned new effect, let us note that the value of the temperature that maximizes $W_1(D)$ can be estimated to be

$$\tau_c \approx \exp\left[\left(V_1(k) - V_1(1)\right)/D\right],$$

i.e. the noise correlation time τ_c is comparable with the Kramers escape time from the right potential well (in the noise state $z_1 = a$). This is a remarkable connection that throws some light on the physics of the effect, namely, it relates two characteristic time scales of the dynamical system (2) and demonstrates that all the three agents (colored noise, thermal noise, and potential configuration) act in unison to generate enhancement of the occupancy of the metastable state at moderate temperatures.

4 Conclusions

In the present work, we analyze the behavior of one-dimensional overdamped Brownian motion in a sawtooth-like asymmetric bistable potential driven by a trichotomous noise and an additive thermal noise. Using the corresponding master equation we have obtained an exact expression of the occupancy probability of the metastable state, and demonstrated the phenomenon of noise induced stability. Our major novel result is the effect of double enhanced stability of a metastable state versus temperature. Notably, enhancement of the stability also occurs at moderate temperatures, i.e., the temperature D is such that the lowest barrier height of the system is just a few D, which is relevant for cell biology [12]. In the case of a dichotomous noise, which is a special case of the trichotomous noise, a qualitatively similar model has been studied in [3]. However, to our knowledge, neither the phenomenon of double temperature-enhanced stability nor the existence of the corresponding resonance-like peak versus temperature at moderate values of D have been recognized or discussed before. The major advantage of this effect is that the control parameter is temperature, which can easily be varied in experiments.

We believe that the results obtained are of interest also in experimental cell biology where the proposed model can be applied [3], [8], [9].

Acknowledgments: The research was partly supported by the Estonian Science Foundation Grant No. 5943 and the International Atomic Energy Agency Grant No. 12026.

References:

[1] R. A. Ibrahim, Excitation-induced stability and phase transition: A review, *Journal of Vibration and Control*, Vol. 12, No. 10, 2006, pp. 1093–1170.

[2] B. Spagnolo, N. Agudov, and A. Dubkov, Noise enhanced stability, *Acta Phys. Pol., B*, Vol. 35, 2004, pp. 1419–1436.

[3] A. Mielke, Noise induced stability in fluctuating, bistable potentials, *Phys. Rev. Lett.*, Vol. 84, No. 5, 2000, pp. 818–821.

[4] R. N. Mantegna and B. Spagnolo, Noise enhanced stability in an unstable system, *Phys. Rev. Lett.*, Vol. 76, 1996, pp. 563–566.

[5] M. Gitterman, Stabilization of metastable states, *Phys. Rev. E*, Vol. 65, 2002, p. 031103.

[6] P. D'Odorico, F. Laio, and L. Ridolfi, Noise-induced stability in dryland plant ecosystems, *Proc. Natl. Acad. Sci. USA*, Vol. 102, No. 31, 2005, pp. 10819–10822.

[7] B. Spagnolo, A. Dubkov, A. Pankratov, E. Pankratova, A. Fiasconaro, and A. Ochab-Marcinek, Lifetime of metastable states and suppression of noise in interdisciplinary physical models, *Acta Phys. Pol., B*, Vol. 38, 2007, pp. 1925–1950.

[8] R. D. Astumian and B. Robertson, Nonlinear effect of an oscillating electric field on membrane proteins, *J. Chem. Phys.*, Vol. 91, No. 8, 1989, pp. 4891–4901.

[9] D. Petracchi, M. Pellegrini, M. Pellegrino, M. Barbi, and F. Moss, Periodic forcing of a K+ channel at various temperatures, *Biophys. J.*, Vol. 66, 1994, pp. 1844–1852.

[10] R. Mankin, A. Ainsaar, and E. Reiter, Trichotomous noise-induced transitions, *Phys. Rev. E*, Vol. 60, 1999, pp. 1374–1380.

[11] C. R. Doering, W. Horsthemke, and J. Riordan, Nonequilibrium fluctuation-induced transport, *Phys. Rev. Lett.*, Vol. 72, 1994, pp. 2984–2987.

[12] R. Astumian and M. Bier, Mechanochemical coupling of the motion of molecular motors to ATP hydrolysis, *Biophys. J.*, Vol. 70, 1996, pp. 637–653.