Travelling wave solutions for a CBS equation in $2 + 1$ dimensions

MARIA LUZ GANDARIAS
University of Cádiz
Department of Mathematics
PO.BOX 40, 11510 Puerto Real, Cádiz
SPAIN
marialuz.gandarias@uca.es

MARIA SANTOS. BRUZÓN
University of Cádiz
Department of Mathematics
PO.BOX 40, 11510 Puerto Real, Cádiz
SPAIN
matematicas.casem@uca.es

Abstract: One of the more interesting solutions of the $(2 + 1)$-dimensional integrable Calogero-Bogoyavlenskii-Schiff (CBS) equation are the soliton solutions. We previously derived a complete group classification for the CBS in $(2 + 1)$-dimensions written as a system of partial differential equations. We now consider the classical Lie symmetries of the CBS equation written in a potential form. We obtain travelling-wave reductions with variable velocity depending on the form of an arbitrary function. The corresponding solutions of the $(2 + 1)$-dimensional equation involve arbitrary smooth functions. Consequently the solutions exhibit a rich variety of qualitative behaviours. Indeed by making adequate choices for the arbitrary functions, we exhibit periodic and solitary waves.

Key–Words: Symmetries, partial differential equation, exact solutions

1 Introduction

The study of higher-dimensional integrable systems is one of the main themes in integrability theory. Several models in the context of $(2 + 1)$-dimensional equations, i.e. equations with two spatial and one temporal variables, which are integrable have been developed by Toda and Yu [11]. These equations has been recently derived by using a method proposed by Calogero. That is, by modifying one of the operators of the Lax pair for $(1 + 1)$-dimension. In this way from the Calogero-Bogoyavlenskii-Schiff CBS equation they obtain

$$W_t + WW_z + \frac{1}{2} W_x \partial_x^{-1} W_z + \frac{1}{4} W_{xxz} = 0 \quad (1)$$

where $\partial_x^{-1} f = \int f \, dx$.

Although this $(2 + 1)$-dimensional CBS equation (1) arises in a non-local form, it can be written as the system of differential equations:

$$v_x - u_z = 0$$
$$2u_x v + u_{xxx} + 4u_t + 4uw_z = 0 \quad (2)$$

In a previous paper we applied the Lie group method of infinitesimals transformations to system (2). By using this method we bring out the unexplored invariance properties and similarity reduced systems of $(1 + 1)$ partial differential equations (PDE’s) of the above system (2). First we obtain a point transformation group which leaves the system (2) invariant. In order to find all invariant solutions with respect to $s$-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order $s$. The set of invariant solutions obtained in this way is called an optimal system of invariant solutions. We only consider one-parameter subgroups. For further details [9].

It was shown in [3] that the $(2 + 1)$-dimensional CBS equation which can be written in the potential form

$$4u_t x + 4u_x u_{xz} + 2u_{xx} u_z + u_{xxxz} = 0, \quad (3)$$

with $W = u_x$, admits a Lax representation and is integrable by the one-dimensional inverse scattering transform. In [3] Bogoyavlenskii proved that an equation equivalent to Eq. (3) has an overturning soliton

By using the classical Lie method, we derive exact solutions for the $(2 + 1)$-dimensional integrable potential breaking soliton equation (BS) (3), as well as for the $(2 + 1)$-dimensional integrable generalization of the CBS equation (1). Some of these solutions are soliton solutions, localized on a curve and that decays exponentially apart from the curve.

2 Lie symmetries.

In this section we perform Lie symmetry analysis for the $(2+1)$-dimensional CBS in potential form (3). Let us consider a one-parameter Lie group of infinitesimal
transformations in \((x, t, z, u)\) given by

\[
\begin{align*}
x^* &= x + \varepsilon \xi(x, z, t, u) + O(\varepsilon^2), \\
z^* &= z + \varepsilon \eta(x, z, t, u) + O(\varepsilon^2), \\
t^* &= t + \varepsilon \tau(x, z, t, u) + O(\varepsilon^2), \\
u^* &= u + \varepsilon \phi(x, z, t, u) + O(\varepsilon^2),
\end{align*}
\]  

(4)

where \(\varepsilon\) is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of the equation (3). This yields to the overdetermined, linear system of eighteen equations for the infinitesimals \(\xi(x, z, t, u), \eta(x, z, t, u), \tau(x, z, t, u)\) and \(\phi(x, z, t, u)\). The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

\[
v = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial z} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}.
\]  

(5)

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

\[
\Phi = \xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial z} + \tau \frac{\partial u}{\partial t} - \phi = 0.
\]  

(6)

Applying the classical method to the (3) yields a system of eighteen equations

\[
\begin{align*}
\xi_x &= 0, \\
\xi_u &= 0, \\
\eta_x &= 0, \\
\eta_u &= 0, \\
\tau_x &= 0, \\
\tau_z &= 0, \\
\tau_u &= 0, \\
\phi_{ux} &= 0, \\
\phi_{uz} &= 0, \\
\phi_{uu} &= 0, \\
\xi_{xx} &= 0, \\
\phi_{xx} &= 0, \\
\phi_t &= 0, \\
\phi_z - 2\xi_t &= 0, \\
\phi_x - \tau_z &= 0, \\
\eta_t - \tau_x - 2\xi_x &= 0, \\
\phi_{xz} + \phi_{tu} - \xi_{tx} &= 0, \\
\phi_u + \tau_t - \eta_z - \xi_x &= 0.
\end{align*}
\]  

(7)

By solving this system we get that \(\xi = \xi(x, t), \eta = \eta(z, t), \tau = \tau(t) \phi = \alpha(t) u + \beta(x, z, t)\) where \(\xi, \eta, \tau, \alpha \) and \(\beta\) must satisfy the following equations

\[
\begin{align*}
\beta_{tx} &= 0, \\
\beta_{xx} &= 0, \\
\beta_x - 2\xi_t &= 0, \\
-\xi_{tx} + \beta_{xz} + \alpha_t &= 0, \\
\xi_{xx} &= 0, \\
\beta_x - \tau_t &= 0, \\
\tau_t - \eta_z - 2\xi_x &= 0, \\
\tau_t - \eta_z - \xi_x + \alpha &= 0.
\end{align*}
\]  

(8)

(9)

Solving this system lead to a six-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the generators, these generators are :

\[
\begin{align*}
v_1 &= \frac{\partial}{\partial t}, \\
v_2 &= \frac{\partial}{\partial z}, \\
v_3 &= t \frac{\partial}{\partial z} + x \frac{\partial}{\partial u}, \\
v_4 &= z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t}, \\
v_5 &= t x \frac{\partial}{\partial x} + 2 t z \frac{\partial}{\partial z} + 2 t^2 \frac{\partial}{\partial t} + (2 x z - t u) \frac{\partial}{\partial u}, \\
v_6 &= x \frac{\partial}{\partial x} - 2 z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u},
\end{align*}
\]

and the infinite-dimensional

\[
\begin{align*}
v_\alpha &= \alpha(t) \frac{\partial}{\partial x} + 2 \alpha'(t) z \frac{\partial}{\partial u}, \\
v_\beta &= \beta(t) \frac{\partial}{\partial u}.
\end{align*}
\]

3 Optimal systems and reductions

In order to construct the one-dimensional optimal system, following Olver, we construct the commutator table and the adjoint table which shows the separate adjoint actions of each element in \(v_i, i = 1 \ldots 6\), as it acts on all other elements. This construction is done easily by summing the Lie series.

The corresponding generators of the optimal system
of subalgebras are

\[ a_1 v_1 + b_6 v_6, \]
\[ v_5, \]
\[ a_4 v_4 + b_6 v_6, \]
\[ a_1 v_1 + b_3 v_3, \]
\[ a_1 v_1 + b_2 v_2. \]

where \( a \in R \), and \( b \in R \) are arbitrary.

Our aim in this paper is to use the theory of symmetry reductions to find traveling-wave solutions for the \((2+1)\)-dimensional CBS equation.

In order to obtain these solutions, we consider the following reductions arising from translations and the infinite-dimensional vector field, i.e., \( v_1, v_2, v_\alpha \) and \( v_\beta \).

**Reduction 1** By using the generator \( v_1 + \lambda v_2 + v_\alpha + v_\beta \) we obtain the similarity variables and similarity solution

\[ z_1 = x - \int \alpha(t)dt, \quad z_2 = z - \lambda t, \tag{10} \]

\[ u = 2z_2 \alpha(t) + 2 \lambda \int (t \alpha'(t) - \beta(t))dt + h(z_1, z_2) \tag{11} \]

and the PDE \( E_1 \)

\[ 2h z_1 z_2 + h z_1 z_1 z_2 + 4h z_1 h z_2 - 4\lambda h z_1 z_2 = 0 \tag{12} \]

**Reduction 2** By using the generator \( v_2 + v_\alpha + v_\beta \) we obtain the similarity variables and similarity solution

\[ z_1 = x - z \int \alpha(t)dt, \quad z_2 = t, \tag{13} \]

\[ u = \alpha(t)z^2 + \beta(t)zdt + h(z_1 z_2) \tag{14} \]

and the PDE \( E_2 \)

\[ 2h z_1 z_1 \beta(z_2) + 4h z_1 z_2 = 0. \tag{15} \]

4 Symmetry reductions to ordinary differential equations (ODE’s)

The reduced PDE’s in \((1 + 1)\) variables admit symmetries which lead to further reductions to ODE’s, we shall use again the techniques of Lie group theory.

1. Equation \( E_1 \), admits the following symmetries

\[ v_{11} = \frac{\partial}{\partial z_1}, \]
\[ v_{12} = \frac{\partial}{\partial h}, \]
\[ v_{13} = z_1 \frac{\partial}{\partial z_1} + (2\lambda z_1 - h) \frac{\partial}{\partial h}, \]
\[ v_{\delta} = \delta(z_2) \frac{\partial}{\partial z_2}, \tag{16} \]

where \( \gamma(z_2) \) is an arbitrary function of \( z_2 \). By using \( v_{11} + v_{\gamma} \), we obtain the similarity variable and similarity solutions

\[ w = z_1 - k_2 \delta(z_2), \tag{17} \]
\[ h = k_3 \delta(z_2) + g(w), \]

where \( \delta(z_2) = \int \frac{dz_2}{\gamma(z_2)} \) and the autonomous ODE

\[ k_2 (g'' + 3(g')^2 - 4\lambda g') - 2k_3 g' - k_4 = 0. \tag{18} \]

Setting \( g' = y \), can be reduced to the following second order autonomous ODE

\[ k_2 (y'' + 3y^2 - 4\lambda y) - 2k_3 y - k_4 = 0. \tag{19} \]

By multiplying by \( 2y' \) and integrating once we get

\[ (y')^2 + 2y^2 - \left( \frac{2k_3}{k_4} + 4\lambda \right)y^2 - \frac{k_4}{k_2} y = 0. \tag{20} \]

Setting \( y = \theta^n \) we get

\[ (\theta')^2 = -\frac{2}{n^2} \theta^{n+2} + \left( \frac{2k_3}{k_2} + \frac{4\lambda}{n^2} \right) \theta^2 - \frac{k_4}{k_2 n^2} \theta^{2-n}. \tag{21} \]

For \( n = 2 \) we get

\[ (\theta')^2 = -\frac{1}{2} \theta^4 + \left( \frac{2k_3}{k_2} - \lambda \right) \theta^2 + \frac{k_4}{2k_2}. \tag{22} \]

For \( n = -2 \) we get

\[ (\theta')^2 = -\frac{1}{2} \theta^4 - \left( \frac{2k_3}{k_2} - \lambda \right) \theta^2 + \frac{k_4}{2k_2} \theta^4. \tag{23} \]
5 Some travelling wave solutions

In the following we present some explicit solutions of the second order ODE’s as well as the corresponding travelling solution of the (2 + 1) BS equation (3), as well as for the (2 + 1) CBS equation (1). Equations (22) and (23) can be integrated in terms of elliptic functions. Consequently, after making the change of variables \( y = \theta^n \), some exact solutions for (19) and (20) are

\[
y = \text{sn}^2(kw|p) \tag{24}
\]

with

\[
2k^2p + 1 = 0, \\
k_1 = 2k^2k_2, \\
k_3 = k_2(1 - 2k^2 - 2\lambda) \\
y = \text{sn}^{-2}(kw|p) \tag{26}
\]

with

\[
k_2(4p^2 + 2p) - 3 = 0, \\
k_4 = -3k_2(p^2 - 1), \\
k_3 = -k_2[(8\lambda - 9)p^2 + (4\lambda - 3)p + 6]/4p^2 + 2p \\
y = \text{cn}^{-2}(kw|p) \tag{28}
\]

with

\[
k_2(2p^2 - 2) - 1 = 0, \\
k_4 = -k_2p(2p - 1), \\
k_3 = -k_2[4p^2 + 2p + 4\lambda + 2)]/2p^2 - 2 \\
y = \text{dn}^{-2}(kw|p) \tag{30}
\]

with

\[
k_2(4p + 2) - 3p^2 = 0, \\
k_4 = 3k_2(p^2 - 1), \\
k_3 = -k_2[3p^3 + 3p^2 + (2\lambda - 6)p + (4\lambda - 6)]/4p + 2 \\
y = \text{dn}^{-2}(kw|p) \tag{32}
\]

with

\[
k_2^2(2p^2 - 2) + p^2 = 0, \\
k_4 = -2k_2(2p - 3), \\
k_3 = -k_2[p^3 + (4\lambda - 3)p^2 - 2p + 6 - 4\lambda)]/2p^2 - 2 \tag{35}
\]

Clearly any of the rational, hyperbolic or trigonometric degenerations of the Jacobi sn functions also give solutions. In particular, solitary waves result:

Setting in (24) \( k^2 = -\frac{1}{2p} \), and \( p = 1 \) we get

\[
y = -\tan^2\left(\frac{w}{\sqrt{2}}\right), \quad h = z - \sqrt{2}\tan\left(\frac{w}{\sqrt{2}}\right) + w \tag{36}
\]

By considering (36), and the corresponding symmetry reductions (11) and (17) we obtain that a solution for the breaking soliton equation and for the CBS equation in (2 + 1) dimensions can be written as

\[
W = -\frac{\sec^2\left(\frac{w}{\sqrt{2}}\right)}{3} + \frac{2d + 1}{3} \tag{37}
\]

\[
u = -\sqrt{2}\tan\left(\frac{w}{\sqrt{2}}\right) + \frac{2d + 1}{3}w + g(t) \tag{38}
\]

with

\[
w = (x - k_2\delta(z_2)), \quad z_2 = z - \lambda t, . \tag{39}
\]

In figure 1 we can see solution (37). \( \lambda = \frac{1}{2}, \delta(z - \lambda t) = (z - \frac{t}{2})^2 \) for \( t = 1 \).

![Figure 1: Solution (37) with \( \delta = (z - \frac{t}{2})^2, t = 1 \).](image)

Setting in (26) \( k^2 = -\frac{1}{2} \) and \( p = 0 \) we get

\[
y = -\cosech^2\left(\frac{w}{\sqrt{2}}\right), \quad h = \sqrt{2}\coth\left(\frac{w}{\sqrt{2}}\right). \tag{40}
\]

Setting in (26) \( k^2 = -\frac{1}{2} \) and \( p = 1 \) we get
We can see (44) with the CBS equation in can see (44) with the soliton.

In figure 3 we can see solution (43) and in figure 4 we can see solution (43) with $p = 1$ we get

$$y = \text{sech}^2\left(\frac{w}{\sqrt{2}}\right), \quad h = \sqrt{2}\tanh\left(\frac{w}{\sqrt{2}}\right).$$

By considering (42), and the corresponding symmetry reductions (11) and (17) we obtain that solutions for the breaking soliton equation BS and for the CBS equation in $(2+1)$ dimensions can be written as

$$W = \text{sech}^2\left(\frac{w}{\sqrt{2}}\right)$$

$$u = \sqrt{2}\tanh\left(\frac{w}{\sqrt{2}}\right).$$

In this work we have discussed symmetry reductions and similarity variables and similarity solutions for the $(2+1)$-dimensional integrable BS equation as well as for the $(2+1)$-dimensional integrable generalization of the CBS equation. By using the classical Lie method, we obtained solutions which have a rich variety of qualitative behaviours. This is due to the freedom in the choice of the arbitrary function $\delta(z - \lambda t)$.

We have obtained soliton solutions. Because of these arbitrary functions are included in the single soliton solution (43) the solution is localized on a curve and the curve may have quite a free form.

### 6 Conclusions

In this work we have discussed symmetry reductions and similarity variables and similarity solutions for the $(2+1)$-dimensional integrable BS equation as well as for the $(2+1)$-dimensional integrable generalization of the CBS equation. By using the classical Lie method, we obtained solutions which have a rich variety of qualitative behaviours. This is due to the freedom in the choice of the arbitrary function $\delta(z - \lambda t)$. We have obtained soliton solutions. Because of these arbitrary functions are included in the single soliton solution (43) the solution is localized on a curve and the curve may have quite a free form.

### References:


Figure 5: Curve soliton with $\delta = (z - \frac{t}{2})^2$, $t = 1$.

Figure 6: Soliton solution (43) $t=1, \delta = \sin(z - \frac{t}{2})^2$, $t = 1$.

Figure 7: Kink solution (44) with $\delta = \sin(z - \frac{t}{2})^2$, $t = 1, g(t) = 0$.

Figure 8: Solution (24) with $w = x - \delta, \delta = z - \frac{t}{2}$, $f = t$, $k = 1$, $p = -\frac{1}{2}$, $t = 1$.

Figure 9: Solution (24) with $w = x - \delta, \delta = \sin(z - \frac{t}{2})^2$, $f = t$, $k = 1$, $p = -\frac{1}{2}$, $t = 1$.


