Analytical PDE Solid Modelling

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Abstract: - In this paper we study analytical solutions of a fourth order partial differential equation subjected to different boundary conditions of solid modelling, and investigate the effects of various factors in the PDE and boundary conditions on the shape of the generated solids. With the proposed modelling technique, we can easily manipulate and modify the shape of the generated solid by altering a set of vector-valued shape parameters and a force function, included in the formulation of the PDE, and / or by altering the positional and tangential constraints, included in the formulation of the boundary conditions of the PDE. We present some examples to demonstrate the capacity of these factors which act as an effective shape manipulation and deformation tool.

Key-Words: - Solid modelling, analytical solutions, fourth order partial differential equation, shape manipulation

1 Introduction

Solid modelling is an important aspect of geometric modelling. Partial differential equations based solid modelling has a potential in introducing more shape manipulation techniques.

Surface-modelling techniques based on the solution of partial differential equations were introduced two decades ago and have recently become more actively investigated.

Bloor and Wilson were the first to use PDEs to perform surface blending [1] and free-form surface modelling [2]. Most recently, Monterde and Ugail presented a new technique to create surfaces from prescribed boundaries using elliptic partial differential equation operators [3]. Some numerical methods such as the finite element method [4, 5], the finite difference method [6], and the collocation method [7] were also developed. However, these numerical methods require expensive computation, which hinders their wider application to interactive computer graphics and CAD. In order to overcome this limitation, Bloor and Wilson proposed the spectral method [8].

Since the vector-valued parameters of the PDE have a strong influence on the generated surface shape, You and Zhang have proposed a more general PDE [9] for surface modelling. Later, they investigated surface blending using the solution of a fourth order PDE [10] and a sixth order PDE [11]. They also investigated vase design, using a fourth order PDE [12], and surface modelling, using a sixth order PDE [13].

Due to the complexity of solid modelling, only a limited number of publications on physically-based volume modelling have appeared to date. As early as 1984, Barr developed a set of hierarchical solid modelling operations [14]. In 1992, Requicha and Rossignac conducted a survey of the field of solid modelling [15]. The same year, Metaxas and Terzopoulos developed a systematic approach to deriving dynamic models [16]. The following year, Bloor and Wilson presented examples of some specific solid volumes generated by solving a second order PDE [17]. In 2000, Ferley et al. presented a sculpture metaphor for rapid shape prototyping [18]. The following year, Breen and Whitaker presented an approach to 3D shape metamorphosis [19]. Du and Qin developed a technique for modelling dynamic solids [20]. Later on, McDonnell and Qin presented a sculptured solid modelling system [21]. Hua and Qin presented a novel interactive solid modelling framework [22]. They also proposed a versatile efficient and intuitive scalar-field-guided adaptive-shape-deformation (SFD) technique [23]. Guo et al. presented a scalar-field-driven editing paradigm [24]. McDonnell and Qin proposed a new volumetric subdivision scheme [25]. Duan and Qin presented a surface reconstruction algorithm [26]. Du and Qin coupled PDEs with volumetric implicit functions [27].

In this paper, we introduce a new fourth order PDE, discuss its analytical solutions, and investigate the effects of the factors in the PDE and boundary conditions on solid shapes.

2 Partial Differential Equations and Boundary Conditions

Allowing for the effect of the boundary tangent on the shape of the solid volume, a PDE solid model can be generated from the solution of a fourth order PDE,
shown in equation (1), which involves three parametric variables and is subject to the boundary conditions, given in equation (2).

\[ a \frac{d^4x}{d\omega^4} + b \frac{d^3x}{d\omega^3} + c \frac{d^2x}{d\omega^2} = p(u, v, w) \]  

(1)

\[ w = 0 \quad x = G_1(u, v) \]  

(2)

\[ w = 1 \quad x = G_2(u, v) \]

where \( a = [a_x \ a_y \ a_z]^T \), \( b = [b_x \ b_y \ b_z]^T \), and \( c = [c_x \ c_y \ c_z]^T \) are the vector-valued parameters of the PDE, \( x(u, v, w) = [x(u, v, w), y(u, v, w), z(u, v, w)]^T \) is a vector-valued positional function, \( G_1(u, v) \) and \( G_2(u, v) \) are the functions of the boundary surfaces of the generated solid volume, \( G_1(u, v) \) and \( G_2(u, v) \) are the functions of the boundary tangents of this volume, and \( p(u, v, w) \) is a force function.

For many solid modelling tasks, PDE (1) subjected to boundary conditions (2) can be solved analytically. In the sections below, we will explore these analytical solutions and investigate the effects of various factors of Eqs. (1) and (2) on solid modelling.

3 Effects of Geometric Parameters of Boundary Conditions

In this example, we generate a solid volume defined by two planar surfaces whose boundary curves are square or rectangular. This example was selected to show how the geometric parameters of the boundary conditions affect the shape of generated PDE solid.

Here, the boundary conditions of the solid are given by:

\[ \begin{align*}
  w = 0 & \quad x = a_0 + a_1u \quad \frac{dx}{dw} = a_0 + a_1u \\
  & \quad y = b_0 + b_1v \quad \frac{dy}{dw} = b_0 + b_1v \\
  & \quad z = h_0
\\
  w = 1 & \quad x = a_2 + a_3u \quad \frac{dx}{dw} = a_2 + a_3u \\
  & \quad y = b_2 + b_3v \quad \frac{dy}{dw} = b_2 + b_3v \\
  & \quad z = h_1
\end{align*} \]  

(3)

According to boundary conditions (3), we take the solution of PDE (1) to have the following forms

\[ \begin{align*}
  x &= c_1 + c_2w + c_3w^2 + c_4w^3 \\
  y &= c_5 + c_6w + c_7w^2 + c_8w^3 \\
  z &= c_9 + c_{10}w + c_{11}w^2 + c_{12}w^3
\end{align*} \]

(4)

Substituting equation (4) into PDE (1), we found that Eq. (1) is exactly satisfied. Introducing Eq. (4) into the boundary conditions (3), all the unknown constants were determined.

Initially, we set the geometric parameters to the following values:

\[ a_0 = a_2 = b_2 = -0.5, \quad a_1 = a_3 = b_3 = h_1 = 1, \quad h_0 = a_0, \quad a_1 = a_2 = b_2 = h_2 = h_0 = h_1 = 0 \]

then, we obtain the cuboid shown in Figs. 1a and 1b. Then, we alter the values of some of the geometric parameters to the following values:

\[ a_1 = b_3 = h_1 = 2 \]

The generated solid now becomes the irregular hexahedron shown in Figs. 1c and 1d. Setting the geometric parameters to the values:

\[ a_0 = -0.5, \quad a_1 = 1.5, \quad a_2 = b_2 = h_1 = 1, \quad a_3 = b_3 = 3, \quad b_2 = -0.4, \quad h_1 = 1, \quad h_0 = 0, \quad h_1 = 2, \quad a_1 = a_2 = a_3 = b_2 = h_0 = h_1 = 3, \quad b_2 = b_3 = h_0 = h_1 = 2 \]

we obtain the solid depicted in Figs. 1e and 1f. Finally, changing the boundary tangents to \( a_0 = 2.5a_0, \ a_1 = 2.5a_1, \ a_2 = 2.5a_2, \ a_3 = 2.5a_3, \ b_2 = b_3 = h_0 = h_1 = 2, \) resulted in the solid depicted in Figs. 1g and 1h.

From Fig. 1, it is clear that varying the geometric parameters including the tangents of the solid in the boundary conditions can greatly affect the shape of the generated PDE solid.
4 Effects of Different Functions of Boundary Conditions

In our second example, we generate a solid volume defined by two planar or 3D boundary surfaces. This example was selected to show how different functions in boundary conditions affect the shape of the generated PDE solids.

Here, the boundary conditions of the solid are given by:

\[
\begin{align*}
  w &= 0 \\
  x &= u(r_0 \sin 2\pi n + r_1 \sin n_{xy} \pi) \\
  \frac{\partial x}{\partial w} &= u(r_0 \sin 2\pi n + r_1' \sin n_{xy} \pi) \\
  y &= v(r_0 \cos 2\pi n + r_1 \cos n_{xy} \pi) \\
  \frac{\partial y}{\partial w} &= v(r_0 \cos 2\pi n + r_1' \cos n_{xy} \pi) \\
  z &= h_0 + h_1 \sin n_{xy} \pi \\
  \frac{\partial z}{\partial w} &= h_0' + h_1' \sin n_{xy} \pi
\end{align*}
\]

\[
\begin{align*}
  w &= 1 \\
  x &= r_2 u \sin 2\pi n \\
  \frac{\partial x}{\partial w} &= r_2 u \sin 2\pi n \\
  y &= r_2 v \cos 2\pi n \\
  \frac{\partial y}{\partial w} &= r_2 v \cos 2\pi n \\
  z &= h_2 + h_3 u \sin n_{xy} \pi \\
  \frac{\partial z}{\partial w} &= h_2' + h_3' u \sin n_{xy} \pi
\end{align*}
\]

(5)

where \( r_0, r_1, h_0 \) and \( h_1 \) are geometric parameters determining the size of the top boundary surface, \( r_0' \), \( r_1' \), \( h_0' \) and \( h_1' \) determine the direction and size of the tangent of the solid at the top boundary surface, \( r_2 \), \( h_2 \) and \( h_3 \) are the geometric parameters determining the size of the bottom boundary surface, and \( r_2', h_2' \) and \( h_3' \) determine the direction and size of the tangent of the solid at the bottom boundary surface.

When \( n_z \) in equation (5) is set to 0, two planar boundary surfaces are defined. Otherwise two three-dimensional boundary surfaces are described.

The functions in the above boundary conditions are \( u \sin 2\pi n \) and \( u \sin n_{xy} \pi \) for the \( x \) component, \( u \cos 2\pi n \) and \( u \cos n_{xy} \pi \) for the \( y \) component, and constants and \( u \sin n_{xy} \pi \) for the \( z \) component. According to these different functions, we found the solution of PDE (1) to be

\[
\begin{align*}
  x &= c_{x1} e^{\gamma_1 w} + c_{x2} e^{\gamma_2 w} + c_{x3} \cos t_1 w + c_{x4} \\
  \sin t_1 w + (c_{x2} e^{\gamma_2 w} + c_{x3} \cos t_1 w + c_{x4} \\
  + c_{x2} \cos t_1 w + c_{x3} \sin t_1 w) \sin n_{xy} \pi \\
  y &= (c_{y1} e^{\gamma_1 w} + c_{y2} e^{\gamma_2 w} + c_{y3} \cos t_1 w + c_{y4} \\
  \sin t_1 w + (c_{y2} e^{\gamma_2 w} + c_{y3} \cos t_1 w + c_{y4} \\
  + c_{y2} \cos t_1 w + c_{y3} \sin t_1 w) \sin n_{xy} \pi \\
  z &= c_{z1} + c_{z2} w^2 + c_{z3} w^3 + (c_{z2} e^{\gamma_2 w} + c_{z3} \\
  e^\gamma_1 w, c_{z3} \cos t_1 w + c_{z4} \sin t_1 w) \sin n_{xy} \pi
\end{align*}
\]

(6)

where

\[
\begin{align*}
  r_1 &= 2\pi \sqrt{\frac{b_1}{c_1}} \\
  r_2 &= n_{xy} \pi \sqrt{\frac{b_2}{c_2}} \\
  (i = x, y) \\
  r_3 &= n_{z} \pi \sqrt{\frac{b_3}{c_3}}
\end{align*}
\]

(7)

In this example, the vector-ordered parameters were set to \( a_x = a_y = a_z = 1 \), \( b_x = b_y = b_z = 2.5 \) and \( c_x = c_y = c_z = -10 \), and the geometric parameters were set to \( r_0 = 0.5 \), \( r_1 = h_0 = 0.1 \), \( r_2 = 0.5 \), \( h_0 = 2 \), \( h_2 = 0 \), \( h_3 = -0.1 \), and \( r_0' = r_1' = r_2' = h_0' = h_1' = h_3' = 0 \). The solid depicted in Fig. 2a was generated by setting \( n_{xy} = 20 \) and \( n_z = 0 \). Changing \( n_{xy} \) to 30, we generated the solid depicted in Fig. 2b. Setting \( r_2 = 0.8 \), \( n_{xy} = 8 \) and \( n_z = 10 \), resulted in the solid depicted in Fig. 2c. Finally, setting \( n_{xy} = 14 \) and \( n_z = 6 \), led to the solid depicted in Fig. 2d.

Examining these images we can conclude that selecting different functions of boundary conditions (i.e. \( u \sin 20\pi n \), \( u \sin 30\pi n \), \( u \sin 8\pi n \) and \( u \sin 14\pi n \) for the \( x \) component, \( u \cos 20\pi n \), \( u \cos 30\pi n \), \( u \cos 8\pi n \) and \( u \cos 14\pi n \) for the \( y \) component, and \( u \sin 10\pi n \) and \( u \sin 6\pi n \) for the \( z \) component) greatly influences the shape of the generated solid.
Fig. 2 PDE solids defined by two planar or three-dimensional surfaces.

5 Combined Effects of Different Factors

In our third example, we generate a solid volume defined by two boundary surfaces confined within two circles and subjected to a vector-valued force function. This example was selected to show how different combinations of the vector-valued parameters, force functions, boundary positions and tangents have the capacity to generate solids with widely diverse shapes.

Here, the boundary conditions of the solid are given by:

\[ w = 0 \]
\[ x = r_0 \sin \eta \cos 2\pi \]
\[ y = r_0 \cos \eta \cos 2\pi \]
\[ z = h_0 \]

\[ w = 1 \]
\[ x = r_1 \sin \eta \cos 2\pi \]
\[ y = r_1 \cos \eta \cos 2\pi \]
\[ z = h_1 \]

where \( r_0 \) and \( h_0 \) determine the size of the upper boundary surface, \( r_1 \) and \( h_1 \) determine the direction and size of the tangent of the solid at this boundary, \( \eta \) and \( h_0 \) determine the size of the bottom boundary surface, and \( r_1 \) and \( h_1 \) determine the direction and size of the tangent of the solid at this boundary.

The different functions in the above boundary conditions are given by \( u \sin 2\pi \) for the \( x \) component, \( u \cos 2\pi \) for the \( y \) component, and constants for the \( z \) component.

According to these different functions, we take the solution of PDE (1) to have the following forms and find that they meet the PDE exactly

\[ x = c_{11} e^{i\omega} + c_{12} e^{-i\omega} + c_{13} \cos t \omega + c_{14} \sin t \omega \]
\[ u \sin 2\pi \]
\[ y = c_{11} e^{i\omega} + c_{12} e^{-i\omega} + c_{13} \cos t \omega + c_{14} \sin t \omega \]
\[ u \cos 2\pi \]
\[ z = c_{11} + c_{12} \omega + c_{13} \omega^2 + c_{14} \omega^3 \]

where

\[ t_j = 2\pi \sqrt{\frac{b_j}{c_j}} \]

As we now wish to consider the effect of the force function on the shape of the generated solid, we represent this function in the following form:

\[ p_{x}(u,v,w) = p_{x0} \sin \xi \nu \cos 2\pi \nu \]
\[ p_{y}(u,v,w) = p_{y0} \sin \xi \nu \cos 2\pi \nu \]
\[ p_{z}(u,v,w) = 0 \]

Substituting equation (11) into PDE (1), the particular solution of the non-homogeneous fourth order PDE is given as:

\[ \bar{x} = \frac{p_0}{(16b_y + c_1 \eta^4)\pi^4} u \sin \xi \nu \cos 2\pi \nu \]
\[ \bar{y} = \frac{p_0}{(16b_y + c_1 \eta^4)\pi^4} u \sin \xi \nu \cos 2\pi \nu \]
\[ \bar{z} = 0 \]

Superimposition of equation (12) onto equation (9) results in the general solution of the non-homogeneous fourth order PDE (1), which has the form:

\[ x = c_{11} e^{i\omega} + c_{12} e^{-i\omega} + c_{13} \cos t \omega + c_{14} \sin t \omega \]
\[ u \sin 2\pi \]
\[ y = c_{11} e^{i\omega} + c_{12} e^{-i\omega} + c_{13} \cos t \omega + c_{14} \sin t \omega \]
\[ u \cos 2\pi \]
\[ z = c_{11} + c_{12} \omega + c_{13} \omega^2 + c_{14} \omega^3 \]

Substituting equation (13) into (8), we determine all the unknown constants in equation (13) and use it to generate the solid.

Different combinations of vector-valued parameters, force functions, positional and tangential parameters for Fig. 3a to Fig. 3f are listed below and various generated solids are shown in Fig. 3.
From the images shown in this figure, it is clear that by combining different force functions and other factors, a large variety of solid shapes can easily be generated.

6 Conclusion
In this paper, we have presented analytical solutions of a fourth order PDE subject to different boundary conditions of solid modelling and investigate the effects of the factors in the PDE and boundary conditions on solid modelling.

We have presented three examples that show how by altering the vector-valued parameters, the force function, the geometric parameters or the functions of the boundary conditions we can control and modify the shape of the generated PDE solid. The technique that we have presented for generating solid models is both accurate and highly efficient, as it is based on analytical solutions of the PDE subject to the boundary conditions representing the generated solids.

References:

Fig. 3b: $b_x = b_y = 2.5$, $c_x = c_y = -10$, $r_0 = 0.5$, $n_0 = 0.5$, $r_0' = 2.5$, $h_0 = 0$, $h_1 = 3$, $h_1' = 3$, $p_0 = 8 \times 10^4$ and $\xi = 5$.

Fig. 3c: $b_x = b_y = 4$, $c_x = c_y = -100$, $r_0 = 0.5$, $n_0 = 0.5$, $r_0' = 2.5$, $h_0 = 0$, $h_1 = 3$, $h_1' = 6$, $p_0 = 1.8 \times 10^5$ and $\xi = 7$.

Fig. 3d: $b_x = b_y = 4$, $c_x = c_y = -10$, $r_0 = 0.5$, $n_0 = 0.5$, $r_0' = 2$, $h_0 = 3$, $h_1 = 3$, $h_1' = 6$, $p_0 = 3 \times 10^5$ and $\xi = 7$.

Fig. 3e: $b_x = b_y = 4$, $c_x = c_y = -10$, $r_0 = 0.5$, $n_0 = 0.5$, $r_0' = 2$, $h_0 = 0$, $h_1 = 3$, $h_1' = 6$, $p_0 = -3 \times 10^5$ and $\xi = 7$.

Fig. 3f: $b_x = b_y = 2.5$, $c_x = c_y = -10$, $r_0 = 0.5$, $n_0 = 0.5$, $r_0' = 1.5$, $h_0 = 0$, $h_1 = 3$, $h_1' = 6$, $h_1'' = 4.5$, $p_0 = 10^3$ and $\xi = 7$.

From the images shown in this figure, it is clear that by combining different force functions and other factors, a large variety of solid shapes can easily be generated.

Fig. 3 Solids generated using different combinations of the factors controlling their shape.

Fig. 3a: $b_x = b_y = 1$, $c_x = c_y = -9.5$, $r_0 = 0.5$, $n_0 = 0.5$, $r_0' = 0$, $h_0 = 0$, $h_1 = 3$, $h_1' = 7.5$, $p_0 = 30000$ and $\xi = 3$.


