Extended Displacement Discontinuity Method of Two-dimensional Magnetoelectroelastic Media

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Abstract: - The extended Green’s functions for extended displacement discontinuity and Crouch fundamental solutions in two-dimensional (2D) magnetoelectroelastic media are obtained by using the counterpart Green’s functions in 3D transversely isotropic magnetoelectroelastic medium [1]. The extended stress filed of a crack is expressed in terms of the extended displacement discontinuity across the crack faces, where the crack is perpendicular to the poling direction of the magnetoelectroelastic medium. The displacement discontinuity method proposed by Crouch for purely elastic material is extended to magnetoelectroelastic medium, and two co-linear and parallel cracks are analyzed under impermeable and permeable boundary conditions.

Key-Words: - Magnetoelectroelastic medium; Crack; Green’s function; Intensity factor; Extended Displacement Discontinuity Method (EDDM);

1 Introduction
Magnetoeletriclastic materials are finding more and more applications in practice. Defects, such as crack, greatly influence the integrity and reliability of structures. So study of analysis method of cracks in magnetoelectroelastic materials and structures has been receiving more and more attention [2-12]. The analytical solution is often difficult to obtain for a general fracture problem. Among the numerical method, the displacement discontinuity method proposed by Crouch in 1976 [13] grasps the basic characteristic of a crack across which the displacement is discontinuous. This method has been proved to be one of the most powerful methods in fracture mechanics of purely elastic media [14,15], as well as piezoelectric media [16] and magnetoelectroelastic media [1,17].

The electric field and the magnetic field, two kinds of loadings in magnetoelectroelastic problem, make the boundary value problem more difficulty to be solved. Similar to fracture analysis in piezoelectric materials [18], two common approaches are often used to describe the electric and magnetic boundary conditions on crack faces. One approach treats the crack as an electrically and magnetically impermeable slit with the assumption that the electric field and the magnetic field within the crack cavity are zero. And the other approach treats the crack as an electrically and magnetically permeable slit which assumes the fields of electric and magnetic potentials, the electric displacement and magnetic induction normal to the crack are continuous [4-9].

In this paper, the extended displacement discontinuity method of 2D magnetoelectroelastic medium is presented. Co-linear and parallel crack problems are studied under impermeable and permeable boundary conditions.

2 Green’s functions
2.1 Green’s functions for unit extended point displacement discontinuity
The Green’s functions for 2D magnetoelectroelastic medium is considered as a special case of 3D magnetoelectroelastic medium [1], in which the uniformly distributed displacement discontinuities are acting along the entire x-axis. Integrating the extended point displacement discontinuity Green’s functions in 3D magnetoelectroelastic medium along the entire x-axis, we can obtain the Green’s functions

\[ \sigma_{yz} = 4 \sum_{i=1}^{4} \alpha_i [c_{ij} D_j + s_i (c_{ij} A_i - e_{ij} B_i - f_{ij} C_i)] \frac{y^2}{(y^2 + z_i^2)^2}, \]  

(1a)

\[ \sigma_{xz} = 4 \sum_{i=1}^{4} \alpha_i [c_{ij} D_j + s_i (c_{ij} A_i - e_{ij} B_i - f_{ij} C_i)] \frac{z^2}{(y^2 + z_i^2)^2}, \]  

(1b)

\[ \sigma_{yx} = 2 \sum_{i=1}^{4} \alpha_i [c_{ij} D_j + s_i (c_{ij} A_i - e_{ij} B_i - f_{ij} C_i)] \frac{z^2 - y^2}{(y^2 + z_i^2)^2}, \]  

(1c)
\[ D_1 = 2\sum_{i=1}^{4} a_{i1}[e_{i1} D_1 s_i + e_{i1} A_i + e_{i1} B_i + g_{i1} C_i] \frac{y_i^2 - z_i^2}{(y_i^2 + z_i^2)^2}, \]  
(1d)

\[ D_2 = 4\sum_{i=1}^{4} a_{i2}[e_{i2} D_2 s_i + e_{i2} A_i + e_{i2} B_i + g_{i2} C_i] \frac{y_i^2 - z_i^2}{(y_i^2 + z_i^2)^2}, \]  
(1e)

\[ B_1 = 2\sum_{i=1}^{4} \theta_{i1}[f_{i1} D_1 s_i + f_{i1} A_i + g_{i1} C_i] \frac{y_i^2 - z_i^2}{(y_i^2 + z_i^2)^2}, \]  
(1f)

\[ B_2 = 4\sum_{i=1}^{4} \theta_{i2}[f_{i2} D_2 s_i + f_{i2} A_i + g_{i2} C_i] \frac{y_i^2 - z_i^2}{(y_i^2 + z_i^2)^2}, \]  
(1g)

for unit point displacement discontinuity in the \( y \)-direction, and

\[ \sigma_{y2} = 2\sum_{i=1}^{4} \theta_{21}[c_{i1} D_1 s_i - s_{i1} e_{i1} A_i - e_{i1} B_i - f_{i1} C_i] \frac{y_i^2 - z_i^2}{(y_i^2 + z_i^2)^2}, \]  
(2a)

\[ \sigma_{y1} = 2\sum_{i=1}^{4} \theta_{11}[c_{i1} D_1 s_i - s_{i1} e_{i1} A_i - e_{i1} B_i - f_{i1} C_i] \frac{y_i^2 - z_i^2}{(y_i^2 + z_i^2)^2}, \]  
(2b)

\[ \sigma_{z2} = 4\sum_{i=1}^{4} \theta_{21}[c_{i2} D_2 s_i + c_{i2} A_i - e_{i2} B_i - f_{i2} C_i] \frac{y_i^2 - z_i^2}{(y_i^2 + z_i^2)^2}, \]  
(2c)

for unit point displacement discontinuity in the \( z \)-direction, where \( c_{ij}, e_{ij}, f_{ij}, \xi_j, \psi_j, \gamma_j \) and \( \mu_j \) are the material constants, \( \theta_{ij}, A_i, B_i, C_i \) and \( D_i \) are the coefficients related to material constants, and \( z_i = S_i Z [1,1,17] \).

Green’s functions for unit point electric and magnetic potential discontinuities can be obtained simply by taking \( \theta_{12} \) and \( \theta_{13} \), respectively, instead of \( \theta_{ij} \) in Eqs. (2).

### 2.2 Extended Crouch fundamental solution for extended displacement discontinuity

The Green’s functions for uniformly distributed displacement discontinuities on a line segment, which are called Crouch fundamental solutions, are useful in purely elasticity [13]. The extended Crouch fundamental solutions in magnetoelectroelasticity will be obtained as an extension.

Consider a linear element of length \( 2l \) parallel to the \( y \)-axis centered at the point \((\xi, \psi)\), as shown in Fig.1, along which uniformly distributed extended displacement discontinuities \( \|v^e\|, \|w^e\|, \|\sigma^e\| \) and \( \|\varphi^e\| \) are applied. Integrating the Green’s functions for extended point displacement discontinuity derived in last subsection along the element gives the extended Crouch fundamental solutions

\[ \sigma_{y2} = \sum_{i=1}^{4} \left[ L_{y2j} G \|v^e\| + L'_{y2j} \|w^e\| + L''_{y2j} \|\sigma^e\| + L'''_{y2j} \|\varphi^e\| \right] G_j, \]  
(3a)

\[ \sigma_{y1} = \sum_{i=1}^{4} \left[ L_{y1j} G \|v^e\| + L'_{y1j} \|w^e\| + L''_{y1j} \|\sigma^e\| + L'''_{y1j} \|\varphi^e\| \right] G_j, \]  
(3b)

\[ \sigma_{z2} = \sum_{i=1}^{4} \left[ L_{z2j} G \|v^e\| + L'_{z2j} \|w^e\| + L''_{z2j} \|\sigma^e\| + L'''_{z2j} \|\varphi^e\| \right] G_j, \]  
(3c)

\[ D_y = \sum_{i=1}^{4} \left[ L_{y2j} G \|v^e\| + L'_{y2j} \|w^e\| + L''_{y2j} \|\sigma^e\| + L'''_{y2j} \|\varphi^e\| \right] G_j, \]  
(3d)

\[ D_z = \sum_{i=1}^{4} \left[ L_{z2j} G \|v^e\| + L'_{z2j} \|w^e\| + L''_{z2j} \|\sigma^e\| + L'''_{z2j} \|\varphi^e\| \right] G_j, \]  
(3e)

\[ B_y = \sum_{i=1}^{4} \left[ L_{y1j} G \|v^e\| + L'_{y1j} \|w^e\| + L''_{y1j} \|\sigma^e\| + L'''_{y1j} \|\varphi^e\| \right] G_j, \]  
(3f)

\[ B_z = \sum_{i=1}^{4} \left[ L_{z1j} G \|v^e\| + L'_{z1j} \|w^e\| + L''_{z1j} \|\sigma^e\| + L'''_{z1j} \|\varphi^e\| \right] G_j, \]  
(3g)

where

\[ G_j = \frac{(y - \eta_j)^{-2} - (y - \xi)^{-1}}{[(y - \xi)^{-1}]^2 + (z - \eta_j)^{-2}}, \]  
(4a)

\[ G_j = \frac{1}{2} \frac{z_j}{[(y - \xi)^{-1}]^2 + (z - \eta_j)^{-2}}, \]  
(4b)

and \( L_{y1j} \) with different subscripts “sub” are material constants given by

\[ L_{y1j} = -4\alpha_{T_{y1j}}, \quad L'_{y1j} = 2\beta_{T_{y1j}}, \quad L''_{y1j} = 2\gamma_{T_{y1j}}, \quad L'''_{y1j} = 2\delta_{T_{y1j}}, \]  
(4b)
\[ L_{ij} = 2 \alpha_i T_{ij}, \quad L_{ij} = 4 \theta_i T_{ij}, \quad L_{ij} = 4 \theta_i T_{ij}, \quad L_{ij} = 2 \theta_i T_{ij}, \]
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\[ L_{ij} = 2 \theta_i T_{ij}, \quad L_{ij} = 2 \theta_i T_{ij}, \]

\[ \text{where} \]
\[ T_{ij} = C_i D_j - s(c_i A_j - e_i B_j - f_i C_j), \]
\[ T_{ij} = c_i D_j - s(c_i A_j - e_i B_j - f_i C_j), \]
\[ T_{ij} = e_i D_j - s(c_i A_j - e_i B_j - f_i C_j), \]
\[ T_{ij} = f_i D_j, \quad T_{ij} = f_i D_j, \]
\[ \text{and} \]
\[ F_i = \frac{(y - \eta)(z - \xi)}{[(y - \eta)^2 + (z - \xi)^2]^{1/2}}, \quad F_i = \frac{(y - \eta)^2 - (z - \xi)^2}{[(y - \eta)^2 + (z - \xi)^2]^{1/2}}, \]
\[ \|v\| = \psi(S^+) - \psi(S^-), \quad \|w\| = \psi(S^+) - \psi(S^-), \]
\[ \|\varphi\| = \psi(S^+) - \psi(S^-), \quad \|\varphi\| = \psi(S^+) - \psi(S^-). \]

It has been proven that the extended stress has the classical singularity of \(1/\sqrt{r}\) near the crack tip. Thus the extended displacement discontinuities at the neighborhood of the right crack tip \((y_c, z_c)\) can be expressed

\[ \sigma_{\alpha \beta} = \frac{1}{\sqrt{r}} \sum_{i=1}^{4c} [L_{ij}^i f_i^r (\theta) + (L_{ij}^i) f_i (\theta) + (L_{ij}^i) f_i (\theta)] \]
The extended intensity factors are defined by superposition of two problems. One is the no crack problem with the given applied loadings. And the other is the perturbed problem with the loadings being applied only on crack faces. The first problem is analyzed to obtain the extended tractions on the crack faces of the perturbed problem. Only the perturbed problem is considered in the present paper. Considering the boundary condition on the crack face,

\[ \sigma_k = -p_k, \]  

where \( p_k \) are the prescribed extended tractions on crack faces, one can obtain a set of algebraic equations. Solving these equations yields the extended displacement discontinuity across the crack. Fitting the calculated extended displacement discontinuity on crack, one obtains the asymptotic behavior of the displacement discontinuity near the crack tip

\[ \|u\| = k_1\alpha^{\nu/2} + k_2\alpha^{\nu/2}. \]  

Thus, one obtains

\[ \lim_{\alpha \to 0} \|u\| \theta^{
u/2} = k_1. \]  

Substituting Eq.(18) into Eq.(13) determines the extended intensity factors.

### 4.2 Permeable boundary condition

The electrically and magnetically permeable condition is expressed by

\[ \sigma_k = -p_k + D_k^e, \quad \|\mathbf{D}\| = \varphi(x,y,0^+) - \varphi(x,y,0^-) = 0, \]

\[ \sigma_k = -p_k + B_k^m, \quad \|\mathbf{B}\| = \psi(x,y,0^+) - \psi(x,y,0^-) = 0, \]

where \( D_k^e \) and \( B_k^m \) denotes the electric displacement and the magnetic induction in the z-axis direction in the crack cavity, respectively. Considering this boundary condition on crack face, one can also obtain a set of algebraic equations based on the method in Subsection 4.1 to determine \( \|u\|, D_k^e \) and \( B_k^m \).

### 5 Numerical example

Consider a transversely isotropic magnetoelastic medium made of BaTiO3 as the inclusion and CoFe2O4 as the matrix. The volume fraction of the inclusions is denoted by \( V \), [1,17].

The following mixture rule is used to determine the composite material constants correspondingly from those of the inclusion and matrix

\[ A^c = A_V(1 - V) + A^m(1 - V), \]  

where the superscripts c, i and m represent the composite, inclusion and matrix respectively.
The uniformly distributed extended loadings on the crack faces are:

\[ p_x = 0, \quad p_y = 0, \quad p_z = 100 \text{MPa}, \quad \omega = 0.1 \text{C/m}^2, \quad \gamma = 10 \text{N/Am}. \quad (21) \]

### 5.1 Co-linear cracks

Consider two co-linear cracks paralleling to the y-axis in the oyz plane as shown in Fig 3. The length of each crack is 2L and the distance between the two cracks is 2B. In the present paper, L=25.

For simplicity, each crack is divided into N elements of equivalent length. In the calculation, the element number is taken N=100. The calculated normalized intensity factors \( F_i \) at the crack tips under impermeable conditions are plotted in Fig.4, where

\[ F_i = \frac{K_i}{(\sqrt{\pi} \ \rho_i)}, \quad F_0 = \frac{K_0}{(\sqrt{\pi} \ \omega)}, \quad F_0 = \frac{K_0}{(\sqrt{\pi} \ \gamma)}. \quad (22) \]

It can be seen that the nearer the two crack tips are, the larger the normalized intensity factors are. The difference between the normalized intensity factors \( F_i \) for \( B/L=2.5 \) and \( B/L=100 \) is less than 0.1%. This indicates that the influence between the two cracks can be neglected for \( B/L>2.5 \). The normalized intensity factors of the inner tip b are larger than those of the outer tip a.

![Fig.4 The normalized intensity factors of impermeable co-linear crack versus B/L.](image)

The normalized extend stress intensity factors \( F_i \) for co-linear cracks under permeable boundary conditions are plotted in Fig.5 in case of \( V_i=0.5 \).

The normalized stress intensity factor of permeable crack is equal to that of impermeable crack and is independent of the electric-magnetic boundary condition. But the electric displacement and the magnetic induction intensity factors are both smaller than those under impermeable condition.

### 5.2 Parallel cracks

Consider two parallel cracks of length 2L. The distance between the two cracks is 2H as schematically shown Fig.6.

![Fig.6 Parallel cracks in the oyz plane](image)
Plotted in Fig.7 are the calculated normalized extended stress intensity factors $F_i$ versus $H/L$. It can be seen that the three normalized extended stress intensity factors of the parallel cracks are unequal under both impermeable and permeable boundary conditions. The normalized extended stress intensity factors $F_i$ all increase with the ratio of $H/L$ increasing. But the electric displacement and the magnetic induction intensity factors are smaller than those under impermeable condition. The difference between the normalized intensity factors $F_i$ for $H/L=6$ and $H/L = \infty$ is less than 0.1% under impermeable boundary conditions. So the influence between the two cracks can be neglected for $H/L>6$.

6 Conclusion

The numerical results of the extended intensity factors for two co-linear and parallel cracks show that the proposed extended method is very efficient for crack analysis in magnetoelectroelastic fracture mechanics. It should be pointed that the extended Crouch fundamental solution is dependent on the crack orientation. Though the numerical examples are for cracks perpendicular to the poling direction, the method is versatile provided that the corresponding fundamental solution is available.

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References

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