Non-Linear Systems of Interfaces of Statistical Mechanics Models with a Fixed Intermediate Region

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Abstract: - We consider the fluctuations of the two interfaces' problems of the one-dimensional statistical mechanics models. The relations of the two random interfaces are constructed by assuming that there is a specified value of the large area in the intermediate region of the two random interfaces, and the two random interfaces have fixed endpoints. When $\beta$ large enough, we discuss the statistical limiting properties of the two random interfaces of the model.

Key-Words: - random paths; interfaces; central limit theory; probability measure

1 Introduction

In recent years, some research work has been done to investigate the statistical properties of the random interfaces for some statistical physics models, for example see Refs. [2-6]. In this paper, we consider the statistical properties of the two random paths model. This work originates in an attempt to describe the fluctuations of the interfaces in random interfaces models (e.g. one-dimensional two random interfaces S.O.S. model). In Ref. [3], the statistical properties of random walks and the interface of Widom-Rowlinson model (conditioned by fixing a large area under their paths and conditioned by fixing the terminating point) are considered, and the central limit theorem for these conditional distributions is proved. The similar problems arise in describing the fluctuations of two random interfaces models. In the first part of the present paper, with the conditions 'fixed area' of the intermediate layer and 'fixed end points' in a two random paths model, we study the limiting properties of the two random paths.

At each site $x$ of the one dimensional lattice $Z$, we attach two variables of 'heights' $\omega_x^1, \omega_x^2 \in Z$, therefore the configurations of the random paths model on a horizontal set $L_X = \{x_0, x_0 + 1, \ldots, x_0 + L\} \subset Z$ (with the length of $L$) are represented by sets of heights $\{\omega_x^1, \omega_x^2\}_{x \in L_X}$, for the simplicity, we assume $x_0 = 0$. In this paper, we study a two random paths model, the Hamiltonian of the model on the horizontal set of $L_X$ is given by

$$H_L(\omega^1, \omega^2) = \sum_{\{x\} \times y \in L_X} \left( |\omega_x^1 - \omega_y^1|^\alpha + |\omega_x^2 - \omega_y^2|^\alpha \right)$$

where the constants $\alpha, \alpha > 0$, and the sum extends over nearest-neighbor pairs in $L_X$. For the two random paths models, let $\Omega_L = \{ (\omega^1, \omega^2) : \omega_x^1, \omega_x^2 \in Z, x \in L_X \}$ be the corresponding configuration space. The partition function of this system is

$$Z_{L,\beta} = \sum_{\{\omega^1, \omega^2\} \in \Omega_L} \exp[- \beta H_L(\omega^1, \omega^2)]$$

where $\beta$ is a positive parameter called an inverse temperature. The corresponding Gibbs probability distribution on $\Omega_L$ is given by

$$P_{L,\beta}(\omega^1, \omega^2) = \left(Z_{L,\beta}\right)^{-1} \exp[- \beta H_L(\omega^1, \omega^2)]$$

Next we define the paths of the two random paths model as followings, for $t \in [0,1]$

$$X_L^\omega(j) = \omega_j, j \in L_X$$

$$X_L^\omega(t) = \left( (j+1-Lt)X_L^\omega\left(\frac{j}{L}\right) + (Lt-j)X_L^\omega\left(\frac{j+1}{L}\right) \right),$$

$$j \leq Lt \leq j + 1$$

and $X_L^\omega\left(\frac{j}{L}\right), X_L^\omega(t)$ are defined similarly as
above definitions.

For \( x \in L_x \) and \( x \geq 1 \), let \( \xi_x = \omega_x^1 - \omega_{x-1}^1 \), \( \eta_x = \omega_x^2 - \omega_{x-1}^2 \), so we have \( \omega_x^1 = \sum_{i=0}^{x} \xi_x \) and \( \omega_x^2 = \sum_{i=0}^{x} \eta_x \), where let \( \xi_0 = 0 \), \( \eta_0 = 0 \). Let \( \xi = \{\xi_x, x \in L_x\} \), \( \eta = \{\eta_x, x \in L_x\} \), then rewrite above partition function as
\[
Z_{L, \beta} = \sum_{\xi, \eta} \exp\left[-\beta H_L(\xi, \eta)\right]
\]
where \( H_L(\xi, \eta) \) is the Hamiltonian function for \((\xi, \eta)\). Then we have the corresponding Gibbs probability distribution \( P_{L, \beta}(\xi, \eta) \) for the partition function \( Z_{L, \beta} \). Thus we have the corresponding paths \( X^x_L \left( \frac{j}{L} \right) \), \( X^\eta_L \left( \frac{j}{L} \right) \), \( X^\eta_L \left( \frac{j}{L} \right) \).

From above definitions, \( \xi = \{\xi_x, x \in L_x\} \) and \( \eta = \{\eta_x, x \in L_x\} \) can be seen as the sequences of i.i.d. random variables respectively. So, the two random paths model has two independent random \( \text{SOS} \) paths, that is, the model corresponds to the ensemble of two independent self-avoiding paths in \([0, L] \times Z \) starting from \((0, 0)\) and ending at sites \( z \) in the line \( \{x = L\} \) (where \( z = (x, y) \)), which do not go back in the horizontal direction. Next we introduce the generating function of the height of the endpoints for one 'step', that is, for a fixed \( x \in L_x \), let
\[
Q(\mu, \nu) = \sum_{\xi, \eta} e^{\beta \mu \xi + \beta \nu \eta} / \sum_{\xi, \eta} \exp\left[-\beta H_L(\xi, \eta)\right]
\]
where \( Q(\mu, \nu) \) is independent of \( x \) and \(-\infty < \xi, \eta_x < +\infty\). Due to the independence of the random variables \( \{\xi_x, x \in L_x\} \) and \( \{\eta_x, x \in L_x\} \), thus
\[
Q(\mu, \nu)^L = \sum_{\xi, \eta} \exp\left[\beta \mu \bar{\xi} + \beta \nu \bar{\eta}\right] / Z_{L, \beta}
\]
where \( \bar{\xi} = \sum_{x=1}^{L} \xi_x \) and \( \bar{\eta} = \sum_{x=1}^{L} \eta_x \). For \((\mu, \nu) \in \mathbb{R} \times \mathbb{R} \), we define
\[
\varphi(\mu, \nu) = \lim_{L \to 0} \frac{1}{L}
\]
ln \( \sum_{\xi, \eta} \exp[\beta \mu \bar{\xi} + \beta \nu \bar{\eta}] \exp[-\beta H_L(\xi, \eta)] / Z_{L, \beta} \)
by the Refs. [3][5], it is known that the limit exists if \((\mu, \nu)\) is in some neighborhood of the origin.

The aim of this paper is to study the asymptotes of fluctuations of the two random paths conditioned by fixing a large area between the two random paths. Denote by \( a_{L}^x, a_x^2 \) representing the areas under the paths \( X^x_L \left( \frac{j}{L} \right) \), \( X^\eta_L \left( \frac{j}{L} \right) \), \( X^\eta_L \left( \frac{j}{L} \right) \), respectively, and denote by \( a_{L}^{\eta \Rightarrow} = a_x^2 - a_x^\eta \) representing the area of the intermediate layer between the two random paths. For a real \( \zeta_0 \) and \( 0 \leq s \leq 1 \), assume that
\[
F(\zeta_0, \beta, s) = \frac{d}{ds} \varphi(1-s, 1-s) \left|_{\zeta_0 = 0}^{\zeta_0 = \zeta_0} \right. ,
\]
where \( a > 0 \) is some constant. Above (1) is important condition, we will use this condition to fulfill our proof in the followings. Then we state the main results of this paper.

**Theorem 1** Assume that for some \( \delta(\beta) > 0 \) and \( a > 0 \), there exists a real \( \zeta_0 \) satisfying above condition (1) and \( |\zeta_0| < \delta(\beta) \), then the process
\[
Y_L(t) = \frac{1}{\sqrt{L}} \left\{ X^\eta_L(t) - X^x_L(t) - \frac{L}{\beta} \int_0^t F(\zeta_0, \beta, s) ds \right\}
\]
under \( P_{L, \beta}(\left[ aL \right]) \), converges weakly to the process
\[
Y(t) = \frac{1}{\beta} \int_0^t \sqrt{\varphi''(-1-s, 1-s)} dB(s)
\]
conditioned that \( \int_0^t Y(t) dt = 0 \), where \( \{B(s)\}_{s \geq 0} \) is the one dimensional standard Brownian motion, and \( \left[ aL \right] \) is the integer part of \( aL \).

**Remark 1** In Theorem 1, the model is only conditioned by fixing a large area between the two random paths and having the same starting endpoints. The results can also be proved similarly for the two random paths with fixed value of area and the two same endpoints.
Theorem 2  Let \( \varphi' (\mu, \nu) = \frac{\partial}{\partial \nu} \varphi (\mu, \nu) \), and
\[
F_\mu (x, \beta, s) = -\varphi' (-s) \xi_0 (1-s) \xi_0 .
\]
With the same conditions of Theorem 1, the probability distribution of the random process \(-X^L_\xi (t)/L\), under \( P_{L,\beta} (a) \), converges weakly to the corresponding probability distribution concentrated on the function
\[
Y_\beta (t) = \frac{1}{\beta} \int \frac{F_\mu (\xi_0, \beta, s) ds}{L^{a_t}}
\]

2  Estimation for the Fluctuations of the Two Random Paths Model

In this section, we begin discussing the area between the two random paths. Then we show the some results about the weak convergence of random vector of the two random interfaces for the model. Now we define the areas of \( a^L_1, a^L_2, a^L_3 \) as followings,
\[
a^L_1 = \sum_{x=1}^{L} \omega x / L = \sum_{x=1}^{L} (1-x/L) \xi x,
\]
\[
a^L_2 = \sum_{x=1}^{L} \omega x / L = \sum_{x=1}^{L} (1-x/L) \eta x,
\]
\[
a^L_{\eta-x} = a^L_2 - a^L_1 = \sum_{x=1}^{L} (1-x/L) (\eta x - \xi x)
\]
By the independence of \( \{ \xi x, x \in L_x \} \) and \( \{ \eta x, x \in L_x \} \), the generation function of the area \( a^L_{\eta-x} \) is defined by
\[
Q_{a^L_{\eta-x}} (\zeta) = \sum_{\eta} \exp \{ \beta \zeta a^L_{\eta-x} \} \exp \{ -\beta H_\xi (\zeta, \eta) \} / Z_{L,\beta}
\]
\[
= \prod_{x=1}^{L} Q (\zeta (1-x/L), \zeta (1-x/L))
\]

Let \( q \) be a natural number, and let \( \{ t_i, 1 \leq i \leq q \} \) be any set of real numbers, such that \( 0 < t_1 < \ldots < t_q \leq 1 \).

Set a random vector as
\[
\hat{X}^{(q)} (t_1, \ldots, t_q) = (a^L_{\eta-x}, a^L_{\eta-x}^2 - a^L_{\eta-x}, \ldots, a^L_{\eta-x}^q - a^L_{\eta-x}^q)
\]
Then for \( \zeta = (\zeta_0, \zeta_1, \ldots, \zeta_q) \in \mathbb{R}^{q+1} \), we have
\[
\sum_{\eta} e^{\beta \zeta \hat{X}_L^{(q)} (t_1, \ldots, t_q)} e^{-\beta H_\xi (\zeta, \eta)} / Z_{L,\beta}
\]
\[
\prod_{x=1}^{L} Q (\zeta (1-x/L), \zeta (1-x/L))
\]
where \( \zeta_L (x; \zeta) = \zeta_0 (1-x/L) + \sum_{x=1}^{q} \zeta_x 1_{(x,L]} (x) \).

For the real \( \zeta_0 \) defined in (1) and some small constant \( \alpha > 0 \), let \( \zeta \in \mathbb{R}^{q+1} \) satisfy the following conditions
\[
D_{\alpha,\zeta_0} = \{ \zeta : -\alpha < \zeta < \zeta_0 + \alpha, |\zeta| < \alpha, i = 1,\ldots,q \}
\]
Next we introduce the corresponding quadratic form, a \((q+1) \times (q+1)\) matrix denote by
\[
V (\zeta) = \frac{1}{\beta^2} \text{Hess} \int \prod_{x=1}^{L} Q (\zeta (1-x/L), \zeta (1-x/L)) ds
\]
and \( \zeta (s) = \zeta_0 (1-s) \sum_{x=1}^{q} \zeta_x 1_{(x,L]} (s) \), for \( 0 \leq s \leq 1 \).
Let \( \hat{P}^{(q)} (\zeta) \) be the probability distribution of \( \hat{X}^{(q)} (t_1, \ldots, t_q) \) under \( P_{L,\beta} \), and \( \hat{P}^{(q)} (\zeta) \) be given by
\[
\hat{P}^{(q)} (\zeta) = e^{\beta \zeta \hat{X}^{(q)} (\zeta)} / E_{L,\beta} \left( e^{\beta \zeta \hat{X}^{(q)} (\zeta)} \right)
\]
for all \( \zeta \in D_{\alpha,\zeta_0} \) and \( \zeta \in \mathbb{Z}^{q+1}_q = (L^1 Z) \times \mathbb{Z}^q \).
Denote by \( \hat{P}^{(q)} (\zeta) \) the corresponding expectation function for \( \hat{P}^{(q)} (\zeta) \). By the uniform boundedness of the family of analytical functions \( V_L (\zeta) \) for all \( L \) and all \( \zeta \) in \( D_{\alpha,\zeta_0} \), according to Lemma 2.6 and Proposition 2.7 in Ref. [3], we have the following Lemma 1 and Lemma 2.

Lemma 1  Let \( \zeta_L, \zeta \in D_{\alpha,\zeta_0} \), and \( \zeta_L \to \zeta \) as \( L \to \infty \). Then the random vector
\[ \hat{Y}_L^{(q)}(t_1, ..., t_q) = \frac{1}{\sqrt{L}} \left( \hat{X}_{L-Z}^{(q)}(t_1, ..., t_q) - \hat{E}_{L-Z}^{(q)} \right) \]

converges weakly to a Gaussian random vector \( \hat{Y}_L^{(q)}(t_1, ..., t_q) \) of which covariance matrix is given by \( V(\zeta) \).

Let \( g_\zeta \) be the density function of the Gaussian vector \( \hat{Y}_L^{(q)}(t_1, ..., t_q) \) given in Lemma 1.

**Lemma 2** Let \( Z_L^{(q)} = \left( L^{-1} Z \right) \times Z^q \), then for each \( z_L \in Z_L^{(q)} \) and \( \zeta_L \in D_{\alpha, \zeta} \), define

\[ z_L \in Z_L^{(q)}, \quad g_{\zeta_L} \left( y_L \right) = \frac{1}{\sqrt{L}} \left( z_L - \hat{E}_{L-Z}^{(q)} \right) \]

Then we have

\[ L^{(q+3)/2} \hat{P}_{L-Z}^{(q)}(z_L) - g_{\zeta_L} \left( y_L \right) \to 0 \quad \text{as} \quad L \to \infty \]

uniformly in \( z_L \in Z_L^{(q)} \) and \( \zeta_L \in D_{\alpha, \zeta} \).

### 3 Proof of the Main Results

In this section, we discuss the limiting properties of the random vector \( \hat{X}_L^{(q)}(t_1, ..., t_q) \) defined in Section 2, and show the proofs of Theorem 1. Here we omit the proof of Theorem 2, in fact, by using the proofing method of Theorem 1, we can prove Theorem 2.

**Proof of Theorem 1.** In Section 2, the random vector \( \hat{X}_L^{(q)}(t_1, ..., t_q) \) is given. First we consider the convergence of the finite-dimensional distribution of the random vector \( \hat{Y}_L^{(q)}(t_1, ..., t_q) \) defined in Lemma 1. Let \( \zeta_L^0, \zeta_L^0 \) be a special sequence in \( D_{\alpha, \zeta} \), such that

\[ \zeta_L^0 = (\zeta_{L,0}, 0, ..., 0), \quad \zeta_L^0 = (\zeta_0, 0, ..., 0) \]

where \( \zeta_0 \) is defined in (1), and \( \zeta_{L,0} \) satisfies the following condition

\[ \frac{d}{d\zeta_0} \ln Q_{\alpha, \zeta} \left( \zeta_0 \right) \bigg|_{\zeta_0 = \zeta_{L,0}} = [aL] \]

by (1)(2), it can be proved that \( \zeta_L^0 \to \zeta_0^0 \) as \( L \to \infty \).

Let

\[ \phi_L \left( \zeta, t_1, ..., t_q \right) = \frac{1}{L} \ln \left( \sum_{\zeta, \eta} e^{\beta \zeta \cdot \chi_2^{(q-1)} (\zeta - \eta)} \right) \]

and denote by

\[ \phi \left( \zeta, t_1, ..., t_q \right) = \lim_{L \to \infty} \phi_L \left( \zeta, t_1, ..., t_q \right) \]

for \( \zeta \in D_{\alpha, \zeta} \). By the uniform boundedness of \( \text{Hess}_\zeta \phi_L \), we have

\[ \hat{E}_{L-Z}^{(q)}(t_1, ..., t_q) = \left( a_L, a_L, \cdots, a_L \right) \]

\[ \left( b_L, a_L, \cdots, a_L \right) \]

\[ \left( 0, a_L, \cdots, a_L \right) \]

\[ \left( \zeta_0, \alpha_L, \cdots, \alpha_L \right) \]

By Lemma 2, we have for \( \infty < a_j < b_j < \infty, 1 \leq j \leq q \),

\[ \lim_{L \to \infty} \hat{P}_{L-Z}^{(q)} \left( y_L \right) = \lim_{L \to \infty} \left( a_L, b_L \right) \]

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\[ \lim_{L \to \infty} \hat{P}_{L-Z}^{(q)} \left( y_L \right) = \lim_{L \to \infty} \left( a_L, b_L \right) \]

\[ \frac{1}{L} \sum_{x, y} g_\zeta \left( 0, y_1, ..., y_q \right) dy_1, ..., y_q \]

According to Lemma 1, let

\[ \hat{Y}_L^{(q)}(t_1, ..., t_q) = (Y_0, Y(t_1), ..., Y(t_q)) \]

be a Gaussian random vector with distribution density \( g_\zeta \left( 0, y_1, ..., y_q \right) \). Then its covariance matrix is given by

\[ E \left[ Y(t_j) Y(t_k) \right] \]

\[ = \frac{1}{\beta^2} \int_0^1 \phi \left( (1-s) \zeta_0, (1-s) \zeta_0 \right) ds \]

\[ E \left[ Y_0 Y(t_j) \right] \]

\[ = \frac{1}{\beta^2} \int_0^1 \phi \left( \zeta_0, (1-s) \zeta_0 \right) ds \]

\[ E \left[ Y_0^2 \right] \]

\[ = \frac{1}{\beta^2} \int_0^1 \phi \left( \zeta_0, (1-s) \zeta_0 \right) ds \]

for \( j, k = 1, ..., q \), where \( a \wedge b = \min \{a, b\} \). This means that \( \{Y_0, Y(t)\}_{t \in [0,1]} \) is a Gaussian random process with covariance matrix given above for every \( q \geq 1 \). In above proof, we suppose that

\[ \alpha_0^{(q)} - \alpha_0^{(q)} = \chi_2^{(q)} \left( \frac{L_t}{L} \right) - \chi_2^{(q)} \left( \frac{L_t}{L} \right), \]
Similarly to Ref. [3], the above argument is also true if we replace
\[ X_L^i \left( \frac{L \eta^-}{L} \right) - X_L^i \left( t_i \right) \]
for every \( 1 \leq i \leq q \). Then the distribution of
\[ \hat{X}_L \left( t_1, \ldots, t_q \right) \], under \( L \cdot P_{L, \beta} \left( \left\lfloor \alpha_L^{\eta^-} = \left\lfloor aL \right\rfloor \right) \),
converges weakly to the corresponding distribution of Gaussian random vector \( \hat{Y}_L \left( t_1, \ldots, t_q \right) \).

Secondly, the tightness of above conditional distribution of the random process \( Y_L \left( t \right) \) should be discussed, see [1]. Following the similar argument of Section 3 in Ref. [3], we can prove a sufficient condition for the tightness of the considered process \( Y_L \left( t \right) \). Together with the first part of this proof, this completes the proof of Theorem 1.

4 Conclusion

In this paper, we studied the statistical properties of the two random paths model. Under some conditions, that there is a specified value of the large area in the intermediate region of the two random interfaces, Theorem 1 shows the weak convergence of the fluctuations for the two random interfaces.

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References: