Error estimates of weighted basis finite element method for convection dominated flow problems

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Abstract: In this paper, we investigate the weighted basis finite element method for solving convection dominated flow problems by applying different weights to different subregions. The stability of this finite element method is proved. An upper bound for convergence in the sense of the energy norm is obtained on the triangular Bakhvalov-type mesh. It is also shown that the error bound is almost independent of the small parameter.

Key-Words: Error estimate, Singular perturbation, Convection-diffusion equation, Finite element, Convergence

1 Introduction

In solving convection dominated flow problems, it is well known that the standard finite element methods or finite difference methods result in spurious oscillation because of the existence of boundary layers. To avoid non-physical numerical solutions, many special finite element techniques have been developed, including upwind finite element [1, 2], streamline diffusion methods [4], and exponentially fitted finite element [11]. However, some methods don’t always give accurate results, especially when diffusion coefficients have the same magnitude as that of mesh size. Li et al [5] presented a weighted basis finite element method. As the basis functions with weighted factors are consistent with the direction of flow and have exponentially fitted properties near the boundary layers, numerical solutions obtained by applying such a finite element method is non-oscillatory. In this work, a combination of the standard linear finite element method and the weighted basis finite element method is subsequently investigated for solving two-dimensional convection-dominated problems. In this approach, the solution domain is divided into two types of subregions, one contains boundary layers and another does not. The convection-diffusion equation is then discreted by the weighted finite element method on an anisotropic mesh for the first-type subregion and by the standard finite element method on a regular triangulation of the second-type subregion. In fact, the standard basis function can be regarded as a special case of the weighted basis function. Therefore, this combination of two methods remains a weighted basis finite element method, which merely uses different weights in different subregions. The resulting finite element space is still conforming. Unlike the mixed nonconforming method, a sophisticated technique is adopted to deal with the nonconforming problem. The stability of this finite element method is proved. Also an upper bound for convergence in the sense of the energy norm is established on the triangular Bakhvalov-type mesh, which is based on prior estimates to the solutions of 2-D singularly perturbed problems. Furthermore, it is shown that the error bound is almost independent of $\varepsilon$. Although the problem considered in this work is two-dimensional and linear, the idea can be extended to solve high-dimensional and nonlinear perturbed problems [6].

The rest of our paper is organized as follows. Section 2 describes the continuous problems. The weighted basis finite element is presented in Section 3. In Section 4, the stability is shown and the error estimate is established in the sense of energy norm.

2 weighted basis functions on the triangular mesh

We consider the following singularly perturbed problem with a small positive parameter $\varepsilon$ in two-dimensional space,

$$\nabla \cdot (-\varepsilon \nabla w + b(X)w) + \mu(X)w = f(X), \quad X \in \Omega,$$

$$w|_{\partial \Omega} = 0,$$

where $\mu(X) = \mu_0 + \mu_1 \varepsilon$.

The whole domain $\Omega$ is divided into two subregions $(\Omega = \Omega_1 \cup \Omega_2)$, the first-type subregion $\Omega_1$ contains perturbation layers while the second-type subregion $\Omega_2$ has an isotropic mesh. The problem on $\Omega_1$ is approximated using weighted basis functions whereas the problem on $\Omega_2$ is approximated using standard finite element method.
where $X = (x,y)^T$, $\Omega = (0,1) \times (0,1)$.

For the coefficient functions, we assume that $b(X) \in C^1(\Omega)$, $\mu(X) \in C(\Omega)$ and $f(X) \in L^\infty(\Omega)$. We also assume that $b(X)$ satisfies

$$
\frac{1}{2} \nabla \cdot b + \mu(X) \geq \alpha > 0, \quad X \in \Omega,
$$

where $\alpha$ is a positive constant. Although the existence and uniqueness of the solutions to both of the continuous problem and the finite element problem do not need this condition, it will be used in the definition of the energy norm and the proof of the error estimates. For simplicity, we also assume that two components of $b$ are bounded below by two positive constants $b_1, b_2$ such that

$$
b_1(X) \geq b_1 > 0, \quad b_2(X) \geq b_2 > 0, \quad \text{in } \Omega.
$$

In this case, the solution to (1) and (2) has two exponential boundary layers with width $O(\varepsilon)$ at boundaries $x = 1$ and $y = 1$. The variational problem corresponding to (1) and (2) is illustrated below.

**Problem 1** Find $w \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$
A(w,v) = (f,v),
$$

where $A(\cdot,\cdot)$ is a bilinear form on $(H^1(\Omega))^2$ defined by

$$
A(w,v) = (\varepsilon \nabla w - b w, \nabla v) + (\mu(X) w, v).
$$

Let $|| \cdot ||_e$ be the function defined on $H^1(\Omega)$ by

$$
||v||_e = [\varepsilon (\nabla v, \nabla v) + (\frac{1}{2} \nabla \cdot b + \mu(X)) v, v)]^{1/2}.
$$

It is easy to see that $|| \cdot ||_e$ is a norm on $H^1(\Omega)$ due to the condition (3). For any $v \in H^1(\Omega)$, we have

$$
A(v,v) = \varepsilon (\nabla v, \nabla v) + (\frac{1}{2} \nabla \cdot b + \mu(X)) v, v) = ||v||_e^2.
$$

This implies that bilinear function $A(\cdot,\cdot)$ is coercive on $H^1(\Omega)$. Then by the Lax-Milgram Lemma, Problem 1 has a unique solution in $H^1(\Omega)$.

For a triangle $T$ with vertices $X_i, X_j, X_k$ in the anti-clockwise direction, the standard linear basis functions satisfy

$$
\varphi_l(X_m) = \delta_{lm},
$$

where $\delta_{lm}$ is the Kronecker delta function. By virtue of the Bernoulli function, for any function $\tilde{b}(X)$ we can define the weighted factor $m_l(X)$ corresponding to $\varphi_l(X)$ as

$$
m_l(X) = B(-\tilde{b}(X - X_l)/\varepsilon).
$$

Using these weighted factors, one obtains weighted basis functions on $T$

$$
\tilde{\varphi}_l(X) = \frac{m_l(X) \varphi_l(X)}{\sum_{r=1,j,k} m_r(X) \varphi_r(X)}
$$

where $\tilde{\varphi}_l(l = i, j, k)$ has the same support as $\varphi_l$.

By the definition of $\tilde{\varphi}_l$, $\tilde{\varphi}_l (l = i, j, k)$ satisfy the following properties, (see [5]).

$$
\tilde{\varphi}_l(X_i) = \delta_{il}; \quad 0 \leq \tilde{\varphi}_i \leq 1,
$$

and on $T$

$$
\tilde{\varphi}_i + \tilde{\varphi}_j + \tilde{\varphi}_k = 1.
$$

For any smooth function $u$, we define a flux $\tilde{g}(u)$ corresponding to the function $\tilde{b}(X)$ as

$$
\tilde{g}(u) = -\varepsilon \nabla u + \tilde{b} u.
$$

As shown in [5], we can give the approximations $\tilde{g}_i$ to $\tilde{g}(\tilde{\varphi}_l)$ ($l = i, j, k$).

$$
\tilde{g}_i = \frac{\varepsilon}{2S_T} \begin{pmatrix}
    y-y_k & -(y-y_j) \\
    -(x-x_k) & x-x_j
\end{pmatrix}
\begin{pmatrix}
    B(\tilde{b}(X - X_j)/\varepsilon) \\
    B(\tilde{b}(X - X_k)/\varepsilon)
\end{pmatrix} \begin{pmatrix}
    \tilde{\varphi}_l(X)/\varphi_l(X)
\end{pmatrix},
$$

Similarly, $\tilde{g}_j$ and $\tilde{g}_k$ are defined. In the above definitions, $S_T$ is the measure of the element $T$. It can be shown that fluxes and their approximations satisfy the following [5]:

$$
\tilde{g}(\tilde{\varphi}_i) + \tilde{g}(\tilde{\varphi}_j) + \tilde{g}(\tilde{\varphi}_k) = \tilde{b}, \quad \tilde{g}_i + \tilde{g}_j + \tilde{g}_k = \tilde{b}.
$$

### 3 The Galerkin finite element formulation

Due to Condition (4), the solution to (1) with the boundary condition (2) has two boundary layers of width $O(\varepsilon)$ at $x = 1$ and $y = 1$, respectively.

Let
\[ \delta_1 = \frac{\beta}{2} \varepsilon \ln(1/\varepsilon), \quad \delta_2 = \frac{\beta}{2} \varepsilon \ln(1/\varepsilon), \]

where the constant \( \beta \) is greater than 2. As shown in Fig. 1, we divide the region \( \Omega \) into four subregions given respectively by

\[
\begin{align*}
\Omega_1 &= (0, 1 - \delta_1) \times (0, 1 - \delta_2), \\
\Omega_2 &= (0, 1 - \delta_1) \times (1 - \delta_2, 1), \\
\Omega_3 &= (1 - \delta_1, 1) \times (1 - \delta_2, 1), \\
\Omega_4 &= (1 - \delta_1, 1) \times (0, 1 - \delta_2).
\end{align*}
\]

The triangulation of \( \Omega \) is as follows. The subregions \( \Omega_1 \) and \( \Omega_2 \cup \Omega_3 \cup \Omega_4 \) are triangulated separately. We assume that the triangulation of \( \Omega_1 \) with the mesh size \( h \) is regular. To triangulate the L-shaped subregions \( \Omega_2 \cup \Omega_3 \cup \Omega_4 \), we first divide it into rectangles using lines parallel or perpendicular to one of the axes. Note that, in this partition, the \( y \)-coordinates of the latitude lines in \( \Omega_2 \) and the \( x \)-coordinates of the longitude lines in \( \Omega_4 \) are determined by the mesh nodes of the triangulation for \( \Omega_1 \) on the boundary segments \( \Omega_1 \cap \Omega_2 \) and \( \Omega_1 \cap \Omega_4 \). As shown in Fig. 2, each of the rectangles is then divided into two triangles by connecting the diagonals. The triangulations for \( \Omega_1 \) and \( \Omega_2 \cup \Omega_3 \cup \Omega_4 \) form the mesh \( T_h \) on \( \Omega \). This global triangulation satisfies that it is regular on \( \Omega_1 \) and \( \Omega_3 \), it contains long, thin triangles on \( \Omega_2 \) and \( \Omega_4 \). A typical case is displayed in Fig. 2. Moreover, the triangular refinement in boundary layers \( \Omega_2 \cup \Omega_3 \cup \Omega_4 \) must be of Bakhvalov-type such that the projection of the diameter of any triangle in \( \Omega_2 \cup \Omega_3 \) onto the \( y \)-direction is \( O(\varepsilon h \ln(1/\varepsilon)) \), and the projection of the diameter of any triangle in \( \Omega_3 \cup \Omega_4 \) onto the \( x \)-direction is \( O(\varepsilon h \ln(1/\varepsilon)) \).

As the width \( \delta_2 \) of \( \Omega_2 \cup \Omega_3 \) is defined in (13), the projection of the diameter of any triangle in \( \Omega_2 \cup \Omega_3 \) onto the \( y \)-direction is smaller than \( \delta_2 \). Similarly, the projections of the diameters of triangles in \( \Omega_4 \cup \Omega_3 \) onto the \( x \)-direction are smaller than \( \delta_1 \).

Although the weighted finite element method adopted in [5] can deal with boundary layers well, it costs more CPU time than the standard finite element method in smooth solution subregions. Then we give weights \( m_i(X) \) in (7) by choosing different \( \bar{b}(X) \) in the four subregions.

\[
\bar{b}(X) = 0, \quad \text{if } X \in \Omega_1, \\
\bar{b}(X) = (0, b_2(X))^t, \quad \text{if } X \in \Omega_2, \quad (14) \\
\bar{b}(X) = b(X), \quad \text{if } X \in \Omega_3, \\
\bar{b}(X) = (b_1(X), 0)^t, \quad \text{if } X \in \Omega_4.
\]

**Lemma 1** Basis functions (8) with the weights (14) are continuous.

**Proof:** If the node \( X_i \) is in subregions, it is easy to see that the basis function \( \bar{\varphi}_i(X) \) defined by (8) with the weights (14) is continuous. Therefore, only node \( X_i \) on the interface between different subregions is shown. Without loss of generality, we consider that \( X_i \) is on the interface between \( \Omega_2 \) and \( \Omega_3 \). Let triangle \( T_1 \) with vertices \( X_i, X_j, X_{k_1} \) and \( T_2 \) with vertices \( X_i, X_{k_2}, X_j \) belong to \( \Omega_2 \) and \( \Omega_3 \), respectively. The two triangles has a common edge \( X_iX_j \). From (8) and (14) we have

\[
\bar{\varphi}_i(X)|_{T_1} = \frac{B[b_2(y - y_1)/\varepsilon]\varphi_i}{\sum_{l=i,j,k_1} B[b_2(y - y_1)/\varepsilon]\varphi_l}
\]

and

\[
\bar{\varphi}_i(X)|_{T_2} = \frac{B[b(X - X_i)/\varepsilon]\varphi_i}{\sum_{l=i,j,k_2} B[b(X - X_l)/\varepsilon]\varphi_l}.
\]
Lemma 2

The flux \( g(\tilde{\varphi}_i) \) and its approximation \( \tilde{g}_l \) satisfy

\[
|g(\tilde{\varphi}_i) - \tilde{g}_l| \leq Ch, \quad l = i, j, k.
\]

Proof: If \( X_i \in \tilde{T}_1 \) and \( X \in \Omega_1 \), then \( \tilde{b} = 0 \) and \( \tilde{\varphi}_i \) is reduced to \( \varphi_i \). By computation we get \( \tilde{g}_l = -\varepsilon \nabla \varphi_i = -\varepsilon \nabla \tilde{\varphi}_i \). So \( g(\tilde{\varphi}_i) = \tilde{g}_l \), i.e. the inequality (17) holds.

If \( X_i \in \tilde{T}_3 \) and \( X \in \Omega_3 \), then \( b = 0 \) and \( g_l = \tilde{g}_l \). Following Theorem 4.3 in [5], we have \( |g(\tilde{\varphi}_i) - \tilde{g}_l| \leq Ch \).

Furthermore, for the case in \( \Omega_2 \), we have \( \tilde{b} = b \) and \( \tilde{g}_l = \tilde{g}_l \). By the definition of flux \( g(\tilde{\varphi}_i) \) and \( \tilde{g}_l \) in (16), we get

\[
|g(\tilde{\varphi}_i) - \tilde{g}_l| = |(-\varepsilon \nabla \tilde{\varphi}_i + b \tilde{\varphi}_i) - (\tilde{g}_l + \tilde{b} \tilde{\varphi}_i)| = |(-\varepsilon \nabla \tilde{\varphi}_i + \tilde{b} \varphi_i) - \tilde{g}_l| = |g(\tilde{\varphi}_i) - \tilde{g}_l|
\]

Again using Theorem 4.3 in [5], we get

\[
|g(\tilde{\varphi}_i) - \tilde{g}_l| \leq |g(\tilde{\varphi}_i) - \tilde{g}_l| \leq Ch.
\]

For the case in \( \Omega_4 \), the proof is similar.

Using (8) and (12), we can get

\[
g(\tilde{\varphi}_i) + g(\tilde{\varphi}_j) + g(\tilde{\varphi}_k) = b,
\]

and

\[
g_i + g_j + g_k = b.
\]

Let \( T_h \) denote a triangular mesh on \( \Omega \). The set of vertices of \( T_h \) not on \( \partial \Omega \) is denoted by \( \{X_i\}^N \). Corresponding to the partition \( T_h \), the finite element space is \( V_h = \text{span} \{\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_N\} \subset H^1_0(\Omega) \).

Problem 2. The finite element method for solving (1)-(2) is to find a \( w_h \in V_h \) such that for any \( v_h \in V_h \), we have

\[
A(w_h, v_h) = (g(w_h), \nabla v_h) + (\mu(X) w_h, v_h) = (f, v_h),
\]

where \( A(\cdot, \cdot) \) is the same bilinear function as in Problem 1.

Assumption 1. The solution \( w \) to (1)-(2) can be decomposed into four parts \( w_l(l = 1, 2, 3, 4) \), i.e.,

\[
w = w_1 + w_2 + w_3 + w_4,
\]

where \( w_1 \) satisfies

\[
||w_1||_{k, \Omega} \leq C \quad \text{for} \quad k = 0, 1, 2, \quad (21)
\]

and \( w_l(l = 2, 3, 4) \) satisfy

\[
\frac{\partial^{i+j} w_2}{\partial x^i \partial y^j} \leq \frac{C}{\varepsilon^j} \exp\left(\frac{b_3(y - 1)}{\varepsilon}\right), \quad (22)
\]

\[
\frac{\partial^{i+j} w_3}{\partial x^i \partial y^j} \leq \frac{C}{\varepsilon^j} \exp\left(\frac{b_4(x - 1) + b_2(y - 1)}{\varepsilon}\right),
\]

\[
\frac{\partial^{i+j} w_4}{\partial x^i \partial y^j} \leq \frac{C}{\varepsilon^j} \exp\left(\frac{b_4(x - 1)}{\varepsilon}\right),
\]

for \( 0 \leq i + j \leq 2 \) and the positive constant \( C \) is independent of \( \varepsilon \).

In the above assumption, \( w_1 \) is globally smooth and uniformly bounded in \( \Omega \), while \( w_l \) contain boundary layers in \( \Omega_l(i = 2, 3, 4) \). Sufficient conditions for the existence of this decomposition have been discussed in many literatures [3, 7, 8, 10, 11]. Let \( C \) be a positive constant independent of \( w \) and \( \varepsilon \). The following lemma shows that \( w \) and all its first and second partial derivatives are uniformly bounded in \( \Omega_1 \).

Lemma 3 If \( \beta \geq 2 \), then the solution \( w \) to (1)-(2) satisfies

\[
||w_l||_{k, \Omega_1} \leq C, \quad l = 1, 2, 3, 4, \quad k = 0, 1, 2, \quad (23)
\]

and

\[
||w||_{k, \Omega_1} \leq C, \quad k = 0, 1, 2. \quad (24)
\]

Proof: For a given \( \varepsilon > 0 \), To show (23) and (24), we only take \( \partial^2 w_2 / \partial y^2 \) as an example. By (22), we have
\[
\begin{align*}
|\frac{\partial^2 w}{\partial y^2}|_{\Omega_1} & \leq C \varepsilon^{-2} \exp\left(-\frac{b_0(1-y)}{\varepsilon}\right) \\
& \leq C \varepsilon^{-2} \exp\left(-\frac{b_0 \delta_2}{\varepsilon}\right) \leq C.
\end{align*}
\]

4 Error estimates

Let \( C \) be a generic positive constant independent of \( \varepsilon \) and \( h \). Let \( w_h \) denote \( w(X_i) \) and \( w_f \) the interpolation from \( V_h \) to the exact solution \( w \) of the problem (1)-(2), i.e.,

\[
w_f(X) = \sum_{l=1}^{N} w_l \bar{\varphi}_l.
\]

Then we have

\[
A(w - w_h, w - w_h) = A(w - w_h, w - w_f) + A(w - w_h, w_f - w_h).
\]

Furthermore, as \( w \) and \( w_h \) satisfy the variational problems (5) and (19) respectively, it is easy to show the following statement

\[
A(w - w_h, w_f - w_h) = 0,
\]

by noting the fact \( (w_f - w_h) \in V_h \). Then we only need to estimate \( A(w - w_h, w - w_f) \).

We have

\[
A(w - w_h, w - w_f) = \langle \varepsilon \nabla (w - w_h), \nabla (w - w_f) \rangle - \langle b(w - w_h), \nabla (w - w_f) \rangle + \langle \mu(X)(w - w_h), (w - w_f) \rangle
\]

\[
+ \langle \nabla (w - w_h), \nabla (w_f - w_h) \rangle + \langle b(w - w_f), (w - w_f) \rangle
\]

\[
\nabla (w - w_h) + \langle \mu(X) + \nabla \cdot b \rangle (w - w_h), (w - w_f) \rangle.
\]

**Lemma 4** The interpolation error \( w - w_f \) satisfies

\[
|| w - w_f ||^2_{\varepsilon} \leq C h.
\]

**Proof:** Considering \( (w - w_f) \in V_h \), we have

\[
|| w - w_f ||^2_{\varepsilon} = \langle \nabla [-\varepsilon \nabla (w - w_f) + b(w - w_f)] + \mu(X)(w - w_f), (w - w_f) \rangle
\]

\[
= \langle f, (w - w_f) \rangle - \langle \nabla \cdot g(w_f), (w - w_f) \rangle - \langle \mu(X) w_f, (w - w_f) \rangle
\]

where the flux \( g(w_f) \) is defined by (15). By Bramble-Hilbert lemma and the fact that \( f, \mu(X) w_f \) are continuous and uniformly bounded, one gets

\[
|| (f, (w - w_f)) ||_{\Omega_1} \leq Ch, \quad || (\mu(X) w_f, (w - w_f) ||_{\Omega_1} \leq Ch, \quad i = 1, 2, 3, 4.
\]

Furthermore, we get

\[
|| (\nabla \cdot g(w_f), (w - w_f)) ||_{\Omega_1} \leq Ch.
\]

The flux \( g(w_f) \) in \( \Omega_3 \) can be decomposed into two parts \( \bar{g}(w_f) \) and \( R(w_f) \). Due to equalities (18) and the fact that \( w_f \mid_T = w_i \bar{\varphi}_i + w_j \bar{\varphi}_j + w_k \bar{\varphi}_k \), \( \bar{g}(w_f) \) in \( T \) with vertices \( X_i, X_j \) and \( X_k \) can be written as

\[
\bar{g}(w_f) = w_i \bar{\varphi}_i + w_j \bar{\varphi}_j + w_k \bar{\varphi}_k
\]

The remainder \( R(w_f) = g(w_f) - \bar{g}(w_f) \). By Lemma 2, it satisfies

\[
\| R(w_f) \| \leq Ch.
\]

Combining (31) and (32), we have

\[
\begin{align*}
|| (\nabla \cdot g(w_f), (w - w_f)) ||_{\Omega_1} & \leq Ch. \\
& \leq Ch,
\end{align*}
\]

\[
R(w_f)(X_i) = 0, \quad l = i, j, k,
\]

\[
|| R(w_f) ||_{0, \Omega_3} \leq Ch.
\]

In the above deduction, we use the fact that \( || \nabla (w - w_f) ||_{0, \Omega_3} \) is bounded, see [7].

By Assumption 1, we have

\[
| w_l - w_i | = \left| \int_{X_i \cdot X_l} \frac{dw}{dx} \right| ds \leq Ch,
\]

where \( l = j, k \).
By computation, one can obtain that $\nabla \cdot (w_1 b_j), (w_j - w_i)\nabla \cdot (g_j)$ and $(w_k - w_i)\nabla \cdot (g_k)$ are bounded. Therefore,

$$|(\nabla \cdot g(w^I), (w - w^I))_{\Omega_1}| \leq Ch. \quad (33)$$

Similarly, we can show that

$$|(\nabla \cdot g(w^I), (w - w^I))_{\Omega_l}| \leq Ch, \quad l = 2, 4. \quad (34)$$

Combining (30), (33) and (34), we have

$$|(\nabla \cdot g(w^I), (w - w^I))|_{\Omega} \leq Ch. \quad (35)$$

Substituting (29) and (35) into (28), one obtains the estimate (27).

By Lemma 4, the error estimate for the first term in (26) can be rewritten

$$|\varepsilon(\nabla(w - w_h), \nabla(w - w^I))| \leq Ch + (\varepsilon/4)||w - w_h||_{L^2, \Omega}^2. \quad (36)$$

We continue the error analysis in (26) and turn to the convection term. Let $\Omega_1 = \Omega_2 \cup \Omega_3 \cup \Omega_4$.

$$|(b \nabla (w - w_h), (w - w^I))_{\Omega_1}| + |(b \nabla (w - w_h), (w - w^I))_{\Omega_2}|$$

By Assumption 1, one get

$$|(b \nabla (w - w_h), (w - w^I))_{\Omega_1}| \leq Ch. \quad (38)$$

Furthermore, we have

$$|(b \nabla (w - w_h), (w - w^I))_{\Omega_2}| \leq C ||w - w^I||_{L^\infty, \Omega_2} ||\nabla (w - w_h)||_{L^1, \Omega_2}$$

$$\leq C ||w - w^I||_{L^\infty, \Omega_2} (\varepsilon \ln \varepsilon^{-1})^{1/2} ||\nabla (w - w_h)||_{L^2, \Omega_2}$$

$$\leq C \varepsilon^{-1} ||w - w^I||_{L^\infty, \Omega_2}^2 + (\varepsilon/4)||\nabla (w - w_h)||_{L^2, \Omega_2}^2$$

$$\leq C \varepsilon^{-1} ||w - w^I||_{L^\infty, \Omega_2}^2 + (\varepsilon/4)||w - w_h||_{L^1, \Omega_2}^2 \quad (39)$$

To obtain the error estimate on the convection term, we need the following lemma

**Lemma 5** A function $w$ be the exact solution to (1) and $w^I$ be the interpolation of $w$ in $V_h$, then

$$||w - w^I||_{L^\infty, \Omega_2} \leq Ch. \quad (40)$$

**Proof:** Without loss of generality, we assume that $X$ belongs a triangle $T$ in $\Omega_2$. The proof for other cases are similar. Let $X_i, X_j$ and $X_k$ denote the three vertices of the triangle, we have $|w(X) - w^I(X)| \leq |w(X) - w_i| + |w_j - w^I(X)| + |w_k - w_i| + |w_j - w^I(X)| + |w_k - w_i| + |w_j - w^I(X)| + |w_k - w_i| + |w_j - w^I(X)| + |w_k - w_i|$. By properties of weighted basis functions, we know that $0 \leq \overline{\varphi}_l \leq 1(l = i, j, k)$. Therefore we have

$$|w(X) - w^I(X)| \leq |w(X) - w_i| + |w_j - w_i| + |w_k - w_i|.$$

As shown in Fig.3, we assume that $X, X_j$ is the horizontal edge in $T$ and $XX'$ is perpendicular to $X_iX_j$. Let $X'$ denote the point of intersection. By Assumption 1, we have

$$|w(X) - w_i| \leq |w(X) - w(X')| + |w(X') - w_i| \leq |\int_{X'X} (\partial w/\partial y) dy| + |\int_{X'X} (\partial w/\partial x) dx| \leq C||X'X|| (1/\varepsilon) + ||X_iX'|| \leq Ch.$$

Applying the above result to two special cases when $X = X_j$ and $X = X_k$, we obtain

$$|w_j - w_i| \leq Ch, \quad (43)$$

$$|w_k - w_i| \leq Ch. \quad (44)$$

Substituting (42), (43) and (44) into (41), we get

$$|w(X) - w_i| \leq Ch.$$
\begin{equation}
|\langle b \cdot \nabla (w - w_h), (w - w^f) \rangle_\Omega| \leq Ch + Ch^2 \ln(1/\varepsilon) + (\varepsilon/4)\|w - w_h\|_{1,\Omega}^2.
\end{equation}

Furthermore, combining (46), (36) with (26) and using the condition (3), we obtain

\begin{align*}
|A(w - w_h, w - w^f)| &\leq Ch + Ch^2 \ln(1/\varepsilon) + (\varepsilon/2)\|w - w_h\|_{1,\Omega}^2 \\
&\quad + |(\mu + \nabla \cdot b)(w - w_h), (w - w^f)| \\
&\leq Ch + Ch^2 \ln(1/\varepsilon) + (\varepsilon/2)\|w - w_h\|_{1,\Omega}^2 \\
&\quad + (\alpha/2)\|w - w_h\|_{0,\Omega}^2 + C\|w - w^f\|_{0,\Omega}^2 \\
&\leq Ch + Ch^2 \ln(1/\varepsilon) + 1/2\|w - w_h\|_\varepsilon^2 + Ch^2.
\end{align*}

Finally, by the equality (25) and the definition of energy norm \(\|\cdot\|_\varepsilon\), we obtain

\begin{equation}
(1/2)\|w - w_h\|_\varepsilon^2 \leq Ch(1 + h \ln(1/\varepsilon) + h).
\end{equation}

Based on the above discussion, we have

**Theorem 6** Let \(w\) and \(w_h\) be the solution to Problems 1 and 2, respectively. If \(w\) satisfies Assumption 1, the error estimate in energy norm is

\begin{equation}
\|w - w_h\|_\varepsilon \leq C h^{1/2} (1 + h \ln(1/\varepsilon) + h)^{1/2},
\end{equation}

where \(C\) is a constant independent of \(\varepsilon\) and \(h\).

In terms of computation, \(\ln(1/\varepsilon)\) can be approximately treated as a bounded amount, for example, \(\varepsilon = 10^{-15}\), \(\ln(1/\varepsilon) < 34.6\). Therefore, the above theorem implies that the error of \(w - w_h\) is almost uniformly bounded.

## 5 Conclusion

In this work, we consider a weighted basis finite element method on the triangular Bakhvalov-type mesh for the two-dimensional singularly perturbed problem. This method is based on choosing different weights in the smooth solution domain and the boundary layers. The convergent result is obtained and an error bound is given, which is almost independent of the diffusion coefficient \(\varepsilon\).

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## References


