On a Parabolic Inclusion with Integral Boundary Conditions

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Abstract: - This paper deals with the investigation of a one-dimensional parabolic equation with a discontinuous nonlinearity and subjected to integral boundary conditions. The problem is transformed into an integral inclusion. We provide sufficient conditions that guarantee the existence of at least one solution.

Key–Words: - parabolic equation, discontinuous nonlinearity, Green’s function, multivalued maps, fixed point theorems.

1 Introduction

Consider the following one dimensional parabolic inclusion subjected to integral boundary conditions.

\[ u_t - u_{xx} \in F(u), \quad 0 < x < \pi, \quad 0 < t < 1, \] (1)
\[ u(x, 0) = u_0(x), \quad 0 \leq x \leq \pi, \] (2)
\[ u(0, t) = \int_0^\pi g(u(x, t))dx, \] (3)
\[ u(\pi, t) = \int_0^\pi h(u(x, t))dx, \] (4)

where \( u_0, g, h \) are given functions and \( F \) is a multivalued map satisfying some conditions that will be specified later.

Parabolic problems with discontinuous nonlinearities arise in the description of many phenomena in the applied sciences. We can mention, for instance, chemical reactor theory, porous medium combustion. See [11] and [12] and the references therein. Problems with integral boundary conditions appear in the modeling of concrete problems, such as heat conduction [3], [15], [5], thermoelasticity [7]. Several papers have been devoted to the study of parabolic problems with integral conditions [6], [19], [21]. A good account on numerical treatment of parabolic problems with integral conditions can be found in [8]. The paper [11] and [12] consider the case \( F(u) = \lambda H(u - 1) \), where \( H(\cdot) \) is the maximal monotone graph generated by the Heaviside function. Here the set \( \{(x, t) \in D; u(x, t) = 1\} \) plays an important role. Their approach is based on the theory of semi-group of linear operators. The paper [4] deals with a general parabolic problem with a right hand side given by a general maximal monotone graph generated by an increasing function of bounded variations. The technique used by these authors is based on the method of upper and lower solutions.

In this paper we consider a nonlocal problem for the heat equation with a convex multivalued right hand side. We shall convert Problem (1), (2), (3), (4) to an integral inclusion using the properties of the Green’s function corresponding to the linear problem. We, then, provide sufficient conditions on the data that
linear nonhomogeneous problem

We will enable us to obtain a priori bounds on possible solutions of a one-parameter family of problems related to the original one. Our approach is based on fixed point theorems for suitable multivalued operators.

The outline of the paper is as follows. Section 2 is devoted to the study of the linear nonhomogeneous problem and the properties of the Green’s function. In section 3, we shall recall the main properties of multivalued maps. We state and prove our main results in section 4.

2 Linear Nonhomogeneous Problem

In this section we consider the linear nonhomogeneous problem

\[ u_t - u_{xx} = f(x, t), \quad 0 < x < \pi, \quad 0 < t < 1, \]

(5)

\[ u(x, 0) = u_0(x), \quad 0 \leq x \leq \pi, \]

(6)

\[ u(0, t) = a(t), \quad 0 \leq t \leq 1, \]

(7)

\[ u(\pi, t) = b(t), \quad 0 \leq t \leq 1, \]

(8)

Assume that the functions \( f, u_0 \), are Hölder continuous, and the functions \( a \) and \( b \) are continuous. Then, Problem (5), (6), (7), (8) has a unique solution given by for each \((x, t) \in D = (0, \pi) \times (0, 1)\),

\[
  u(x, t) = \int_0^t \int_0^\pi G(x, t; y, s) f(y, s) dyds + \int_0^\pi G(x, t; y, 0) u_0(y) dy + \int_0^t \frac{\partial G}{\partial y}(x, t; 0, s) a(s) ds - \int_0^t \frac{\partial G}{\partial y}(x, t; \pi, s) b(s) ds,
\]

(9)

where \( G(x, t; y, s) \) is the Green’s function corresponding to the linear homogeneous problem. This function satisfies the following

(i) \( G_t - G_{xx} = \delta(t - s) \delta(x - y) \quad s < t, \quad 0 < x, y < \pi \)

(ii) \( G(0, t; y, s) = 0 \quad s > t, \quad 0 < x, y < \pi \)

(iii) \( G(0, t; y, s) = G(\pi, t; y, s) = 0 \quad s < t \)

(iv) \( G(x, t; y, s) \geq 0 \) for \((x, t) \in (0, \pi) \times (0, 1)\)

(v) \( G, G_t, G_x, G_{xx} \) are continuous functions of \((x, t; y, s) \in ((0, \pi) \times (0, 1))^2, \quad t - s > 0\).

Moreover, \( v = G(f + u_0) \) solves the problem

\[
  v_t - v_{xx} = f, \quad v(x, 0) = u_0(x) \quad 0 \leq x \leq \pi, \quad 0 < t < 1.
\]

(10)

We write (9) in the following convenient form, for each \((x, t) \in D\),

\[
  u(x, t) = G(f + u_0)(x, t) + \gamma(a, b)(x, t)
\]

where

\[
  G(f + u_0)(x, t) = \int_0^t \int_0^\pi G(x, t; y, s) f(y, s) dyds + \int_0^\pi G(x, t; y, 0) u_0(y) dy,
\]

(11)

\[
  \gamma(a, b)(x, t) = \int_0^t \frac{\partial G}{\partial y}(x, t; 0, s) a(s) ds - \int_0^t \frac{\partial G}{\partial y}(x, t; \pi, s) b(s) ds.
\]

(12)

The operators \( G, \gamma \) map \( C(\overline{D}; \mathbb{R}) \) into \( C^{2,1}(D; \mathbb{R}) \). Moreover, \( v = G(f + u_0) \) solves the problem

(13)

\[
  u_t - u_{xx} = f, \quad u(x, 0) = u_0(x) \quad 0 \leq x \leq \pi, \quad 0 < t < 1.
\]

(14)

and

\[
  w(x, 0) = w(x, 0) = 0, \quad w(0, t) = a(t), \quad w(\pi, t) = b(t).
\]

3 Multivalued Functions

We, now, introduce some useful definitions and properties from set-valued analysis. For complete details on multivalued maps we refer the interested reader to the books [1], [2] and [9].

Let \((Y, |\cdot|)\) be a normed space. We shall denote the set of all subsets of \( Y \) having property \( \ell \) by \( P_\ell(Y) \). For instance, \( U \in P_{1\ell}(Y) \) means \( U \) closed in \( Y \); when \( \ell = b \) we have the bounded subsets of \( Y \), \( \ell = cv \) for convex subsets, \( \ell = cp \) for compact subsets and \( \ell = cp, cv \) for compact and convex subsets. A multivalued map \( R : Y \to 2^Y \) is convex (closed) valued if \( R(z) \) is convex (closed) for each \( z \in Y \). \( R \) is bounded on bounded sets if \( R(B) = \cup_{z \in B} R(z) \) is bounded
in $Y$ for all $B \in P_b(Y)$ (i.e. $\sup_{z \in B} \sup_{y \in R(z)} |y| < \infty$). The multivalued map $R$ is called upper semicontinuous (usc) on $Y$ if for each $z \in Y$ the set $R(z) \in P_d(Y)$ and is nonempty, and for each open subset $\Lambda$ of $Y$ containing $R(z)$, there exists an open neighborhood $\Pi$ of $z$ such that $R(\Pi) \subset \Lambda$. The set-valued map $R$ is called completely continuous if $R(B)$ is relatively compact for every $B \in P_b(Y)$. If $R$ is completely continuous with nonempty compact values, then $R$ is usc if and only if $R$ has a closed graph (i.e. $z_n \to z$, $w_n \to w$, $w_n \in R(z_n) \Rightarrow w \in R(z)$). $R$ has a fixed point if there exists $z \in Y$ such that $z \in R(z)$. A multivalued map $R : \mathcal{D} \to P_d(\mathbb{R})$ is measurable if for every $\theta \in \mathbb{R}$, the function $v \mapsto \text{dist}(\theta, R(v)) = \inf \{|\theta - z|; z \in R(v)\}$ is measurable.

**Definition 1** Let $u \in X$. The set of selections of the set-valued map $F : \mathbb{R} \to \mathcal{D}$ is defined by

$$S_{F,u} := \{w \in X; w(x,t) \in F(u(x,t)), \forall (x,t) \in D\}$$

**Definition 2** Let $F : \mathbb{R} \to \mathcal{D}$ have nonempty compact values. The Nemitsky operator $\mathcal{F}$ of $F$ is the set-valued operator defined by

$$\mathcal{F}(u) = \{w \in X; w(x,t) \in F(u(x,t)), \forall (x,t) \in D\}.$$  

It can be shown (see [14, page 40]) that if $F$ is usc with convex bounded values then the operator $\mathcal{F}$ is well defined, usc, bounded on bounded sets in $C(\mathbb{R})$, and has convex values.

**Definition 3** $u \in C(D;\mathbb{R})$ is a solution of (1), (2), (3), (4) if there exists a Lipschitz selection $f \in S_{F,u}$ and $u$ has the integral representation (9).

**Remark.** If $F$ is a Lipschitz multifunction then it admits a Lipschitz selection. See [16].

**Definition 4** Let $(Z,d)$ be a metric space and let $A,B$ be two nonempty subsets of $Z$. The Hausdorff distance between $A$ and $B$ is defined by

$$d_H(A,B) = \max \{\sup_{a \in A} \sup_{b \in B} d(a,b), \sup_{b \in B} \sup_{a \in A} d(a,b)\}.$$  

Here $d(a,B) = \inf \{d(a,b); b \in B\}$. Then $(P_d,b(D), d_H)$ is a metric space.

**Definition 5** A multivalued operator $\mathcal{L} : Z \to P_d(Z)$ is called

(i) a Lipschitz if and only if there exists $\delta > 0$ such that $d_H(\mathcal{L}(u), \mathcal{L}(v)) \leq \delta d(u,v)$ for all $u,v \in Z$

(ii) a contraction if and only if it is a Lipschitz with $\delta < 1$.

The following theorems play an important role in our existence results.

**Theorem 6** [18] Let $E$ be a Banach space and $\mathcal{L} : E \to P_{c,b}(E)$ a condensing map. If the set $S := \{z \in E; \lambda z \in \mathcal{L}(z) \text{ for some } \lambda > 1\}$ is bounded, then $\mathcal{L}$ has a fixed point.

We remark that a compact map is the simplest example of condensing map.

**Theorem 7** [10] Let $B_r(0)$ and $\overline{B}_r(0)$ denote respectively the open and closed balls in a Banach space $(E, \|\cdot\|)$ centered at 0 and having radius $r$. Let $\mathcal{L}_1 : \overline{B}_r(0) \to P_{c,b}(E)$ and $\mathcal{L}_2 : B_r(0) \to P_{c,b}(E)$ be two multivalued operators satisfying

(i) $\mathcal{L}_1$ is a contraction,

(ii) $\mathcal{L}_2$ is compact and usc.

Then either

(j) the operator inclusion $u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ has a solution in $B_r(0)$, or

(jj) there exists $u \in E$ with $\|u\| = r$ such that $\lambda u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ for some $\lambda > 1$.

**4 Main Results**

In this section, we shall state and prove our main results.

Our first result is based on the following assumptions.

(H1) $u_0 \in C([0, \pi]; \mathbb{R})$

(H2) $g, h : C(D;\mathbb{R}) \to \mathbb{R}$ are continuous and bounded

(H3) $F$ is usc, has convex and compact values, and maps bounded sets into relatively compact sets. Moreover, there are positive constants $c_1, c_2$ such that $|F(u)| \leq c_1 + c_2 |u|$

**Theorem 8** Suppose (H1), (H2) and (H3) are satisfied. Then Problem (1), (2), (3), (4) has at least one solution.
Proof. It follows from (9), (10), and (11) that \( u \) is a solution of (1), (2), (3), (4) if and only if \( u \) is a fixed point of the multivalued operator \( \mathcal{L} \), defined by

\[
\mathcal{L} u = G ( Fu + u_0 ) + \gamma ( u ),
\]

where \( \mathcal{F} \) is the Nemitski operator of \( F \). In fact, we have

\[
\mathcal{L} u ( x, t ) = \int_0^t \int_0^\pi G ( x, t ; y, s ) F ( u ( y, s ) ) dy ds
\]

\[+ \int_0^t g ( x, t ; y, 0 ) u_0 ( y ) dy\]

\[+ \int_0^t \partial_y G ( x, t ; 0, s ) \int_0^\pi g ( u ( y, s ) ) dy ds\]

\[− \int_0^t \partial_y G ( x, t ; \pi, s ) \int_0^\pi h ( u ( y, s ) ) dy ds,
\]

where

\[
\int_0^t \int_0^\pi G ( x, t ; y, s ) ( \mathcal{F} u ) ( y, s ) dy ds
\]

is the Aumann integral of \( \mathcal{F} \). We see that \( \mathcal{L} \) is the sum of a multivalued operator \( G ( \mathcal{F} \cdot + u_0 ) \) and a single valued operator \( \gamma ( \cdot ) \). We apply Theorem 6 to the operator \( \mathcal{L} \).

Let \( u \in X \). We show that \( \mathcal{L} u \in P_{cp,cv}(X) \).

(a) \( \mathcal{L} u \) is a convex subset of \( X \) for each \( u \in X \).

Let \( v_1, v_2 \in \mathcal{L} u \). Then there exists \( w_1, w_2 \in S_{F,u} \) such that for each \( (x,t) \in D \) we have for \( i=1,2 \)

\[
v_i ( x, t ) = \int_0^t \int_0^\pi G ( x, t ; y, s ) w_i ( y, s ) dy ds
\]

\[+ \int_0^t G ( x, t ; y, 0 ) u_0 ( y ) dy\]

\[+ \int_0^t \partial_y G ( x, t ; 0, s ) \int_0^\pi g ( u ( y, s ) ) dy ds\]

\[− \int_0^t \partial_y G ( x, t ; \pi, s ) \int_0^\pi h ( u ( y, s ) ) dy ds.
\]

Since \( S_{F,u} \) is convex, it is clear from the above relation that any convex combination of \( v_1, v_2 \) is an element of \( \mathcal{L} u \).

(b) \( \mathcal{L} u \) is a compact subset of \( X \) for each \( u \in X \).

Let \( (w_n)_{n \in \mathbb{N}} \) be a bounded sequence in \( S_{F,u} \). By (H3) the Nemitski operator \( \mathcal{F} \) of \( F \) is well defined, use and maps bounded sets into relatively compact, the sequence \( (v_n)_{n \in \mathbb{N}} \) given by, for each \( n \in \mathbb{N} \)

\[
v_n ( x, t ) = \int_0^t \int_0^\pi G ( x, t ; y, s ) w_n ( y, s ) dy ds
\]

\[+ \int_0^\pi G ( x, t ; y, 0 ) u_0 ( y ) dy\]

\[+ \int_0^t \partial_y G ( x, t ; 0, s ) \int_0^\pi g ( u ( y, s ) ) dy ds\]

\[− \int_0^t \partial_y G ( x, t ; \pi, s ) \int_0^\pi h ( u ( y, s ) ) dy ds.
\]

is relatively compact in \( \mathcal{L} u \). This implies that \( \mathcal{L} u \) is a compact subset of \( X \).

(c) We show that \( \mathcal{L} = G ( \mathcal{F} \cdot + u_0 ) + \gamma ( \cdot ) \) is a compact operator. To achieve this, we show that \( \mathcal{L} \) is uniformly bounded and maps bounded sets into equicontinuous sets.

Let \( B \) be a bounded subset of \( X \), and let \( u \in B \). Then there is \( M > 0 \) such that \( \| u \|_\infty \leq M \).

Now, for each \( v \in \mathcal{L} u \) there exists \( w \in S_{F,u} \) such that

\[
v ( x, t ) = \int_0^t \int_0^\pi G ( x, t ; y, s ) w ( y, s ) dy ds
\]

\[+ \int_0^\pi G ( x, t ; y, 0 ) u_0 ( y ) dy\]

\[+ \int_0^t \partial_y G ( x, t ; 0, s ) \int_0^\pi g ( u ( y, s ) ) dy ds\]

\[− \int_0^t \partial_y G ( x, t ; \pi, s ) \int_0^\pi h ( u ( y, s ) ) dy ds.
\]

Hence, if \( m_g \) and \( m_h \) denote the bounds on \( g \) and \( h \) respectively

\[
|v ( x, t )| \leq \int_0^t \int_0^\pi G ( x, t ; y, s ) |c_1 + c_2 | u ( y, s ) | dy ds
\]

\[+ \int_0^\pi G ( x, t ; y, 0 ) | u_0 ( y ) | dy\]

\[+ \pi m_g \max_D \int_0^1 | \partial_y G ( x, t ; 0, s ) | ds\]

\[+ \pi m_h \max_D \int_0^1 | \partial_y G ( x, t ; \pi, s ) | ds\]

\[\leq \pi ( c_1 + \| u_0 \|_\infty ) \| G \|_\infty
\]

\[+ \pi m_g \max_D \int_0^1 | \partial_y G ( x, t ; 0, s ) | ds\]

\[+ \pi m_h \max_D \int_0^1 | \partial_y G ( x, t ; \pi, s ) | ds\]

\[+ c_2 \int_0^1 \int_0^\pi G ( x, t ; y, s ) | u ( y, s ) | dy ds
\]

\[\leq M_0 + c_2 \int_0^1 \int_0^\pi G ( x, t ; y, s ) | u ( y, s ) | dy ds
\]
where

\[
M_0 = \pi \left( c_1 + \|u_0\|_\infty \right) \|G\|_\infty \\
+ \pi m_g \max_D \int_0^1 \left| \frac{\partial G}{\partial y} (x, t; 0, s) \right| ds \\
+ \pi m_h \max_D \int_0^1 \left| \frac{\partial G}{\partial y} (x, t; \pi, s) \right| ds.
\]

Then

\[
|v(x, t)| \\
\leq M_0 + \pi c_2 \|G\|_\infty \|u\|_\infty \cdot \leq M_0 + \pi c_2 \|G\|_\infty M.
\]

This shows that \(Lu \) is uniformly bounded.

Next, let \((x, t), (\xi, \tau) \in D\). Then

\[
|v(x, t) - v(\xi, \tau)| \leq (c_1 + c_2 M) \int_0^\pi \int_0^\pi |G(x, t; y, s) - G(\xi, \tau; y, s)| dyds \\
+ \|u_0\|_\infty \int_0^\pi G(x, t; y, 0) - G(\xi, \tau; y, 0)dy \\
+ \pi m_g \int_0^1 \left| \frac{\partial G}{\partial y} (x, t; 0, s) \right| ds \\
+ \pi m_h \int_0^1 \left| \frac{\partial G}{\partial y} (x, t; \pi, s) \right| ds.
\]

It follows from the properties of the Green’s function that, as \(|x - \xi| + |t - s| \to 0\), the right hand of the last inequality tends to zero. This shows that \(Lu \) is equicontinuous.

(d) Now, consider the set \(S = \{u \in X; \lambda u \in Lu, \text{ for some } \lambda > 1\}\). We show that this set is bounded.

We proceed as before to obtain

\[
|\lambda u(x, t)| \leq M_0 + c_2 \int_0^\pi \int_0^\pi G(x, t; y, s) |u(y, s)| dyds
\]

Since \(\lambda > 1\) it follows from the above inequalities that,

\[
|u(x, t)| \leq M_0 + c_2 \int_0^\pi \int_0^\pi G(x, t; y, s) |u(y, s)| dyds.
\]

Gronwall’s inequality implies

\[
\|u\|_\infty \leq M_0 \exp \left( \pi c_2 \|G\|_\infty \right).
\]

Therefore the set \(S\) is bounded and consequently, \(L\) has a fixed point in \(X\). This fixed point is the solution to our original problem. □

For our second result, we shall assume, in addition to (H1), that the following conditions are satisfied.

(H4) \(g\) and \(h\) are Lipschitz continuous, with Lipschitz constants \(k_g\) and \(k_h\) respectively, with \(\Delta < 1\), where

\[
\Delta = k_g \max_D \int_0^1 \left| \frac{\partial G}{\partial y} (x, t; 0, s) \right| ds \\
+ k_h \max_D \int_0^1 \left| \frac{\partial G}{\partial y} (x, t; \pi, s) \right| ds,
\]

and further \(g(0) = h(0) = 0\).

(H5) \(F\) has compact, convex values and there exists \(\Psi : [0, \infty) \to (0, \infty)\) continuous and nondecreasing such that \(|F(u)| \leq \Psi(|u|)\)

\[
\sup_{r \in (0, \infty)} \frac{1}{\pi} \frac{r (1 - \pi \Delta)}{\|G\|_\infty \left(\|u_0\|_\infty + \pi \Psi(r)\right)} > 1
\]

Theorem 9 If the conditions (H1), (H4), (H5) and (H6) are satisfied then Problem (1), (2), (3), (4) has at least one solution.

Proof. Condition (H6) implies that there exists \(r > 0\) such that

\[
\frac{r (1 - \pi \Delta)}{\pi \|G\|_\infty \left(\|u_0\|_\infty + \pi \Psi(r)\right)} > 1. \tag{14}
\]

Consider the closed ball \(\overline{B_r(0)}\) in the Banach space \(X\). Let \(u \in \overline{B_r(0)}\). Write \(Lu\) as \(L_1u + L_2u\), with

\[
L_1u(x, t) = \\
\int_0^t \frac{\partial G}{\partial y} (x, t; 0, s) \int_0^\pi g(u(y, s)) dy ds \\
- \int_0^t \frac{\partial G}{\partial y} (x, t; \pi, s) \int_0^\pi h(u(y, s)) dy ds,
\]

and

\[
L_2u(x, t) = \\
\int_0^t \int_0^\pi G(x, t; y, s) F(u(y, s)) dy ds \\
+ \int_0^\pi G(x, t; y, 0) u_0(y) dy.
\]

Claim 1. \(L_1 : \overline{B_r(0)} \to P_{cl,cv,b}(X)\) is a contraction.

Notice that \(L_1\) is a single valued operator. The continuity of the functions \(g\) and \(h\) implies that \(L_1u \in P_{cl,cv,b}(X)\).

Now, let \(u, v \in \overline{B_r(0)}\). Then
\[ |L_1 u(x, t) - L_1 v(x, t)| \leq \]
\[ \int_0^t \left| \frac{\partial g}{\partial y} (x, t; 0, s) \right| \int_0^\pi |g(u(y, s)) - g(v(y, s))| \, dy \, ds \]
\[ + \int_0^t \left| \frac{\partial g}{\partial y} (x, t; \pi, s) \right| \int_0^\pi |h(u(y, s)) - h(v(y, s))| \, dy \, ds. \]

Hence
\[ \|L_1 u - L_1 v\|_\infty \leq \]
\[ k_1 \int_0^t \left| \frac{\partial g}{\partial y} (x, t; 0, s) \right| \int_0^\pi |u(y, s) - v(y, s)| \, dy \, ds \]
\[ + k_2 \int_0^t \left| \frac{\partial g}{\partial y} (x, t; \pi, s) \right| \int_0^\pi |u(y, s) - v(y, s)| \, dy \, ds. \]

Condition (H4) implies that
\[ d_H(L_1 u, L_1 v) = \|L_1 u - L_1 v\|_\infty \leq \Delta \|u - v\|_\infty. \]

Since \( \Delta < 1 \) it follows (see Definition 5) that \( L_1 \) is a contraction. \( \square \)

**Claim 2.** \( L_2 : \overline{B_r(0)} \to P_{cp,cv}(X) \) is compact and use.

Let \( u \in \overline{B_r(0)} \). We proceed as in the proof of the previous theorem to show that \( L_2 u \) is a compact and convex subset of \( X \).

We show that \( L_2 \) is a compact operator on \( \overline{B_r(0)} \).

For each \( v \in L_2 u \), there exists \( w \in S_{F,u} \) such that for each \( (x, t) \in D \) we have
\[ v(x, t) = \int_0^t \int_0^\pi G(x, t; y, s) w(y, s) \, dy \, ds \]
\[ + \int_0^\pi G(x, t; y, 0) u_0(y) \, dy. \]

Condition (H5) implies that
\[ |v(x, t)| \leq \int_0^t \int_0^\pi G(x, t; y, s) |w(y, s)| \, dy \, ds \]
\[ + \int_0^\pi G(x, t; y, 0) |u_0(y)| \, dy \]
\[ \leq \int_0^t \int_0^\pi G(x, t; y, s) \Psi(|u(y, s)|) \, dy \, ds \]
\[ + \int_0^\pi G(x, t; y, 0) |u_0(y)| \, dy \]
\[ \leq \int_0^t \int_0^\pi G(x, t; y, s) \Psi\left(\|u\|_\infty\right) \, dy \, ds \]
\[ + \pi \|G\|_\infty \|u_0\|_\infty. \]

Thus,
\[ \|v\|_\infty \leq \pi \|G\|_\infty \left[ \Psi(r) + \|u_0\|_\infty \right]. \]

Next, we show that \( L_2 \) maps bounded sets into equicontinuous subsets of \( X \).

Let \( (x, t) \) and \( (\xi, \tau) \in D \). For each \( v \in L_2 u \) there is \( w \in S_{F,u} \) such that
\[ |w(y, s)| \leq \Psi(|u(y, s)|). \]

Thus,
\[ \int_0^t \int_0^\pi |G(x, t; y, s) - G(\xi, \tau; y, s)| |w(y, s)| \, dy \, ds \]
\[ \leq \Psi(r) \int_0^t \int_0^\pi |G(x, t; y, s) - G(\xi, \tau; y, s)| |w(y, s)| \, dy \, ds. \]

The continuity of the Green’s function implies that the right hand side of the above inequality tends to zero as \( |x - \xi| + |t - \tau| \) tends to zero. By the Ascoli-Arzelà theorem, we conclude that the operator \( L_2 \) is compact.

\( L_2 \) has a closed graph. Let \( (u_n, v_n) \in Gr(L_2) \) converge to \( (u, v) \). We must show that \( v \in L_2 u \). We have \( v_n \in L_2 u_n \), and there exists \( w \in S_{F,u} \) such that for each \( (x, t) \in D \)
\[ v_n(x, t) = \int_0^t \int_0^\pi G(x, t; y, s) w_n(y, s) \, dy \, ds \]
\[ + \int_0^\pi G(x, t; y, 0) u_0(y) \, dy. \]

Obviously,
\[ \|v_n - v\|_\infty \to 0 \text{ as } n \to \infty. \]

Consider the continuous operator \( \Gamma : X \to X \), defined by
\[ (\Gamma w)(x, t) = \int_0^t \int_0^\pi G(x, t; y, s) w(y, s) \, dy \, ds \]
\[ + \int_0^\pi G(x, t; y, 0) u_0(y) \, dy. \]

Then \( \Gamma \circ S_F \) has a closed graph (see [16, Theorem 2]). Also,
\[ v_n \in \Gamma \circ S_{F,u_n}. \]

Since \( u_n \to u \), uniformly, it follows that
\[ v \in \Gamma \circ S_{F,u}. \]

Hence, there exists \( w \in S_{F,u} \) such that
\[ v(x, t) = \int_0^t \int_0^\pi G(x, t; y, s) w(y, s) \, dy \, ds \]
\[ + \int_0^\pi G(x, t; y, 0) u_0(y) \, dy. \]
This shows that \( v \in \mathcal{L}_2u \), and hence \( \mathcal{L}_2 \) has a closed graph.

Since \( \mathcal{L}_2 \) has compact values, it follows that \( \mathcal{L}_2 \) is usc. □

Claim 3. The second alternative in Theorem 7 does not hold.

Suppose, on the contrary, that there exists \( u \in X \) with \( \|u\|_\infty = r \) and \( \lambda > 1 \) such that \( \lambda u \in \mathcal{L}_1u + \mathcal{L}_2u \). There exists \( z \in S_{F,u} \) such that

\[
\lambda u(x, t) = \int_0^t \int_0^\pi G(x, t; y, s)z(y, s) \, dy \, ds \\
+ \int_0^\pi G(x, t; y, 0) u_0(y) \, dy \\
+ \int_0^t \partial_G (x, t; 0, s) \int_0^\pi g(u(y, s)) \, dy \\
- \int_0^t \partial_G (x, t; s, \pi) \int_0^\pi h(u(y, s)) \, dy.
\]

Then, by (H4)

\[
\|u(x, t)\| \\
\leq \int_0^1 \int_0^\pi G(x, t; y, s) |z(y, s)| \, dy \\
+ \int_0^\pi G(x, t; y, 0) |u_0(y)| \, dy \\
+ \int_0^t \partial_G (x, t; 0, s) \int_0^\pi |g(u(y, s))| \, dy \\
+ \int_0^t \partial_G (x, t; \pi, s) \int_0^\pi |h(u(y, s))| \, dy \\
\leq \int_0^1 \int_0^\pi G(x, t; y, s) |\Psi(u(y, s))| \, dy \\
+ \pi \|G\|_\infty \|u_0\|_\infty \\
+ k_g \int_0^1 \partial_G (x, t; 0, s) \, ds \|u\|_\infty \\
+ \pi k_h \int_0^1 \partial_G (x, t; \pi, s) \, ds \|u\|_\infty \\
\leq \pi \Delta r + \pi \|G\|_\infty \|u_0\|_\infty \\
+ \int_0^1 \int_0^\pi G(x, t; y, s) |\Psi(u(y, s))| \, dy \\
\pi \Delta r + \pi \|G\|_\infty \|u_0\|_\infty + \pi \|G\|_\infty \Psi(r).
\]

This last inequality infer that

\[
r \leq \pi \Delta r + \pi \|G\|_\infty \|u_0\|_\infty + \pi \|G\|_\infty \Psi(r),
\]

which, in turn, implies that

\[
r (1 - \pi \Delta) \leq \pi (\|G\|_\infty \|u_0\|_\infty + \|G\|_\infty \Psi(r)).
\]

This contradicts the definition of \( r \) (see (14)). Therefore the first alternative holds, which means that \( u \in \mathcal{L}_1u + \mathcal{L}_2u \) has a solution in \( \overline{B}_r(0) \). This proves that our problem has at least one solution. □

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References


