COMPARISON OF CONSTRAINT DESIGN OF FIR FILTER 
ON COMPLEX ERROR FUNCTION AND ON MAGNITUDE 
AND PHASE INDEPENDENTLY 
*RAGHVENDRA KUMAR, **LILLIE DEWAN 
*M.Tech Student **Asst. Professor 
Department of Electrical Engineering 
National Institute of Technology 
Kurukshetra 
INDIA 
e-mail: raghvendra_kumar80@rediffmail.com, l_dewanin@yahoo.com 

Abstract: An important feature of this paper is that both the algorithm presented in this paper allow for 
design constraints which often arise in practical filter design problems. Meeting required minimum stopband 
attenuation or a maximum deviation from the desired magnitude and phase responses in the passbands are 
common design constraints that can be handled by both the methods compared here. Constraints designing 
can be done either by constraints on complex error function or constraints on magnitude and phase of error. 
This paper shows the comparison of these two techniques. 

Key Words: Constraints, Linear Quadratic, Exchange Algorithm, Complex error function, Magnitude and Phase 
error 

1. Introduction 

Digital filters are integral parts of many digital signal 
processing systems, including control systems, systems 
for audio and video processing, communication systems 
and systems for medical applications. Due to the 
increasing number of applications involving digital 
signal processing and digital filtering, the variety of 
requirements that have to be met by digital filters has 
increased as well. Consequently, there is a need for 
flexible techniques that can design digital filters 
satisfying sophisticated specifications. The design 
specifications are formulated in the frequency domain 
by choosing a complex desired frequency response 
$D(e^{j\omega})$ which prescribes the desired magnitude and 
phase response. The complex function $D(e^{j\omega})$ is 
defined on $\Omega$ the domain of approximation. Which is a 
subset of the interval $[0, 2\pi ]$. In most cases the domain 
$\Omega$ is the union of several disjoint frequency bands 
which are separated by transition bands where no 
desired response is specified. The constraints designing of FIR Filter can be done by 
constraints on complex error function or by applying 
constraints on magnitude and phase of the error. These 
constraints can be put while using exchange algorithm 
as well. In this paper comparison of these two methods 
is discussed in detail. 

Cortezzao and Lightner [3] apply a multiple criterion 
optimization technique to a specification of both gain 
and group delay of FIR filter. But they state that their 
design method requires considerable computing time and 
is reliable only for orders not higher than five for FIR 
filter. 

Chen and Parks [12] investigate an approach in which a 
large computer memory requirement and CPU time 
increases exponentially with increasing grid density, 
the complex valued response is converted into a real-
valued function which is nearly equivalent to the 
complex function. They also state that their methods has 
using a linear programming technique the grid density 
governs the accuracy with which the solution 
approaches the optimum. 

Xiapoping Lai [13] applied PLS algorithm to the 
constrained least square design of FIR filter directly. 
But there was no method for non convex problem. 
The method presented here intends to solve the problem 
of computation time and memory requirement. 
The paper is organized as, after the brief introduction in 
section I, section II gives insight into Least square 
Approximation. Section IIIa deals with the Constrained 
Designing of filter with constraints on complex error function without using exchange approach while section 
III b deals exchange approach. Section IVa shows the 
constraints on magnitude and phase error independently 
without using exchange approach while IVb shows with 
using exchange approach. Section Vth gives the results 
obtained for the given example and section VIth deals 
conclusion followed by references.
2. Constrained Least Square approximation:

The union of all passbands is denoted by \( \Omega^p \):
\[
\Omega^p = \{ \omega \in \Omega \mid |D(e^{j\omega})| > 0 \}.
\]
The union of all stopbands is denoted by \( \Omega^s \):
\[
\Omega^s = \{ \omega \in \Omega \mid |D(e^{j\omega})| = 0 \}.
\]

If the designed filter is to have real valued coefficients, only the domain \( \Omega \cap [0, \pi] \) is considered. In this case the symmetry \( D(e^{j\omega}) = D^*(e^{-j\omega}) \) is assumed implicitly. This paper focus will be on the design of filters with real-valued coefficients. It is, however straightforward to extend the proposed methods to the design of filters with complex coefficients.

For formulating the design problems it is useful to define a complex error function by:
\[
E_c(\omega) = H(e^{j\omega}) - D(e^{j\omega})
\]  
(1)

Where \( H(e^{j\omega}) \) is the actual frequency response of the filter. Often \( h \), a real valued positive weighting function \( W(\omega) \) is used, e.g. by considering a weighted error function \( W(\omega)E_c(\omega) \), in order to give different weights on the approximation error in different frequency regions.

Considering the design of FIR filters. In this case the frequency response is given by
\[
H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}
\]  
(2)

Where \( N \) denotes the length of the impulse response \( h(n) \). The degree of the FIR filter is \( N-1 \). For further discussions it will be advantageous to use vector/matrix notation. Define a column vector of FIR filter coefficients by:
\[
h = [h(0), h(1), \ldots, h(N-1)]^T
\]

Let define a column vector of complex exponentials:
\[
e(\omega) = [1, e^{j\omega}, e^{2j\omega}, \ldots, e^{(N-1)j\omega}]^T
\]  
(3)

With these vectors the frequency response \( H(e^{j\omega}) \) of an FIR filter given by (2) can be written as:
\[
H(e^{j\omega}) = e^H(\omega)h,
\]  
(4)

Where the superscript \( H \) denotes conjugate transposition. The notation \( H(e^{j\omega}, h) \) will be used to explicitly express the dependence of \( H(e^{j\omega}) \) on the coefficients \( h \) whenever necessary. Likewise, the complex error function will be written as \( E_c(\omega, h) \) whenever its dependence on the filter coefficients is to be emphasized. Note the linear dependence of \( H(e^{j\omega}, h) \) and of \( E_c(\omega, h) \) on the coefficients \( h \) in the FIR case. For this reason, FIR filter design problems are much simpler to solve.

The constrained least squares filter design problem can be posed as an optimization problem where approximation error energy is minimized subject to constraints on error functions of interest.[8] Phase errors are used to define error energy the resulting objective function is non-convex which may cause severe difficulties when solving the design problem and prevents the use of simple and efficient algorithms. Moreover the minimization of error energy is most important in the stopbands of the filter. The use of the complex error for defining the error energy leads to a simple and meaningful objective function. These considerations lead to two constrained least squares design problems those are of practical interest and that result in tractable optimization problems[10]. This can be formulated by
\[
\min_{h} \int_{\Omega_{\text{min}}} W(\omega)|E_c(\omega, h)|^2 d\omega
\]  
(5)

Subject to \( |E_c(\omega, h)| \leq \delta_c(\omega), \omega \in \Omega_B \)
And by \( |E_m(\omega, h)| \leq \delta_m(\omega), \omega \in \Omega_B \)
\( \Omega_{\text{min}} \) contains the frequency bands where the mean squared error is to be minimized. \( \Omega_B \) contains the frequency bands where constraints are to be imposed and \( \Omega^p \) is the passband region. The positive function \( \delta_c(\omega) \), defining the constraints are part of the design specifications.

2.1 Constraints on Complex Error Function using Linear and quadratic programming approach

In this section we consider the constrained least squares problems equation (5) with constraints on the complex error function. Problems with constraints on the complex error function are convex problems. In order to formulate the design problems as (finite) linear or quadratic programming problems two approximations have to be made[6]. The approximation is the linearization of all constraints that are nonlinear functions of the filter coefficients \( h \). For convex problems such as equation (5) and the errors introduced
by linearizing the constraints can in principle be made arbitrarily small by increasing the number of linear constraints. Discussing about Constraints on complex error function:

Consider the feasible region $K$ defined by

$$K = \{ h \in \mathbb{R}^N \mid \text{Re}[E_c(\omega, h)e^{j\alpha}] \leq \delta_c(\omega), \omega \in \Omega, \alpha \in [0,2\pi) \}$$  \hspace{1cm} (6)

Replacing the feasible set $K$ by a finite number of linear inequality constraints corresponds to replacing the continuous angle variable $\alpha$ in (7) by a finite number of angles. The discrete angle is chosen as:

$$\alpha_k = \frac{k\pi}{p}, \quad k = 0,1,\ldots,2p-1, \quad p \geq 2 \hspace{1cm} (7)$$

This replaces the circular error region in the complex plane by a regular polygon with $2p$ vertices. As opposed to complex Chebyshev approximation we are free to choose the size of this polygon since its size is fixed instead of being minimized. Two reasonable choices for the size of the polygon are shown in Figure (1). The polygon denoted by ‘1’ in Figure (1) corresponds to the feasible region given by:

$$K_1 = \{ h \in \mathbb{R}^N \mid \text{Re}[E_c(\omega, h)e^{j\alpha}] \leq \delta_c(\omega), \omega \in \Omega, k = 0,1,\ldots,2p-1 \}$$

and the smaller polygon denoted by ‘2’ in fig. (1) is associated with the set:

$$K_2 = \{ h \in \mathbb{R}^N \mid \text{Re}[E_c(\omega, h)e^{j\alpha}] \leq \delta_c(\omega)\cos\frac{\pi}{2p}, \text{where } \omega \in \Omega, k = 0,1,\ldots,2p-1 \}$$

With angle $\alpha_k$ given by (7). The following relationship holds:

$$K_2 \subseteq K \subseteq K_1 \hspace{1cm} (8)$$

Denote the optimum solutions corresponding to the feasible sets $K$, $K_1$ and $K_2$ by $h_0$, $h_{01}$ and $h_{02}$. If $f(h)$ denotes the objective function to be minimized, then due to (8) $f(h_{01}) \leq f(h_0) \leq f(h_{02})$ must hold. Note that $h_{02}$ is guaranteed to be feasible with respect to the original feasible region $K$, but the corresponding value of the objective function may be larger than $f(h_0)$. Choosing $K_1$ may result in a smaller objective function but $h_{01}$ may be infeasible with respect to $K$. The maximum violation of the original constraints by solution $h_{01}$ is:

$$\Delta(\omega, p) = \delta_c(\omega)\left[\frac{1}{\cos(\pi/2p)} - 1\right]$$

$$= \delta_c(\omega)\left[\frac{\pi}{2p}\right]^2 + 5\left(\frac{\pi}{2p}\right)^4 + \ldots$$

\hspace{1cm} (9)

**Figure 1:** Constraints on the complex error function: replacing the circular error region in the complex plane by a regular polygon with $2p$ vertices. 1: new feasible region is larger than the original feasible region. 2: New feasible region is smaller than the original feasible region.

### 2.2-Constraints on complex error function using Exchange approach:

Method discuss in last section has two drawback. The first is the large number of constraints resulting in a high computational effort and in high memory requirements. The second drawback is the fact that replacing the nonlinear magnitude constraints by linear constraints introduces errors. These errors can be made arbitrarily small in the stopbands by increasing the number of vertices of the polygon shown in Figure (1). However this also increases the number of linear constraints. In the passbands the linearization errors depend on second and higher order terms involving the constraint function $\delta_c(\omega)$.\[6\]

The set given by equation (6) is convex and its representation by infinite number of constraints is given in form of factor $\alpha$. The optimum solution $h_0$ to the constrained least squares problem equation (5) with the constraint $h \in K$ will satisfy the inequality $|E_c(\omega, h)| \leq \delta_c(\omega), \forall \omega \in \Omega$ with equality only at a finite number of frequency points $\omega \in \Omega$ corresponding to the active set of constraints. Hence only a finite number of linear constraints in is satisfied with equality where for each frequency point $\omega \in \Omega$ there is an associated angle $\alpha: \alpha(\omega) = -\varphi_E(\omega, h_0)$, $\omega \in \Omega$ with
\( \varphi_E(\omega,h) = \arg\{E_c(\omega,h)\} \). It is sufficient to find the set \( \Omega_{\text{act}} \) and the associated arguments of the complex error function in order to compute the solution to the constrained optimization problem by solving a quadratic programming problem with a relatively small number of linear equality constraints. It is however very difficult to directly identify the set \( \Omega_{\text{act}} \) and the corresponding angles \( \alpha \). Hence the constraints of the sub problems will not be formulated as equality constraints but as inequality constraints. This allows to include more constraints in every iteration step and the set of constraints that will be active at the optimum solution need not be exactly identified. These constraints must only be a subset of the inequality constraints of the last sub problem. An important question is how to identify frequency points that are likely to correspond to active constraints at the solution. Assume that for a given coefficient vector \( h^{(k)} \) the constraint |
\[ E_c(\omega, h^k) \leq \delta_\varphi(\omega), \forall \omega \in \Omega_B \]

is violated. It is reasonable to use the local maximizers of |
\[ |E_c(\omega, h^k)| \leq -\delta_\varphi(\omega) \]

that violate the constraint as candidate frequencies for active constraints at the solution. Using this strategy | \( Ec(\omega, h^k) \) | is evaluated in every iteration step with \( h^k \) being the solution of the actual sub problem the local maximizers of | | \( Ec(\omega, h^k) | - \delta_\varphi(\omega) \) at which the constraint is violated are identified and new linear constraints are formulated according to

\[ | \text{Re} E_c(\omega, h) \leq \delta_\varphi(\omega), \omega \in \Omega_{\text{viol}} \] (10)

Where \( \Omega_{\text{viol}} \) is the set of frequencies that correspond to local maximizers of | \( Ec(\omega, h^k) \) | violating the constraint. The angles \( \alpha(\omega) \) are estimates of the negative argument of the complex error function at the next iteration step. A reasonable choice is \( \alpha(\omega) = -\varphi_E(\omega, h^k) \). The expression \( \text{Re} [E_c(\omega, h) e^{i\alpha(\omega)}] \) serves as a linearization of the nonlinear function | \( Ec(\omega, h) \) |. A first order Taylor expansion of | | \( Ec(\omega, h) \) | about some \( h^k \) exists for all frequencies satisfying | | \( Ec(\omega, h^k) | > 0 \) and is given by

\[ |E_c(\omega, h^k) | + (h - h^k)^T \nabla_h |E_c(\omega, h^k) | \] (11)

With the gradient vector \n
\[ \nabla_h |Ec(\omega, h)| \]

Writing | \( Ec(\omega, h) \) | as

\[ |E_c(\omega, h)| = \text{Re} \{E_c(\omega, h)e^{-j\varphi_E(\omega, h)}\} = \text{Re} \{e^{j\omega h} - D(e^{j\omega})e^{-j\varphi_E(\omega, h)}\} \]

The gradient vector \( \nabla_h |E_c(\omega, h)| \) can be written as

\[
\nabla_h |E_c(\omega, h)| = \text{Re} \{e^{j\omega h}e^{-j\varphi_E(\omega, h)}\}

- jE_c(\omega, h)e^{-j\varphi_E(\omega, h)}\nabla \varphi_E(\omega, h)\]

= \text{Re} \{e^{j\omega h}e^{-j\varphi_E(\omega, h)} - \text{Re} [j |E_c(\omega, h)| \nabla \varphi_E(\omega, h)] \} \]

(12)

For \( |Ec(\omega, h)| > 0 \), \( \nabla \varphi_E(\omega, h) \) is the gradient of \( \varphi_E(\omega, h) = \arg \{E_c(\omega, h)\} \) with respect to the coefficient vector \( h \). Since the term in brackets on the right-hand side of (12) is purely imaginary the gradient vector can finally be written as

\[ \nabla_h |E_c(\omega, h)| = \text{Re} \{e^{j\omega h}e^{-j\varphi_E(\omega, h)} \} \]

(13)

Where \( e(\omega) \) is an \( N \times 1 \) vector with elements \( e^{j\omega_n} \), \( n = 0,1,\ldots\ldots,N-1 \) as defined by (3). Inserting (13) into (11) yields the following expression for the first order Taylor series of \( |Ec(\omega, h)| \) about \( h^{(k)} \):

\[ |E_c(\omega, h^k) | + (h - h^k)^T \nabla_h |E_c(\omega, h^k) | \]

= \text{Re} \{E_c(\omega, h)e^{-j\varphi_E(\omega, h^k)} \} \]

(14)

Hence the choice \( \alpha(\omega) = -\varphi_E(\omega, h^{(k)}) \) in (10) is not only reasonable as stated earlier but also corresponds to a first order Taylor expansion of \( |Ec(\omega, h)| \) about the current solution \( h^{(k)} \). Another set of frequencies that are candidates for active constraints at the solution are those frequencies corresponding to active constraints at the solution to the respective sub problem. Hence the current active constraints should be reused in the next iteration step. Note that due to convexity of the feasible region linear constraints that have been derived from a first order Taylor expansion of the original constraints will never cut away parts of the original feasible region.

**Algorithm:**

1) \( k = 0 \) Solve the unconstrained quadratic minimization problem for \( h^{(0)} \).
2) Determine the local maxima of

\[ |E_c(\omega, h^{(k)}) | - \delta_\varphi(\omega), \omega \in \Omega_B \]

if \( |E_c(\omega, h^k) | \leq \delta_\varphi(\omega), \omega \in \Omega_B \), is satisfied up to some specified tolerance stop. Otherwise go to 3.
3) Determine the set \( \Omega_{\text{viol}} \) of local maximizer of

\[ |E_c(\omega, h^k) | - \delta_\varphi(\omega), \]

that satisfy

\[ |E_c(\omega, h^k) | \leq \delta_\varphi(\omega), \]
4) Formulate a new set of constraints by using the current active constraints and by imposing new constraints
\[ \text{Re}\left[ E_c(\omega, h)e^{-j\phi_k(\omega, h^{(i)})}\right] \leq \delta_\epsilon(\omega) \quad \omega \in \Omega_p \]

5) \(k: k+1\), Compute \(h^{(k)}\) by solving the quadratic programming problem subject to the constraints determined in step 4. Go to 2.

The algorithm presented in this section solves an unconstrained least squares problem as an initial problem and then adds all constraints that are necessary to compute the optimum solution to the semi-infinite programming problem.

3. Constraints on Magnitude and Phase Errors using Linear and quadratic programming approach:

As shown constraints on the phase error are exactly represented by linear inequality constraints if the function \(\delta_\phi(\omega)\) constraining the phase response satisfies \(\delta_\phi(\omega) \leq \pi/2, \forall \omega \in \Omega^p\). Hence only the nonlinear magnitude constraints must be replaced by a finite set of linear constraints. In the stopbands the error region is a circle around the origin of the complex plane. Consequently all results from the previous section still apply to the stopbands. The modifications of the passband error region due to magnitude constraint linearization are shown in Figure (2). The original magnitude constraints are shown by the dashed arcs. The solid vertical lines correspond to linear constraints replacing the original magnitude constraints. The constraints denoted by ‘1’ in Figure (2) correspond to a feasible region which is the smallest set described by two linear inequality constraints per frequency point containing the original feasible set. The constraints denoted by ‘2’ correspond to a feasible region being the largest set described by two linear inequality constraints per frequency point that is contained in the original set. A better approximation to the upper magnitude constraint could be achieved by using more than one linear upper magnitude constraint per frequency point. However this increases the size of the resulting optimization problem. Note that improving the approximation by using more than one linear constraint per frequency point is not possible for the non-convex lower magnitude Constraint. Choosing the linear constraints denoted by ‘1’ in Figure (2) results in the following maximum violations of the original passband magnitude constraints if \(\delta_\phi(\omega) < \pi / 2\) holds

\[ \Delta_u(\omega) = \left[ D(e^{j\omega}) \right] + \delta_u(\omega) \left[ \frac{1}{\cos \delta_\phi(\omega)} - 1 \right] \]

\[ \omega \in \Omega^p \]

\[ \Delta_u(\omega) = \left[ D(e^{j\omega}) \right] + \delta_u(\omega) \left[ \frac{1}{2} \delta_\phi^2(\omega) + \frac{5}{24} \delta_\phi^4(\omega) + \ldots \ldots \right] \]

\[ \omega \in \Omega^p \]

Where \(\Delta_u(\omega)\) and \(\Delta_l(\omega)\) are the maximum violations of the upper and lower magnitude constraints respectively. It is straightforward to show that \(\Delta_l(\omega) \leq \Delta_u(\omega), \forall \omega \in \Omega^p\). If \(0 < \delta_\phi(\omega) < \pi / 2, \forall \omega \in \Omega^p\) holds. From (9) it is clear that the violations \(\Delta_u(\omega)\) and \(\Delta_l(\omega)\) consist only of mixed and higher order terms of the specified functions \(\delta_m(\omega)\) and \(\delta_m(\omega)\). Hence for small \(\delta_\phi(\omega)\) the errors introduced by using linearized magnitude constraints are small as well. If the linear constraints denoted by ‘2’ in Figure (2) are used no violations with respect to the original feasible region occur. Using the linearized passband magnitude constraints shown in Figure (2) linear and quadratic programming problem formulation can be done with approximating the constrained least squares problems (5). If the new feasible region is to completely contain
the original feasible region (constraints denoted by ‘1’ in Figure (2). define positive functions \( U(\omega) \) and \( L(\omega) \) according to:

\[
U(\omega) = |D(e^{j\omega})| + \delta_m(\omega), \quad \omega \in \Omega_p \cap \Omega_B
\]

\[
\delta_m(\omega) \quad \omega \in \Omega^s \cap \Omega_B
\]

\[
L(\omega) = |D(e^{j\omega})| - \delta_m(\omega)|\cos \delta_p(\omega), \quad \omega \in \Omega_p \cap \Omega_B
\]

If no constraint violations with respect to the original feasible region are tolerated (constraints denoted by ‘2’ in Figure (1) then \( U(\omega) \) and \( L(\omega) \) must be chosen according to:

\[
U(\omega) = |D(e^{j\omega})| + \delta_m(\omega)|\cos \delta_p(\omega), \quad \omega \in \Omega_p \cap \Omega_B
\]

\[
\delta_m(\omega)\cos(\pi/2p) \quad \omega \in \Omega^s \cap \Omega_B
\]

\[
L(\omega) = |D(e^{j\omega})| - \delta_m(\omega), \quad \omega \in \Omega_p \cap \Omega_B
\]  \hspace{1cm} (16)

The quadratic programming problem approximating the constrained least squares problem reads

\[
\text{minimize} \quad \int_{\Omega_{\text{min}}} W(\omega) |E(\omega, h)|^2 \, d\omega \quad \text{subject to}
\]

\[
\text{Re}[H(e^{j\omega}, h)e^{-j\delta(\omega)}] \leq U(\omega), \quad \omega \in \Omega_p \cap \Omega_B
\]

\[
\text{Re}[H(e^{j\omega}, h)e^{-j\delta(\omega)}] \geq L(\omega), \quad \omega \in \Omega_p \cap \Omega_B
\]  \hspace{1cm} (18)

\[
\text{Im}[H(e^{j\omega}, h)e^{-j\delta(\omega)}] \leq \tan \delta(\omega)\text{Re}[H(e^{j\omega}, h)e^{-j\delta(\omega)}],
\]

\[
\omega \in \Omega_p \cap \Omega_B
\]

\[
\text{Im}[H(e^{j\omega}, h)e^{-j\delta(\omega)}] \geq -\tan \delta(\omega)\text{Re}[H(e^{j\omega}, h)e^{-j\delta(\omega)}],
\]

\[
\omega \in \Omega_p \cap \Omega_B
\]

\[
\text{Re}[H(e^{j\omega}, h)e^{j\delta(\omega)}] \leq U(\omega), \quad \omega \in \Omega^s \cap \Omega_B
\]

Where \( \Omega_p \) and \( \Omega^s \) are passbands and stopbands respectively, \( \Omega_0 \) is the union of all bands where constraints are to be imposed and \( \Omega_{\text{min}} \) is the union of all bands

### 3.1. Constraints on Magnitude and Phase using Exchange approach:

In the passbands the linearization errors depend on second and higher order terms involving the constraint functions \( \delta_m(\omega) \) and \( \delta\phi(\omega) \). Hence these errors are small if \( \delta_m(\omega) \) and \( \delta\phi(\omega) \) are small. However especially in situations where \( \delta\phi(\omega)/\text{rad} >> \delta_m(\omega) \) holds in parts of the passbands the linearization errors may become quite large. Exchange algorithms can solve optimization problems with an arbitrarily large or even infinite number of constraints. During the iteration process a set of constraints containing those constraints that will be active at the optimum solution is identified. The solution of the last sub problem subject to these constraints equals the solution of the original problem. The efficiency of these methods strongly depends on the efficiency of the algorithms used for solving the sub problems. Hence it is desirable to solve sub problems with linear constraints since there exist fast algorithms for linearly constrained problems. The amount of memory required by exchange algorithms is independent of the total number of constraints. Hence the first drawback of the standard quadratic programming formulation is eliminated. Also the linearization errors can be eliminated. We have to distinguish between problems with convex and non-convex feasible regions. Since any convex feasible region can be represented by an infinite number of linear constraints such problems can directly be handled by exchange algorithms that solve a sequence of linearly constrained sub problems. These algorithms can be viewed as generalizations of cutting plane methods for convex programming problems. Linearization of nonlinear constraints is no longer necessary. The only requirement for algorithms based on cutting plane methods to be applicable is the convexity of the feasible region is satisfied for upper bounds on the magnitude response and for phase constraints if \( \delta\phi(\omega) \leq \pi/2 \) holds. Only lower bounds on the magnitude response as used in the passbands result in a non-convex feasible region. Hence for exchange algorithms based on cutting plane methods to be applicable the lower bounds on the magnitude error have to be replaced by constraints such that the feasible region is convex. Non-convex feasible regions cannot be represented by an infinite number of linear constraints and exchange algorithms based on cutting plane methods cannot be applied.

An exchange algorithm that exactly solves the non-convex constrained least squares problem with magnitude and phase constraints is presented. The algorithms used in literature so far solve a constrained optimization problem in every iteration step where the set of constraints is composed of a part of the constraints used in the previous iteration step and new constraints determined by evaluating the original semi-infinite constraints at the current solution. Reusing a part of the old constraints is crucial for convergence. A nonlinear semi-infinite constraint can be written in the form
\[ c(\omega, h) \leq 0, \quad \forall \omega \in \Omega_B \quad (19) \]

Each constraint used in a certain iteration step is derived from a first order Taylor expansion of \( c(\omega, h) \) about some coefficient vector \( h \), evaluated at some frequency point \( \omega \):

\[
c(\omega, h) + (h - h_j)^T \nabla c(\omega, h_j) \leq 0, \quad (20)
\]

Where \( \nabla c(\omega, h) \) is the gradient vector of \( c(h) \) with respect to \( h \). If \( c(\omega, h) \) is convex, the inequality

\[
c(\omega, h) + (h - h_j)^T \nabla c(\omega, h_j) \leq c(\omega, h) \quad \forall \omega \in \Omega_B \quad (21)
\]

\[
E_p(\omega, h) = \arctan \left[ \frac{\text{Im}\{H(e^{j\omega}, h)\}e^{-j\delta_p(\omega)}}{\text{Re}\{H(e^{j\omega}, h)\}e^{-j\delta_p(\omega)}} \right] \quad (22)
\]

\[
-\tan \delta_p(\omega) \leq \left[ \frac{\text{Im}\{H(e^{j\omega}, h)\}e^{-j\delta_p(\omega)}}{\text{Re}\{H(e^{j\omega}, h)\}e^{-j\delta_p(\omega)}} \right] \leq \tan \delta_p(\omega)
\]

\[
\delta_p(\omega) < \pi/2 \quad (23)
\]

is satisfied for any \( h \). Constraints from previous iteration steps may become unnecessary and could be thrown away but they will never cut away parts of the original feasible region. However if \( c(\omega, h) \) is not convex the feasible set defined by (22) is non-convex in general and linear constraints as formulated by (23) might cut away parts of the original feasible region because (5) is not satisfied in general. Hence old constraints must be removed because they might cut away the part of the feasible region that contains the optimum solution. Instead of reusing old constraints some other constraints related to these old constraints must be used. The exchange algorithm presented in Section (III b) reuses the active constraints of the previous iteration step. A logical extension is to formulate new constraints at the frequency points corresponding to active constraints of the previous iteration step. These constraints are used instead of the old active constraints. Hence in every iteration step all constraints are formulated anew and the propagation of old constraints cutting away parts of the feasible region are prevented.

Note that all these considerations only apply to non-convex constraint functions. In the problem under consideration only the exchange rule for the lower bounds on the magnitude response must be adapted. The exchange rule for upper magnitude bounds and for phase constraints remains unchanged. The modified exchange algorithm for exactly solving the constrained least squares problem with magnitude and phase constraints works as follows:

**Algorithm:**

1) \( k = 0 \) Solve the unconstrained quadratic minimization problem for \( h^{(0)} \).
2) Determine the local maxima of 
\[
E_m(\omega, h^{(k)}) - \delta_m(\omega), \quad \omega \in \Omega_B \quad \text{and of}
\]
\[
E_p(\omega, h^{(k)}) - \delta_p(\omega), \quad \omega \in \Omega_B \quad \text{and the}
\]
local minima of \( E_m(\omega, h^{(k)}) + \delta_m(\omega), \quad \omega \in \Omega_B \), and of \( E_p(\omega, h^{(k)}) + \delta_p(\omega), \quad \omega \in \Omega_B \). If \( E_m(\omega, h^{(k)}) \leq \delta_m(\omega), \quad \omega \in \Omega_B \) and \( E_p(\omega, h^{(k)}) \leq \delta_p(\omega), \quad \omega \in \Omega_B \), is satisfied up to some specified tolerance, stop Otherwise go to 3.

3) Determine the sets of frequencies at which local maxima or minima of the functions considered in step 1 violate the respective constraints.
4) Formulate a new set of constraints for the next iteration step:
   a) Reuse the current active upper magnitude bounds and active phase constraints.
   b) Impose new magnitude constraints at passband frequencies corresponding to active lower magnitude bounds in the current iteration step.
   c) Impose new magnitude and phase constraints at the respective sets of frequencies determined in step 2. Formulate new magnitude constraints using a first order Taylor series of \( \text{Em}(\omega, h) \) about the current solution \( h^{(k)} \):
\[
E_m(\omega, h) = \text{Re}\{H(e^{j\omega}, h)e^{-j\delta_m(\omega)}\} - D(e^{j\omega})
\]
5) \( k: k+1 \), Compute \( h^{(k)} \) by solving the quadratic programming problem subject to the constraints determined in step 4. Go to 2.

The algorithm presented in this section solves an unconstrained least squares problem as an initial problem and then adds all constraints that are necessary to compute the optimum solution to the semi-infinite programming problem.

**4 Design Example:**

A delay low pass filter (\( N=250 \)) having following specifications.
The constraint on the complex error function is given by:

\[ \delta_i(\omega) = \begin{cases} 2.1 \cdot 10^{-4}, & \omega \in \Omega^p \\ 2.1 \cdot 10^{-3}, & \omega \in \Omega^s \end{cases} \]

In order to achieve a Chebyshev behavior in the passband the weighting ‘1’ is chosen in the passband and ‘1000’ in stopband.

Now imposing the constraints on magnitude and phase errors independently. The maximum magnitude error of the optimum Chebyshev filter is bounded by \( \delta = 2.02 \cdot 10^{-4} \) in the passband and by \( \delta / 10 = 2.02 \cdot 10^{-5} \) in the stopband. The maximum phase error of the optimum Chebyshev filter is bounded by \( \arcsin \delta = \delta = 2.02 \cdot 10^{-4} \). These upper bounds for magnitude and phase errors are tight for the filter under consideration.

**5. Results:**

The results of above example are shown in figures below. The results for constraints on complex error function are shown in figure (3) and the results for constraints on magnitude and phase error are shown in figure (4).
The plots of the constraints on complex error function show the magnitude of the frequency response and the passband magnitude and phase errors of the designed filter. It took 15 sec to compute this filter. While in [14] it was reported it took 5 minutes on a comparable computer. This results in a considerably speed up.

The plots in fig (4) shows the response when constraints are applied on magnitude and phase independently. It takes 10 sec to compute. So it is showing faster response. The maximum magnitude and phase errors are the same as the constraints on complex error function. But we get a considerably better stopband behavior and fast computing without relaxing the original tolerance scheme.

6. Conclusion: For solving constrained least square problems exchange algorithm is better than linear programming problem. But as result is showing the responses of two techniques using exchange algorithm one is constraints on complex error function and another constraints on magnitude and phase errors independently. The results obtained for given example in section III b shows that the second technique constrained on magnitude and phase error independently is showing better response in terms of memory and fast response.

References:
