Non-Linear Fluctuations of Interfaces by the Voter Model and Stopping Times

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Abstract: - Applying the theory of voter model and the theory of stopping time, we investigate the statistical properties of the fluctuations of interfaces model that defined from the voter model. We show that the probability distributions of the fluctuations, under some conditions, converge to the corresponding distribution of a geometric Brownian motion.

Key-Words: - Fluctuation; interface; voter model; stopping time; geometric Brownian motion

1 Introduction
In this paper, we consider the statistical properties of the fluctuations of interfaces that defined from the voter model. As the name might suggest that the voter model can model political systems, but rather the fact that the voter model is exactly the class of spin systems which duality can be applied most completely and successfully, the voter model is a continuous Markov process on \( \{0,1\}^\mathbb{Z} \), see [4]. This work originates in an attempt to describe the fluctuations of interfaces of the voter model, and study the convergence of the corresponding probability distributions. Recently, some research work has been done to study the statistical properties of the random interfaces for some statistical physics models, for example the two-dimensional Widom-Rowlinson model, see Refs. [2-6]. Their research work heavy depends on the theory of the cluster expansion and the partition functions expansion. But in the present paper, the stopping time methods will be used to study the fluctuations of the model.

First, we give the brief definitions and properties of the voter model, for details see [4]. One interpretation for the voter model is, for a collection of individuals, each of which has one of two possible positions on a political issue, at independent exponential times, an individual reassesses his view by choosing a neighbor at random with certain probabilities and then adopting his position. Specifically, the voter model is one of the statistical physics models, we think of the sites of the \( d \)-dimensional integer lattice as being occupied by persons who either in favor of or opposed to some issue. To write this as a set-valued process, we let \( \{\xi(s), s \geq 0\} \) the set of voters in favor, we can also think of the sites in \( \xi(s) \) as being occupied by cancer cells, and the other sites as being occupied by healthy cells. We can formulate the dynamics as follows: (i) An occupied site becomes vacant at a rate equal to the number of the vacant neighbors; (ii) An vacant site becomes occupied at a rate equal to \( \lambda \) times the number of the occupied neighbors, where \( \lambda \) is a intensity which is called the "carcinogenic advantage" in voter model. When \( \lambda = 1 \), the model is called the voter model, and when \( \lambda > 1 \), the model is called the biased voter model.

Next we introduce the graphical representation of the model, since the graphical representation is necessary for us to give a good description of the model. For simplicity, we give the construction of graphical representation for 1-dimensional voter model (\( \lambda = 1 \)), for more general cases, see [4]. Thinking of 1-dimensional integer points as being laid out on a horizontal axis, with the time lines being placed vertically, above that axis. Define independent Poisson processes with rate 1 for each time lines, at each event time \((x,s)\), we choose one of its two neighbors with probability 1/2, and draw an arrow from that neighbor point to \((x,s)\), and write a \( \delta \) at \((x,s)\). To construct the process from this "graphical representation", we imagine fluid entering the bottom at the points in \( \xi(0) \) and flowing up the structure. The \( \delta \)'s are the dams and the arrows are pipes which allow the fluid to flow in the indicated direction. Let \( \xi^A(s) (s \in I) \) denote the
2 Notations and Definitions

First we define the interfaces by the voter model on discrete time \( k \in \{1, \cdots, n\} \), suppose that the interfaces fluctuate at each time \( k \). For each \( k \in \{1, \cdots, n\} \), let \( \omega_k \) be random variable such that
\[
    P(\omega_k = +1) = p_k, \quad P(\omega_k = -1) = q_k, \quad P(\omega_k = 0) = r_k,
\]
where \( p_k + q_k + r_k = 1 \), and \( \omega_1, \omega_2, \cdots, \omega_n \) is an independent random sequence.

Let \( l_n \) denote a positive integer, and let \( \Lambda_{l_n} = [-l_n, l_n] \) be a subset of 1-dimensional lattice \( Z \). For a parameter \( \beta_k > 0 \), we define a function of interfaces by the voter model at time \( k \) by
\[
    A(\sigma_k) = \beta_k \omega_k \frac{\xi^{(0)}(s)}{|\Lambda_{l_n}|},
\]
where \(|\Lambda_{l_n}|, |\xi^{(0)}(s)|\) are the cardinality of \( \Lambda_{l_n} \) and \( \xi^{(0)}(s) \), \( s \in I \). Now we define the interfaces of the model by
\[
    G(k) = G(k-1) \exp \{A(\sigma_k)\},
\]
then for \( k \in \{1, 2, \cdots, n\} \)
\[
    G(k) = G(0) \exp \left\{ \sum_{i=1}^{k} A(\sigma_i) \right\}
    = G(0) \exp \left\{ \sum_{i=1}^{k} \beta_i \omega_i \frac{\xi^{(0)}(s)}{|\Lambda_{l_n}|} \right\}
\]
where \( G(0) \) is an initial state at time 0, and let \( A_k = \sum_{i=1}^{k} A(\sigma_i) \).

The interpretation for the interfaces model of (1) is, for example, for a collection of individuals, each of which has one of two possible positions on a political issue, and \( G(k) \) is the number that reflects this political issue at time \( k \). From (1), if \( \omega_k = +1 \), then
\[
    A(\sigma_k) = \beta_k \omega_k \frac{\xi^{(0)}(s)}{|\Lambda_{l_n}|} > 0,
\]
this means that the number of voters in favor at time \( k \) is more than that of voters in favor at time \( k-1 \), so it implies that \( G(k) \) increases; On the contrary, if \( \omega_k = -1 \), then \( G(k) \) decreases.

3 Convergence of the Fluctuations of Interfaces

In this section, first we introduce some results of the voter model (see [4]), then we define the stopping times for the interfaces model, at last we show the results of the present paper. From [4] we have the following Lemma 1. According to above Section 2 and Lemma 1, we can show the following Corollary 1.

Lemma 1 (1) If \( \lambda < \lambda_c \), there is a \( \rho > 0 \) such that
\[
    P(\xi^{(0)}(s) \neq \emptyset) \leq e^{-\rho s}
\]
then the process dies out exponentially fast; (2) If \( \lambda > \lambda_c \), then on \( \{\xi^{(0)}(s) \neq \emptyset, \text{ for all } s \geq 0\} \),
\[
    \frac{\xi^{(0)}(s)}{s} \rightarrow 2(\lambda - 1), \quad \text{a.s.}, \quad \text{as } s \rightarrow \infty.
\]

Corollary 1 For any \( \varepsilon > 0 \) and \( l_n \) large enough,
(1) If \( \lambda < \lambda_c \), for any fixed \( k \),
\[
    E \left( \frac{\xi^{(0)}(s)}{||\Lambda_{l_n}||} \right) < \varepsilon, \quad \text{as } s \rightarrow \infty.
\]
(2) If \( \lambda > \lambda_c \), for any fixed \( k \), there is a \( \rho > 0 \) such that, as \( s \rightarrow \infty \),
\[
    E \left( \frac{\xi^{(0)}(s)}{||\Lambda_{l_n}||} \right) \geq \rho,
    \quad E \left( \frac{\xi^{(0)}(s)}{||\Lambda_{l_n}||} \right)^2 \geq \rho.
\]

Next we define the stopping time for the interfaces model. Let \( \tau_1, \tau_2, \cdots, \tau_m, \cdots \), denote the stopping defined as followings
\[
    \tau_1 = \min \{k \geq 1; \ n \frac{1}{3} \sum_{i=1}^{k} A(\sigma_i) \geq 1\}, \cdots,
    \tau_m = \min \{k \geq 1; \ n \frac{1}{3} \sum_{i=1}^{k} A(\sigma_i) \geq 1\}, \cdots.
\]
For every stopping time intervals \([\tau_{m-1}, \tau_m]\), define a \(\lambda_m > 0\) on this time interval, such that for some \(0 < \alpha < 1\), if \(m \leq n^\alpha / 2\) then \(\lambda_m < \lambda_c\), if \(m > n^\alpha / 2\) then \(\lambda_m > \lambda_c\). Then we have the following results. For any fixed \(k\),

\[
E[A(\sigma_k)] = \beta_k (p_k - q_k) E\left( \frac{|\xi^{(0)}(s)|}{|\Lambda_k|} \right) \tag{3}
\]

\[
E[A(\sigma_k)^2] = \beta_k^2 (p_k + q_k) E\left( \frac{|\xi^{(0)}(s)|}{|\Lambda_k|} \right)^2 \tag{4}
\]

By Lemma 1, Corollary 1, (3) and (4), if \(\lambda < \lambda_c\) (or \(m \leq n^\alpha / 2\)), we can properly choose \(\beta_k, p_k, q_k\) , where \(k\) belongs to some time interval \([\tau_{m-1}, \tau_m]\), such that

\[
E[A(\sigma_k)] = E[A(\sigma_k)^2] = \frac{1}{\sqrt{n}} \tag{5}
\]

If \(\lambda > \lambda_c\) (or \(m > n^\alpha / 2\)), by Lemma 1 and Corollary 1, (3) and (4), we properly choose \(\beta_k, p_k, q_k\), where \(k\) belongs to some time interval \([\tau_{m-1}, \tau_m]\), such that

\[
E[A(\sigma_k)] = E[A(\sigma_k)^2] = c \tag{6}
\]

where \(c\) is a constant positive.

Taking the scaling limit of discrete time model of (1), we will obtain a continuous time process—the continuous time interfaces model, and we discuss the probability distribution of this continuous time model. Let

\[
0 < \nu < 1, \quad [n\nu] \in [1 + \tau_1, \cdots + \tau_{m-1}, \tau_1 + \cdots + \tau_m]
\]

where \([n\nu]\) is the integer part of \(n\nu\). Then \(m\) can be expressed by \(m = m(n, \nu)\), let

\[
A^n \nu = \frac{1}{\sqrt{n}} \sum_{k=1}^{\tau_1 + \cdots + \tau_{n\nu}} A(\sigma_k), \quad 0 < \nu < 1. \tag{7}
\]

Now we define the interfaces of the model in terms of above (7) by (see (1))

\[
G(n, \nu) = G(0) \exp\{A^n \nu\}, \quad 0 < \nu < 1. \tag{8}
\]

**Theorem 1** Suppose that the interfaces model follows (8), when \(n \to \infty\), the probability distribution of the process \(G(n, \nu)\) convergence to the corresponding distribution of

\[
G(0) \exp\{\int_0^\nu \mu(v)dv + \int_0^\nu \sigma(v)B(v)dv\}, \quad 0 < \nu < 1
\]

where \(B(v)\) is the one dimensional standard Brownian motion, \(\mu(v)\) is the trend function and \(\sigma(v)\) is the volatility function.

**Proof of Theorem 1.** In order to show the convergence of the distribution, we consider the convergence of the characteristic function of \(G(n, \nu)\), i.e.,

\[
\phi^n_i(z) = E[\exp\{izG(n, \nu)\}], \quad \text{as} \quad n \to \infty
\]

where \(i = \sqrt{-1}\). \(\phi^n_i(z)\) is divided into two terms as follows

\[
\phi^n_i(z) = E[\exp\{izG(n, \nu)\}], \quad |\tau_m - n^{5/6}| \leq n^{2/3+\epsilon},
\]

for all \(m = 1, \cdots, n^\alpha / 2\),

\[
+ E[\exp\{izG(n, \nu)\}], \quad |\tau_m - n^{5/6}| > n^{2/3+\epsilon},
\]

for some \(m = 1, \cdots, n^\alpha / 2\). \(\tag{9}\)

Next we define the conditional expectations,

\[
K_1 = E[\exp\{izG(n, \nu)\}], \quad |\tau_m - n^{5/6}| \leq n^{2/3+\epsilon}, \quad \text{for all} \quad m = 1, \cdots, n^\alpha / 2, \tag{10}\]

\[
K_2 = E[\exp\{izG(n, \nu)\}], \quad |\tau_m - n^{5/6}| > n^{2/3+\epsilon}, \quad \text{for some} \quad m = 1, \cdots, n^\alpha / 2. \tag{11}\]

(I) Now we estimate the second term \(K_2\). Let

\[
1/12 < \epsilon < 1/6 \quad \text{and} \quad A_k = \sum_{i=1}^{m_i} A(\sigma_i), \quad \text{then}
\]

\[
P(\tau_m - n^{5/6} > n^{2/3+\epsilon}) = P(\tau_m > n^{2/3+\epsilon} + n^{5/6}) + P(\tau_m < n^{5/6} - n^{2/3+\epsilon}) = P(A_{n^{5/6}+n^{2/3+\epsilon}} \leq n^{1/3}) + P(A_{n^{5/6}-n^{2/3+\epsilon}} \geq n^{1/3}) = P(A_{n^{5/6}+n^{2/3+\epsilon}} - E[A_{n^{5/6}+n^{2/3+\epsilon}}] \leq n^{1/6+\epsilon})
\]

\[
+ P((A_{n^{5/6}+n^{2/3+\epsilon}} - E[A_{n^{5/6}+n^{2/3+\epsilon}}]) \geq n^{1/6+\epsilon}) \leq n^{1/3}(n^{1/3} + n^{1/6+\epsilon}) + n^{1/3}(n^{1/3} + n^{1/6+\epsilon}) / n^{1/3+2\epsilon} = 2 / n^{2\epsilon}. \tag{12}\]

By (11) and (12), when \(\alpha = 1/6\), then we have

\[
K_2 \leq 2n^\alpha / n^{2\epsilon}, \quad \text{so that}
\]

\[
K_2 \to 0, \quad \text{as} \quad n \to \infty.
\]

This implies that the length of stopping time \(\tau_m\) is about \(n^{5/6}\).

(II) From above discussion, for \(\alpha = 1/6\), \((n^\alpha / 2) \times n^{5/6} = n / 2\). So, when \(k \leq n / 2, \lambda < \lambda_c\); when \(k > n / 2, \lambda > \lambda_c\).
(a) If $m \leq n^\alpha / 2$, and $k \leq n / 2$, by (5) we have
\[
E[\exp\{\frac{iz}{\sqrt{n}} A(\sigma_k)\}] = 1 + iz \frac{E[A(\sigma_k)]}{n} - \frac{z^2}{2n} E[A(\sigma_k)^2] + o\left(\frac{1}{n^{3/2}}\right)
\]
(b) If $m > n^\alpha / 2$, and $k > n / 2$, by (6) we have
\[
E[\exp\{\frac{iz}{\sqrt{n}} A(\sigma_k)\}] = 1 + iz \frac{E[A(\sigma_k)]}{n} + o\left(\frac{1}{n^{3/2}}\right)
\]

(III) We estimate the first term $K_1$ in two parts.

(a) If $0 < \nu < 1 / 2 \ (k \leq n / 2)$, so $m(n, \nu) = \lfloor n^{1/6} \nu \rfloor$, by (13) we have
\[
K_1 = \sum_{\lfloor n^{1/6} \nu \rfloor}^{[n^\alpha / 2]} E[\exp\{izG(n, \nu)\} | \tau_m = r_m \ , \ \text{for all} \ m = 1, \ldots, n^\alpha / 2 \],
\]
\[
= \sum_{\lfloor n^{1/6} \nu \rfloor}^{[n^\alpha / 2]} \prod_{m=1}^{[n^\alpha / 2]} E[\exp\{\frac{iz}{\sqrt{n}} A(\sigma_k)\}]^{m(n, \nu)}
\]
\[
= \prod_{m=1}^{[n^\alpha / 2]} \left[1 + iz \frac{E[A(\sigma_k)]}{n} + o\left(\frac{1}{n^{3/2}}\right)\right]^{m(n, \nu)}
\]
\[
= \prod_{m=1}^{[n^\alpha / 2]} \exp\left[\sum_{m=1}^{[n^\alpha / 2]} r_m \ln\left(1 + iz \frac{E[A(\sigma_k)]}{n} + o\left(\frac{1}{n^{3/2}}\right)\right)\right]
\]
\[
= \prod_{m=1}^{[n^\alpha / 2]} \exp[iz \nu + o\left(\frac{1}{n^{1/6}}\right)].
\]

On the other hand, by (12) we have
\[
P\left(\tau_m = n^{5/6} | n^{2+3/\nu}, m = 1, \ldots, n^\alpha / 2\right)
\]
\[
= \prod_{m=1}^{[n^\alpha / 2]} P\left(\tau_m = n^{5/6} | n^{2+3/\nu}\right)
\]
\[
= \prod_{m=1}^{[n^\alpha / 2]} (1 - n^{-2\varepsilon}) = (1 - n^{-2\varepsilon})^{n^\alpha / 2}.
\]

So for $\alpha = 1 / 6$ and $1 / 12 < \varepsilon < 1 / 6$, then $2\varepsilon > \alpha$, so that, as $n \to \infty$,
\[
\ln(1 - n^{-2\varepsilon})^{n^\alpha / 2} \approx \frac{1}{2} n^\alpha \frac{1}{n^{2\varepsilon}} = \frac{n^\alpha}{2n^{2\varepsilon}} \to 0.
\]

Then we have
\[
\lim_{n \to \infty} P\left(\tau_m = n^{5/6} | n^{2+3/\nu}, m = 1, \ldots, n^\alpha / 2\right) = 1. \quad (17)
\]

Combining (9)-(11) and (15)-(17), if $0 < \nu < 1 / 2$, then we have
\[
\lim_{n \to \infty} \phi^\nu(\varepsilon) = \lim_{n \to \infty} \exp[iz \nu + o\left(\frac{1}{n^{1/6-\varepsilon}}\right)]
\]
\[
= \exp\{iz \nu\}. \quad (18)
\]

(b) If $1 / 2 \leq \nu < 1 \ (k > n / 2)$, following the similar procedure of above (a), and by (13)(14), we have
\[
E[\exp\{izG(n, \nu)\}] \tau_m = r_m, \ \text{for all} \ m = 1, \ldots, n^\alpha / 2 \]
\[
= \prod_{m=1}^{[n^1/2]} E[\exp\{\frac{iz}{\sqrt{n}} A(\sigma_k)\}]^{m(n, \nu)} \times \exp\{iz \nu + o\left(\frac{1}{n^{1/6}}\right)\}
\]
\[
= \exp\left[iz \nu + o\left(\frac{1}{n^{1/6}}\right)\right].
\]

Then we have, for $1 / 2 \leq \nu < 1$
\[
\lim_{n \to \infty} \phi^\nu(\varepsilon) = \lim_{n \to \infty} \exp[iz \nu + o\left(\frac{1}{n^{1/6-\varepsilon}}\right)]
\]
\[
= \exp[iz \nu + o\left(\frac{1}{n^{1/6-\varepsilon}}\right)]. \quad (19)
\]

(IV) Combining (18) and (19), for $0 < \nu < 1$, we have
\[
\lim_{n \to \infty} \phi^\nu(\varepsilon) = \lim_{n \to \infty} E[\exp\{izG(n, \nu)\}]
\]
\[
= \exp[iz \nu + o\left(\frac{1}{n^{1/6}}\right)]. \quad (20)
\]

where $\nu = 1$, and $\sigma^2(\nu) = 0$ if $0 < \nu < 1 / 2$, $\sigma^2(\nu) = c$ if $1 / 2 \leq \nu < 1$.

By Refs. [1], above (20) shows that the probability distributions of interfaces model of the present paper converge to the corresponding distributions of a geometric Brownian motion, this completes the proof of Theorem 1.

Remark 1 The proof of Theorem 1 can be extended to more complicated interfaces model. For example,
the time $s$ in the voter model $\xi^d(s) \ (s \in I)$, can be defined to be an independent random sequence $\{s_k(\sigma)\} (k = 1, \cdots, n)$, then following the similarly proof methods, we can obtain the different trend function $\mu(v)$ and the different volatility function $\sigma(v)$.

4 Conclusion
In this paper, we studied the statistical properties of the fluctuations of interfaces model given by the voter model and stopping times. Theorem 1 shows that the probability distributions of the fluctuations of interfaces model converge to the corresponding distributions of a geometric Brownian motion.

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References: