Detecting Sliding Areas in Three-Dimensional Filippov Systems using an Integration-Free Method

IVÁN ARANGO
Universidad EAFIT
Department of Mechanics Engineering
Kra 42d 50 sur 40 Medellín
Colombia

JOHN ALEXANDER TABORDA
Universidad Nacional de Colombia
Department Electronics Engineering
Campus La Nubia Manizales
Colombia

Abstract: In this paper, we detect sliding areas in three-dimensional (3D) Filippov systems using an integration-free method denominated Singular Point Tracking (SPT). Many physical applications in engineering can be modelled as Filippov systems. Sliding dynamics due to nonsmooth phenomena as friction, hysteresis or switching are inherent to Filippov systems. The analysis of sliding dynamics has many mathematical and numerical difficulties. Several well-known numerical problems can be avoid using integration-free methods. In this paper, we extend the SPT method to 3D Filippov systems. In comparison with the 2D case, the evaluation of the vector fields on the discontinuity boundary (DB) should be reformulated and new dynamics on DB should be characterized.

Key–Words: Bifurcation theory, numerical analysis, non-smooth bifurcations, Filippov systems.

1 Introduction

Nonsmooth characteristics as sliding, switching or impact cause many mathematical and numerical difficulties in modeling, simulation and analysis stages [1], [2], [3]. The bifurcation theory and the piecewise-smooth approach have been used widely to analyze the dynamics of nonsmooth systems as power converters [4], [5], [6], [7], friction oscillator [8], [9], [3], [10], [11] or impact oscillators [1], [12], [13], [14].

In this paper we concentrate in Filippov systems of three dimensions (3D). A lot of papers have been restricted to 2D Filippov systems or Filippov systems not involving sliding motion because of the analysis is more simplified [9]. When the sliding motion exist and the Filippov system is 3D then the analysis is more complicated.

The analysis of sliding dynamics has many mathematical and numerical difficulties. The number of specialized software in nonsmooth dynamics is reduced [15], [16]. In [17] and [18], two toolboxes are presented for analysis and continuation of nonsmooth bifurcations in Filippov systems. Several well-known numerical problems can be avoid using integration-free methods. In this paper, we analyze sliding dynamics in three-dimensional (3D) Filippov systems using an integration-free method denominated Singular Point Tracking (SPT).

We use the evaluation of the vector fields on the discontinuity boundary (DB) to analyze the dynamics of the Filippov systems without integration of the ODE sets or integrating only in the points computed for the SPT algorithm. We apply a classification of points and events on DB recently proposed in [19], [20], [21].

In comparison with the 2D case, the evaluation of the vector fields on the discontinuity boundary (DB) should be reformulated and new dynamics on DB should be characterized. The existence conditions of the crossing areas, sliding areas and singular sliding lines are formulate using Boolean-valued functions \( B(\cdot) \) based on integration-free geometric criterions. These conditions are easily programmable and they can be used directly in the detection of nonsmooth bifurcations.

The Boolean-valued functions \( B(\cdot) \) return TRUE or FALSE when their arguments are evaluated. The logical functions are composed of logical connectives: AND, OR and NOT denoted by \( \land \), \( \lor \) and \( \neg \), respectively.

The paper is organized as follows. In section II we present the background concepts of Filippov systems and the SPT numerical method. The type of areas on DB and singular lines on DB are summarized in the sections III and IV, respectively. The basic dynamics on DB are presented in the section V while an illustrative example is presented in section VI. Finally, the conclusions are presented in the section VII.
2 Background

Filippov systems are a subclass of discontinuous dynamical systems which can be described by a set of first-order ordinary differential equations with a discontinuous right-hand side [22]. These systems are modelled as piecewise-smooth systems (PWS) where the state space contains two kinds of entities: Smooth Zones ($Z_i$) and Discontinuity Boundaries ($\Sigma$) just they are presented in the figure 1.

The flow of the PWS system can be expressed as:

\[
\dot{x} = \begin{cases} 
F_i(x, \alpha) & \text{if } x \in Z_i \\
F_j(x, \alpha) & \text{if } x \in Z_j 
\end{cases}
\] (1)

where $F_i$ and $F_j$ are sufficiently smooth vector functions and $\alpha \in \mathbb{R}^p$ is the bifurcation parameter vector. The zones $Z_i$ and $Z_j$ depend of the scalar function $H(x, \alpha)$ and they are defined in the equation (2).

\[
\begin{align*}
Z_i := & \{ x \in \mathbb{R}^3 : H(x, \alpha) > 0 \} \\
Z_j := & \{ x \in \mathbb{R}^3 : H(x, \alpha) < 0 \}
\end{align*}
\] (2)

Between $Z_i$ and $Z_j$ the PWS system has the discontinuity boundary (DB) that it is assumed to be a smooth hyperplane. The DB is denoted as $\Sigma$ and it is defined in the equation (3).

\[
\Sigma := \{ x \in \mathbb{R}^3 : H(x, \alpha) = 0 \}
\] (3)

The system (1) is not invertible because of the orbits can overlap on DB with sliding [9]. In sliding situations, a convex combination $G(x, \alpha)$ of the vectors $F_i$ and $F_j$ is defined as the Filippov Method [23].

The vector $G$ can be written as the equation (4) where $\lambda$ is a parameter defined in function of the vector fields projections in the tangent vector $H_t$ defined as $F^i_t = \langle H_t, F_i \rangle$ and $F^j_t = \langle H_t, F_j \rangle$ where $\langle \cdot, \cdot \rangle$ denotes scalar product.

\[
G(x, \alpha) = \lambda F_i(x, \alpha) + (1 - \lambda) F_j(x, \alpha)
\] (4)

with

\[
\lambda = \frac{\langle H_t(x), F_i(x) \rangle}{\langle H_t(x), F_j(x) - F_i(x) \rangle}
\]
fields and \(0 \leq \varphi \leq 2\pi\) is the azimuth angle between the positive x-axis and the vector field projected onto the \(xy\)-plane.

The azimuth angle of the sliding vector \(\mathbf{G}(x)\) denoted by \(\varphi_G\) is used to define the direction of sliding motion in the analysis point \(x_0 \in \Sigma\).

Recently, a methodology to study nonsmooth bifurcations on DB was proposed [19], [20], [21]. In each analysis point \(x_0\) on the DB \((x_0 \in \Sigma)\), we compute the normal vector \((\mathbf{H}_i, j)\) and tangent vector \((\mathbf{H}_j, i)\) to the DB in \(x_0\). Also, we evaluate the vector fields \(\mathbf{F}_i\) and \(\mathbf{F}_j\) that define the orbit flow in the zones \(Z_i\) and \(Z_j\), respectively (see the figure 2). With reference to \(\mathbf{H}_i\), we compute the angles of vector fields \(\mathbf{F}_i\) and \(\mathbf{F}_j\).

Two main ranges of angles are defined \(\Theta_J\) and \(\Theta_I\) in the equation (5) where \(\Delta_\theta\) is the tolerance angle. These ranges of angles are used to study the type of points on DB. In the figure 2 we present the ranges \(\Theta_J\) and \(\Theta_I\).

\[
\begin{align*}
\Theta_J &= \{\theta \in (\pi/2 + \Delta_\theta, \pi - \Delta_\theta)\} \\
\Theta_I &= \{\theta \in (\pi/2 - \Delta_\theta, \Delta_\theta)\}
\end{align*}
\]

The general Boolean-valued conditions \(B(.)\) for the three types of areas on DB and the singular lines on DB are presented in the equation (6) where \(F_i^n = \langle \mathbf{H}_i, \mathbf{F}_i \rangle\) and \(F_j^n = \langle \mathbf{H}_j, \mathbf{F}_j \rangle\) are the vector field projections in the normal vector \(\mathbf{H}_i\) and \(Q\) is the condition for pseudo-equilibrium lines given by the equation (7).

\[
\begin{align*}
C &= B\left(F_i^n F_j^n > 0\right) \\
S &= B\left(F_i^n F_j^n < 0\right) \land (\neg Q) \\
\Omega &= B\left(F_i^n F_j^n = 0\right) \lor Q
\end{align*}
\]

\[
Q = \left\{ \begin{array}{ll}
B((\pi - \Delta_\theta) < (\theta_i + \theta_j) < (\pi + \Delta_\theta)) \land \\
B((\pi - \Delta_\varphi) < (\varphi_i - \varphi_j) < (\pi + \Delta_\varphi))
\end{array} \right.
\]

Crossing and sliding flows are the predominant behaviors of the 3D Filippov systems on the discontinuity boundary (DB). Depending of the direction of the crossing orbits, two crossing (C) areas are defined and two sliding (S) areas are determined depending of the stability condition. Fourteen singular sliding lines (\(\Omega\)) exist in the transition of C and S dynamics on DB.

3.1 Crossing Areas (C)

In the equation (8) we present the Boolean-valued conditions \(B(.)\) for the crossing areas \(C_{ij}\) and \(C_{ji}\). Both zenith angles of the vector fields (\(\theta_i\) and \(\theta_j\)) should be contained in the same range \(\Theta_I\) or \(\Theta_J\). The generic representation of the crossing areas are shown in the figure 3.

\[
\begin{align*}
C_{ij} &= C \land B(\theta_i \in \Theta_J) \land B(\theta_j \in \Theta_J) \\
C_{ji} &= C \land B(\theta_i \in \Theta_I) \land B(\theta_j \in \Theta_I)
\end{align*}
\]

3.2 Stable Sliding Areas (Ss)

In the two-dimensional case, the sliding motion was characterized according to stability and direction properties [19], [20], [21]. In the first attempt of 3D characterization, we will consider only the stability conditions.

A sliding (S) point is stable if the Boolean-valued function \(S_s\) presented in the equation (9) is True.

\[
S_s = S \land B(\theta_i \in \Theta_J) \land B(\theta_j \in \Theta_I)
\]

In the figure 4(a) we present the area \(S_s\) with their respective vector fields. Note that the zenith angles should be contained in the reciprocal angle range (\(\Theta_I\) or \(\Theta_J\)).
3.3 Unstable Sliding Areas ($S_u$)

In the same form, a sliding (S) area is unstable if the Boolean-valued function $S_u$ presented in the equation (10) is True. Note that each B(.) function is excluding for each analysis point $x_b$, i.e. if $C_{ij}(x_b)$ is True then $C_{ij}$, $S$ or $\Omega$ are False in this point. Also, if $S_u$ is True in a point $P_{xy} \in \Sigma$ then $S_u$ is False.

$$S_u = S \land B (\theta_i \in \Theta_T) \land B (\theta_j \in \Theta_T)$$ (10)

In the figure 4 (b) we present the area $S_u$ with their respective vector fields. The zenith angles ($\theta_i$ and $\theta_j$) should be contained in the respective angle range ($\Theta_T$ or $\Theta_T$).

4 Type of Singular Sliding Lines

To analyze the singular sliding points ($\Omega$) we define four subclasses: $T$, $V$, $Q$ and $\Psi$. Next, we explain the general considerations of each subclass. More details can be found in [19].

4.1 Tangent Lines ($T$)

The vector fields $F_i$ and/or $F_j$ are tangent on the analysis point $x_b$. Five lines can be defined: $T^{ij}_i$, $T^{ij}_j$, $T^{ji}_i$, $T^{ji}_j$. The Boolean-valued condition for Tangent (T) singular lines is given in the equation (11) where $\Theta_T = \{ \theta \in (\pi/2 + \Delta_\theta, \pi/2 - \Delta_\theta) \}$ (See figure 5).

$$T = \begin{cases} \Omega \land (T^j \lor T^i) \\ T^i = B (\theta_i \in \Theta_T) \lor B (\theta_j \in \Theta_T) \\ T^j = B (\theta_i \in \Theta_T) \land B (\theta_j \in \Theta_T) \end{cases}$$ (11)

4.2 Vanished Lines ($V$)

The vector fields $F_i$ and/or $F_j$ are vanished on the analysis point $x_b$. Five lines can be defined: $V^{ij}_i$, $V^{ij}_j$, $V^{ji}_i$, $V^{ji}_j$. The Boolean-valued condition for Vanished (V) singular lines is given in the equation (12) where $\theta \not\in \Theta$ implies that the magnitude of the vector field is zero ($r = 0$) (See figure 5).

$$V = \begin{cases} \Omega \land (V^i \lor V^j) \\ V^i = B (\theta_i \not\in \Theta) \lor B (\theta_j \not\in \Theta) \\ V^j = B (\theta_i \not\in \Theta) \land B (\theta_j \not\in \Theta) \end{cases}$$ (12)

4.3 Pseudo-equilibrium Lines ($Q$)

The vector fields $F_i$ and $F_j$ are anti-collinear on the analysis point $x_b$. Two lines can be defined: $Q_s$ and $Q_u$ (See figure 5). The Boolean condition was presented in the equation (7).

4.4 Tangent-Vanished Lines ($\Psi$)

A vector field $F_i$ or $F_j$ is tangent and the other vector field is vanished on the analysis point $x_b$. Two lines can be defined: $\Psi_i$ and $\Psi_j$ (See figure 5). The Boolean-valued condition for Tangent-Vanished ($\Psi$) singular lines is given in the equation (13).

$$\Psi = \begin{cases} \Omega \land (\Psi^i \lor \Psi^j) \\ \Psi^i = B (\theta_i \not\in \Theta) \lor B (\theta_j \not\in \Theta) \\ \Psi^j = B (\theta_i \not\in \Theta) \land B (\theta_j \not\in \Theta) \end{cases}$$ (13)
5 Basic Scenarios on DB of 3D Filippov Systems

The existence of several types of areas on the discontinuity boundary characterizes different scenarios on DB. Eight basic scenarios are considered. In all scenarios, singular sliding lines separate the crossing and sliding areas. In the figure 6 we present the characteristics of each scenario.

- $C_{ij} \leftrightarrow C_{ji}$: Change of direction of crossing points. The singular sliding line should be: $T^{ij}$, $V^{ij}$, $Q_s$, $Q_u$, $\Psi^T_x$ or $\Psi^V_x$.

- $S_s \leftrightarrow S_u$: Change of stability of sliding points. The singular sliding line should be: $T^{ij}$, $V^{ij}$, $\Psi^T_x$ or $\Psi^V_x$.

- $C_{ij} \leftrightarrow S_s$: Change of crossing boundary $C_{ij}$ to stable sliding boundary and vice versa. The singular sliding line should be: $T^j_s$ or $V^j_s$ lines.

- $C_{ji} \leftrightarrow S_s$: Change of crossing boundary $C_{ji}$ to stable sliding boundary and vice versa. The singular sliding line should be: $T^i_s$ or $V^i_s$ lines.

- $C_{ij} \leftrightarrow S_u$: Change of crossing boundary $C_{ij}$ to unstable sliding boundary and vice versa. The singular sliding line should be: $T^j_u$ or $V^j_u$ lines.

- $C_{ji} \leftrightarrow S_u$: Change of crossing boundary $C_{ji}$ to unstable sliding boundary and vice versa. The singular sliding line should be: $T^i_u$ or $V^i_u$ lines.

- $S_s \leftrightarrow S_u$: Change of direction in stable sliding boundary. The singular line should be: $Q_s$ line.

- $S_u \leftrightarrow S_u$: Change of direction in unstable sliding boundary. The singular line should be: $Q_u$ line.

6 Illustrative example

Let the three-dimensional Filippov system presented in the equation (1) with the configuration $(F_i, F_j, H)$ given in (14).

$$
F_i = \begin{bmatrix}
0 & 0 & 1 \\
0 & -0.2 & 0 \\
-1 & -0.1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\alpha
\end{bmatrix}
$$

$$
F_j = \begin{bmatrix}
0 & 0 & 1 \\
0 & -0.2 & 0 \\
-1 & -0.1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
-\alpha
\end{bmatrix}
$$

$H = z$

The configuration (14) can be found in mechanical systems as friction oscillators where $\alpha$ is the bifurcation parameter. In the figure 7 the sliding areas are presented in the plane $xy$ bounded by $x = \pm 20$ and $y = \pm 20$.

For $\alpha < 0$ (figure 7(a)), the DB has crossing areas and unstable sliding areas. For $\alpha = 0$ (figure 7(b)), the system has a DB where the sliding area $S_s$ appears. For $\alpha > 0$ (figure 7(c)), the system has a DB where the unstable sliding area disappears.

7 Conclusions

We have detected sliding areas in three-dimensional (3D) Filippov systems using an integration-free...
method denominated *Singular Point Tracking* (SPT). The analysis of sliding dynamics has many mathematical and numerical difficulties. Several well-known numerical problems can be avoid using integration-free methods. In this paper, we have extended the SPT method to 3D Filippov systems. In comparison with the 2D case, the evaluation of the vector fields on the discontinuity boundary (DB) should be reformulated and new dynamics on DB should be characterized.

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