On Intuitionistic Fuzzy Negations and Intuitionistic Fuzzy Modal Operators with Contradictory Evidence

Chris J Hinde
Loughborough University
Computer Science
Ashby Road
Loughborough LE11 3TU
UK

Krassimir T. Atanassov
CLBME-Bulgarian Academy of Sciences
Acad. G. Bonchev Str.
Bl. 105
Sofia-1113
Bulgaria

Abstract: Some relations between intuitionistic fuzzy negations and intuitionistic fuzzy modal operations (from standard type) are studied. In particular versions of negation are extended to deal with contradictory evidence. Example proofs are presented for both the negations and the modal operators to show that the results from Intuitionistic Fuzzy Sets carry over to the extended versions incorporating contradictory evidence.

Key–Words: Intuitionistic Fuzzy Sets, Negation, Contradiction.

1 On some previous results

The concept of the Intuitionistic Fuzzy Set (IFS, see [Atanassov(1999)]) was introduced in 1983 as an extension of Zadeh's fuzzy set. All operations, defined over fuzzy sets were transformed for the IFS case. One of them - operation "negation" now there is 24 different forms (see [Atanassov and Dimitrov(2007)]. In [Atanassov(1999)] the relations between the "classical" negation and the two standard modal operators "necessity" and "possibility" are given. Here, we shall study the same relations with the rest of the negations, defined over IFSs. Following [Atanassov(1986)] the definition of an intuitionistic L-fuzzy set (ILFS) A^* over a universe of discourse $\mathbb{E}$ has the form:

$$A^* \underline{\triangle} \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in \mathbb{E}\}$$

For two IFSs A and B the following relations are valid:

$$\begin{array}{llll} A \subseteq B & \text{iff} & (\forall x \in \mathbb{E})(\mu_A(x) \leq \mu_B(x) & (1) \\ & & \wedge \nu_A(x) \geq \nu_B(x)), & \\ A \supseteq B & \text{iff} & B \subseteq A, & (2) \\ A = B & \text{iff} & (\forall x \in \mathbb{E})(\mu_A(x) = \mu_B(x) & (3) \\ & & \wedge \nu_A(x) = \nu_B(x)). & \end{array}$$

Where there is contradiction involved, [Hinde and Patching(2007), Cubillo and Castiñeira(2005)], if the measure of contradiction is included in the denotation of the set then we obtain the IIFS as defined:

$$A^{\iota *} = \{\langle x, \mu_A(x), \nu_A(x), \iota_A(x)\rangle \mid x \in \mathbb{E}\}$$

subject to the constraint:

$$(\forall x \in \mathbb{E})(\mu_A(x) + \nu_A(x) + \iota_A(x) \leq 1)$$

rather than

$$(\forall x \in \mathbb{E})(\mu_A(x) + \nu_A(x) \leq 1) \qquad (4)$$

The quantity $\iota_A(x)$ is the contradiction involved in the membership and non-membership function. [Hinde and Patching(2007)] and [Cubillo and Castiñeira(2005)] differ in their treatment of contradiction. Whereas Cubillo computes the contradiction between the membership and non-membership values given those values, Hinde allows contradiction to occur before the membership functions are complete and so whereas the hesitation $\pi_A(x)$ denoting the unknown between membership and non-membership must be zero in Cubillo's analysis, Hinde allows all 4 elements to be non-zero simultaneously giving rise to Equation 5.

$$(\forall x \in \mathbb{E})(\mu_A(x)+\nu_A(x)+\iota_A(x)+\pi_A(x) = 1) \qquad (5)$$

An Interval-valued IFS (IVIFS, see [Atanassov and Gargov(1989), Atanassov(1999)]) A over $\mathbb{E}$ is an object of the form:

$$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in \mathbb{E}\},$$

where $M_A(x) \subset [0, 1]$ and $N_A(x) \subset [0, 1]$
are intervals and for all $x \in \mathbb{E}$:

ISBN: 978-960-6766-86-2

ISSN: 1790-2769

$$\sup(M_A(x)) + \sup(N_A(x)) \leq 1 \qquad (6)$$

There is contradiction between the membership curve and the non-membership curve, however even when the two curves appear to be in agreement there may be contradiction involved. Let us use a voting model to derive a very simple set of curves for the fuzzy set $Tall$. The voters are asked to state:

1. whether they would allow a given height to be described as tall.
   - resulting in the membership curve
2. whether they would not allow a given height to be described as tall.
   - resulting in the non-membership curve

Further let the votes be cast as shown in Table 1.

Table 1: The incomplete table of votes where voter 3 casts votes for membership and non-membership

Height	160		170		180	
Voter	μ	ν	μ	ν	μ	ν
1	-	+	+	-	+	-
2	-	+	-	-	+	-
3	-	+	+	+	+	-
4	-	+	+	-	+	-
5	-	+	-	-	+	-

The resulting curves are shown in Figure 1. Clearly voter number 3 has contradicted themselves but it is not apparent from the membership and non-membership curves.

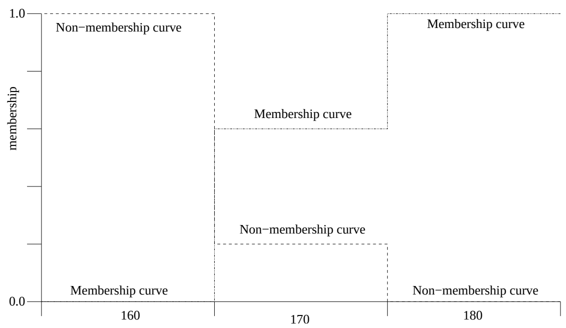

Figure 1: The membership and non-membership curves arising from table 1

So given the membership and non-membership votes resulting in the curves shown in Figure 1 the μ and ν values give no indication of contradiction.

$$\begin{aligned} Tall &= \{\langle x, \mu_{Tall}(x), \nu_{Tall}(x)\rangle \\ &\quad \mid x \in \Re\} \end{aligned}$$

$$\begin{aligned} Tall &= \{\langle 160, 0.0, 1.0\rangle, \langle 170, 0.6, 0.2\rangle, \\ &\quad \langle 180, 1.0, 0.0\rangle\} \end{aligned}$$

If we introduce $\iota_{Tall}(x)$ then the following set results:

$$\begin{aligned} Tall &= \{\langle x, \mu_{Tall}(x), \nu_{Tall}(x), \iota_{Tall}(x)\rangle \\ &\quad \mid x \in \Re\} \end{aligned}$$

$$\begin{aligned} Tall &= \{\langle 160, 0.0, 1.0, 0.0\rangle, \langle 170, 0.6, 0.2, 0.2\rangle, \\ &\quad \langle 180, 1.0, 0.0, 0.0\rangle\} \end{aligned}$$

This tells us that there is contradiction.

If we examine the intervals associated with the memberships and non-memberships of $Tall$ we get the following:

$$\begin{aligned} Tall &= \{\langle 160, [0.0, 0.0], [1.0, 1.0]\rangle, \\ &\quad \langle 170, [0.6, 1.0], [0.2, 0, 6]\rangle, \\ &\quad \langle 180, [1.0, 1.0], [0.0, 0.0]\rangle\} \end{aligned}$$

If we complete the table so voters 2 and 5 choose to vote for membership or non-membership we might get the Table 2

Table 2: The completed Table 1 which show up the hitherto latent contradiction in the membership curves

Height	160		170		180	
Voter	μ	ν	μ	ν	μ	ν
1	-	+	+	-	+	-
2	-	+	-	+	+	-
3	-	+	+	+	+	-
4	-	+	+	-	+	-
5	-	+	+	-	+	-

Table 2 shows a clear contradiction between the membership and non-membership curves, in the manner of [Cubillo and Castiñeira(2005)]. Shown in Figure 2

ISBN: 978-960-6766-86-2 ISSN: 1790-2769

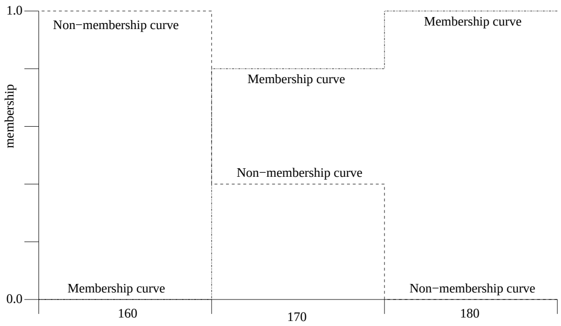

Figure 2: The membership and non-membership curves arising from table 2

The sets that arise, we do not count contradictory votes in the membership or the membership:

$$Tall = \{\langle x, \mu_{Tall}(x), \nu_{Tall}(x), \iota_{Tall}(x)\rangle \mid x \in \Re\}$$

If we just count the votes for membership as an IFS, then the following set arises:

$$Tall = \{\langle 160, 0.0, 1.0\rangle, \langle 170, 0.8, 0.4\rangle, \langle 180, 1.0, 0.0\rangle\}$$

This would give a contradiction of 0.2 using the methods of [Cubillo and Castiñeira(2005)]. But this only occurs after all votes have been cast.

The set that arises using our contradictory notation, since we do not count contradictory votes in the membership or the membership is:

$$Tall = \{\langle 160, 0.0, 1.0, 0.0\rangle, \langle 170, 0.6, 0.2, 0.2\rangle, \langle 180, 1.0, 0.0, 0.0\rangle\}$$

This contradiction was apparent in the earlier set.

If we again examine the intervals associated with the memberships and non-memberships of $Tall$ we get the following:

$$\begin{aligned} Tall = \{&\langle 160, [0.0, 0.0], [1.0, 1.0]\rangle, \\ &\langle 170, [0.8, 0.8], [0.4, 0, 4]\rangle, \\ &\langle 180, [1.0, 1.0], [0.0, 0.0]\rangle\} \end{aligned}$$

This also now shows the contradiction in the evidence for an against height 170 as the membership and non-membership sum to greater than 1.0 breaking constraint 6.

Clearly the definitions of subsets 1 need to be revisited as a difference in contradiction would at least make two sets unequal. Subset is more complex as it is unclear where the contradiction should lie. If contradiction is regarded as part of the uncertainty between membership and non-membership then the equations 1 are adequate. If the two parts ι and π are regarded as different forms of uncertainty then they are inadequate. Equation 7 must then be the extended definition of equality, and the definitions of subset must also satisfy Equation 8.

$$\begin{aligned} A = B \quad \text{iff} \quad & (\forall x \in \mathbb{E})(\mu_A(x) = \mu_B(x) \\ & \wedge \nu_A(x) = \nu_B(x) \wedge \iota_A(x) = \iota_B(x)). \end{aligned} \tag{7}$$

$$A \subseteq B \wedge B \subseteq A \rightarrow A = B \tag{8}$$

Along with Equation 1 some other choices for subset definitions are presented in Equations 9. We maintain Equation 2

$$\begin{aligned} A \subseteq B \quad \text{iff} \quad & (\forall x \in \mathbb{E})(\mu_A(x) + \iota_A(x) \\ & \leq \mu_B(x) + \iota_B(x) \wedge \\ & \nu_A(x) + \iota_A(x) \geq \nu_B(x) + \iota_B(x)), \end{aligned} \tag{9}$$

Each set has 3 degrees of freedom so with only 2 constraints involved in subset-hood so far we are unable to derive Equation 8 from either Equation 7 or Equation 9. But a combination of both gives the requisite constraints in Equation 10.

$$\begin{aligned} A \subseteq_\iota B \quad \text{iff} \quad & (\forall x \in \mathbb{E}) \\ & (\mu_A(x) + \iota_A(x) \leq \mu_B(x) + \iota_B(x) \wedge \\ & \nu_A(x) + \iota_A(x) \geq \nu_B(x) + \iota_B(x) \wedge \\ & \mu_A(x) \leq \mu_B(x) \wedge \\ & \nu_A(x) \geq \nu_B(x)), \end{aligned} \tag{10}$$

In this work the extended definition will be unnecessary as the sets being compared always have the same level of contradiction embodied in them. In all the equations involving $\subseteq$ the original definition can be substituted with no actual change in the logic.

1.1 Definitions

In some definitions we shall use functions sg and $\overline{\text{sg}}$:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

$$\overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

Let A be a fixed IFS. In [Atanassov(1999)] definitions of standard modal operators are given, Equations 11 and 12:

$$\Box A \quad \triangleq \quad \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \,|x \in \mathbb{E}\} \quad (11)$$
$$\Diamond A \quad \triangleq \quad \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle \,|x \in \mathbb{E}\} \quad (12)$$

If the assignment is contradictory in the sense of [Cubillo and Castiñeira(2005)] then constraint 4 will be broken. If the contradiction is moved to the contradiction measure ι, as in Equation 5 then the possibility measure $1-\nu_A(x)$ in Equation 12 needs revision. $\nu_A(x)$ cannot be used for membership which limits the possible values for membership to $1-\nu_A(x)$, Equation 12, however the value $\iota_A(x)$ also cannot be used for membership which then limits the maximum possible value for membership to $1-\nu_A(x)-\iota_A(x)$, Equation 14. A similar argument can be made for the necessity operator. This results in Equations 13 and 14.

$$\Box_\iota A \quad \triangleq \quad \{\langle x, \mu_A(x), 1-\mu_A(x)-\iota_A(x)\rangle \,|x \in \mathbb{E}\} \quad (13)$$
$$\Diamond_\iota A \quad \triangleq \quad \{\langle x, 1-\nu_A(x)-\iota_A(x), \nu_A(x)\rangle \,|x \in \mathbb{E}\} \quad (14)$$

2 Main results

In [Atanassov and Dimitrov(2007), Atanassova(2008), Atanassova(2007), Dimitrov(2008)] the following 24 different negations are described, these are shown in Equations 15.

$$
\begin{aligned}
A &\triangleq \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in \mathbb{E}\}, \qquad (15)\\
\neg_1 A &\triangleq \{\langle x, \nu_A(x), \mu_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_2 A &\triangleq \{\langle x, \overline{\text{sg}}(\mu_A(x)), \text{sg}(\mu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_3 A &\triangleq \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x)+\mu_A(x)^2\rangle \,|x \in \mathbb{E}\},\\
\neg_4 A &\triangleq \{\langle x, \nu_A(x), 1-\nu_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_5 A &\triangleq \{\langle x, \overline{\text{sg}}(1-\nu_A(x)), \text{sg}(1-\nu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_6 A &\triangleq \{\langle x, \overline{\text{sg}}(1-\nu_A(x)), \text{sg}(\mu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_7 A &\triangleq \{\langle x, \overline{\text{sg}}(1-\nu_A(x)), \mu_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_8 A &\triangleq \{\langle x, 1-\mu_A(x), \mu_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_9 A &\triangleq \{\langle x, \overline{\text{sg}}(\mu_A(x)), \mu_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_{10} A &\triangleq \{\langle x, \overline{\text{sg}}(1-\nu_A(x)), 1-\nu_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_{11} A &\triangleq \{\langle x, \text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_{12} A &\triangleq \{\langle x, \nu_A(x)(\mu_A(x)+\nu_A(x)),\\
&\quad \mu_A(x)(\mu_A(x)+\nu_A(x)+\nu_A(x)^2)\rangle | x \in \mathbb{E}\},\\
\neg_{13} A &\triangleq \{\langle x, \text{sg}(1-\nu_A(x)), \overline{\text{sg}}(1-\mu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_{14} A &\triangleq \{\langle x, \text{sg}(\nu_A(x)), \overline{\text{sg}}(1-\mu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_{15} A &\triangleq \{\langle x, \overline{\text{sg}}(1-\nu_A(x)), \overline{\text{sg}}(1-\mu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_{16} A &\triangleq \{\langle x, \overline{\text{sg}}(\mu_A(x)), \overline{\text{sg}}(1-\mu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_{17} A &\triangleq \{\langle x, \overline{\text{sg}}(1-\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle \,|x \in \mathbb{E}\},\\
\neg_{18} A &\triangleq \{x, \langle x, \nu_A(x).\text{sg}(\mu_A(x)),\\
&\quad \mu_A(x)\mu_A(x).\text{sg}(\nu_A(x))\rangle | x \in \mathbb{E}\},\\
\neg_{19} A &\triangleq \{\langle x, \nu_A(x).\text{sg}(\mu_A(x)), 0\rangle \,|x \in \mathbb{E}\},\\
\neg_{20} A &\triangleq \{\langle x, \nu_A(x), 0\rangle \,|x \in \mathbb{E}\},\\
\neg_{21} A &\triangleq \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x)+\mu_A(x)^n\rangle \,|x \in \mathbb{E}\},\\
&\quad \text{where } n \in \Re \wedge n \in [2,\infty),\\
\neg_{22} A &\triangleq \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x)+\overline{\text{sg}}(1-\mu_A(x))\rangle\\
&\quad |x \in \mathbb{E}\},\\
\neg^{\varepsilon} A &\triangleq \{\langle x, \min(1, \nu_A(x)+\varepsilon), \max(0, \mu_A(x)-\varepsilon)\rangle\\
&\quad |x \in \mathbb{E}\}, \text{where } \varepsilon \in [0,1],\\
\neg^{\varepsilon,\eta} A &\triangleq \{\langle x, \min(1, \nu_A(x)+\varepsilon), \max(0, \mu_A(x)-\eta)\rangle\\
&\quad |x \in \mathbb{E}\}, \text{where } 0 \le \varepsilon \le \eta \le 1.
\end{aligned}
$$

Given a degree of contradiction, $\iota_A(x) > 0$, many of remain unchanged, but some require some modification given that the maximum sum of membership and non-membership is now $1-\iota_A(x)$. Equations 16 list only those that are changed, except for $\neg_1$ to show the pattern.

$$
\begin{aligned}
A &\triangleq \{x, \langle \mu_A(x), \nu_A(x), \iota_A(x)\rangle \,|x \in \mathbb{E}\}, \qquad (16)\\
\neg_{\iota 1} A &\triangleq \{\langle x, \nu_A(x), \mu_A(x), \iota_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_{\iota 4} A &\triangleq \{\langle x, \nu_A(x), 1-\iota_A(x)-\nu_A(x), \iota_A(x)\rangle \,|x \in \mathbb{E}\},\\
\neg_{\iota 5} A &\triangleq \{\langle x, \overline{\text{sg}}(1-\iota_A(x)-\nu_A(x)),\\
&\quad \text{sg}(1-\iota_A(x)-\nu_A(x), \iota_A(x))\rangle | x \in \mathbb{E}\},\\
\neg_{\iota 6} A &\triangleq \{\langle x, \overline{\text{sg}}(1-\iota_A(x)-\nu_A(x)), \text{sg}(\mu_A(x)), \iota_A(x)\rangle\\
&\quad |x \in \mathbb{E}\},\\
\neg_{\iota 7} A &\triangleq \{\langle x, \overline{\text{sg}}(1-\iota_A(x)-\nu_A(x)), \mu_A(x), \iota_A(x)\rangle\\
&\quad |x \in \mathbb{E}\},\\
\neg_{\iota 8} A &\triangleq \{\langle x, 1-\iota_A(x)-\mu_A(x), \mu_A(x), \iota_A(x)\rangle\\
&\quad |x \in \mathbb{E}\},\\
\neg_{\iota 10} A &\triangleq \{\langle x, \overline{\text{sg}}(1-\iota_A(x)-\nu_A(x)),\\
&\quad 1-\iota_A(x)-\nu_A(x), \iota_A(x)\rangle | x \in \mathbb{E}\},\\
\neg_{\iota 13} A &\triangleq \{\langle x, \text{sg}(1-\iota_A(x)-\nu_A(x)),\\
&\quad \overline{\text{sg}}(1-\iota_A(x)-\mu_A(x)), \iota_A(x)\rangle | x \in \mathbb{E}\},\\
\neg_{\iota 14} A &\triangleq \{\langle x, \text{sg}(\nu_A(x)),\\
&\quad \overline{\text{sg}}(1-\iota_A(x)-\mu_A(x)), \iota_A(x)\rangle | x \in \mathbb{E}\},
\end{aligned}
$$

$$\begin{aligned}
\neg_{\iota 15}A &\triangleq \{\langle x, \overline{\text{sg}}(1-\iota_A(x)-\nu_A(x)),\\
&\quad \overline{\text{sg}}(1-\iota_A(x)-\mu_A(x)), \iota_A(x)\rangle | x\in\mathbb{E}\},\\
\neg_{\iota 16}A &\triangleq \{\langle x, \overline{\text{sg}}(\mu_A(x)),\\
&\quad \overline{\text{sg}}(1-\iota_A(x)-\mu_A(x)), \iota_A(x)\rangle | x\in\mathbb{E}\},\\
\neg_{\iota 17}A &\triangleq \{\langle x, \overline{\text{sg}}(1-\iota_A(x)-\nu_A(x)),\\
&\quad \overline{\text{sg}}(\nu_A(x)), \iota_A(x)\rangle | x\in\mathbb{E}\},\\
\neg_{\iota 22}A &\triangleq \{\langle x, \nu_A(x),\\
&\quad \mu_A(x).\nu_A(x)+\overline{\text{sg}}(1-\iota_A(x)-\mu_A(x)), \iota_A(x)\rangle\\
&\quad | x\in\mathbb{E}\},\\
\neg_{\iota}^{\varepsilon}A &\triangleq \{\langle x, \min(1-\iota_A(x),\\
&\quad \nu_A(x)+\varepsilon), \max(0,\mu_A(x)-\varepsilon), \iota_A(x)\rangle\\
&\quad | x\in\mathbb{E}\},\\
&\quad \text{where } \varepsilon\in[0,1-\iota_A(x)],\\
\neg_{\iota}^{\varepsilon,\eta}A &\triangleq \{\langle x, \min(1-\iota_A(x),\\
&\quad \nu_A(x)+\varepsilon), \max(0,\mu_A(x)-\eta), \iota_A(x)\rangle\\
&\quad | x\in\mathbb{E}\},\\
&\quad \text{where } 0\le\varepsilon\le\eta\le 1-\iota_A(x).
\end{aligned}$$

Now, following and extending the idea from [Hinde and Patching(2007)] we shall prove the following

Theorem 1: For every IFS A the following properties are valid:

1. $\neg_{\iota 2}\Box_\iota A = \Box_\iota\neg_{\iota 2}A$
2. $\neg_{\iota 2}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 2}A$
3. $\neg_{\iota 3}\Box_\iota A \supset \Box_\iota\neg_{\iota 3}A$
4. $\neg_{\iota 3}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 3}A$
5. $\neg_{\iota 4}\Box_\iota A \supset \Box_\iota\neg_{\iota 4}A$
6. $\neg_{\iota 4}\Diamond_\iota A = \Diamond_\iota\neg_{\iota 4}A$
7. $\neg_{\iota 5}\Diamond_\iota A = \Diamond_\iota\neg_{\iota 5}A$
8. $\neg_{\iota 6}\Diamond_\iota A = \Diamond_\iota\neg_{\iota 6}A$
9. $\neg_{\iota 7}\Box_\iota A \supset \Box_\iota\neg_{\iota 7}A$
10. $\neg_{\iota 7}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 7}A$
11. $\neg_{\iota 8}\Box_\iota A = \Box_\iota\neg_{\iota 8}A$
12. $\neg_{\iota 8}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 8}A$
13. $\neg_{\iota 9}\Box_\iota A \supset \Box_\iota\neg_{\iota 9}A$
14. $\neg_{\iota 9}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 9}A$
15. $\neg_{\iota 10}\Box_\iota A \supset \Box_\iota\neg_{\iota 10}A$
16. $\neg_{\iota 11}\Box_\iota A = \Box_\iota\neg_{\iota 11}A$
17. $\neg_{\iota 11}\Diamond_\iota A = \Diamond_\iota\neg_{\iota 11}A$
18. $\neg_{\iota 13}\Box_\iota A = \Box_\iota\neg_{\iota 12}A$
19. $\neg_{\iota 15}\Box_\iota A \supset \Box_\iota\neg_{\iota 15}A$
20. $\neg_{\iota 15}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 15}A$
21. $\neg_{\iota 16}\Box_\iota A \supset \Box_\iota\neg_{\iota 16}A$
22. $\neg_{\iota 16}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 16}A$
23. $\neg_{\iota 17}\Box_\iota A \supset \Box_\iota\neg_{\iota 17}A$
24. $\neg_{\iota 17}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 17}A$
25. $\neg_{\iota 18}\Box_\iota A \supset \Box_\iota\neg_{\iota 18}A$
26. $\neg_{\iota 18}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 18}A$
27. $\neg_{\iota 19}\Box_\iota A \supset \Box_\iota\neg_{\iota 19}A$
28. $\neg_{\iota 19}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 19}A$
29. $\neg_{\iota 20}\Box_\iota A \supset \Box_\iota\neg_{\iota 20}A$
30. $\neg_{\iota 20}\Diamond_\iota A = \Diamond_\iota\neg_{\iota 20}A,$
31. $\neg_{\iota 21}\Box_\iota A \supset \Box_\iota\neg_{\iota 21}A$
32. $\neg_{\iota 21}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 21}A$
33. $\neg_{\iota 22}\Box_\iota A \supset \Box_\iota\neg_{\iota 22}A$
34. $\neg_{\iota 22}\Diamond_\iota A \subset \Diamond_\iota\neg_{\iota 22}A$
35. $\neg_\iota^\varepsilon\Box_\iota A \supset \Box_\iota\neg_\iota^\varepsilon A$
36. $\neg_\iota^\varepsilon\Diamond_\iota A \subset \Diamond_\iota\neg_\iota^\varepsilon A$
37. $\neg_\iota^{\varepsilon,\eta}\Box_\iota A \supset \Box_\iota\neg_\iota^{\varepsilon,\eta} A$
38. $\neg_\iota^{\varepsilon,\eta}\Diamond_\iota A \subset \Diamond_\iota\neg_\iota^{\varepsilon,\eta} A$

Proof: Let us prove, for example (38). The remaining assertions can be proved similarly.
Let $0\le\varepsilon\le\eta\le 1$ for some ε and η.
Then

$$\begin{aligned}
\neg_\iota^{\varepsilon,\eta}\Diamond_\iota A &= \neg_\iota^{\varepsilon,\eta}\{\langle x, 1-\iota_A(x)-\nu_A(x), \nu_A(x)\rangle\\
&\quad | x\in\mathbb{E}\}\\
&= \{\langle x, \min(1-\iota_A(x), \nu_A(x)+\varepsilon),\\
&\quad \max(0, 1-\iota_A(x)-\nu_A(x)-\eta)\rangle\\
&\quad | x\in\mathbb{E}\}.\\
\Diamond_\iota\neg_\iota^{\varepsilon,\eta}A &= \Diamond_\iota\{\langle x, \min(1-\iota_A(x), \nu_A(x)+\varepsilon),\\
&\quad \max(0,\mu_A(x)-\eta)\rangle | x\in\mathbb{E}\}\\
&= \{\langle x, 1-\iota_A(x)-\max(0,\mu_A(x)-\eta),\\
&\quad \max(0,\mu_A(x)-\eta)\rangle | x\in\mathbb{E}\}.
\end{aligned}$$

Let

$$\begin{aligned} X &\triangleq 1-\iota_A-\max(0,\mu_A(x)-\eta)- \\ &\quad \min(1-\iota_A(x),\nu_A(x)+\varepsilon). \end{aligned}$$

If $\nu_A(x)+\varepsilon \geq 1-\iota_A(x)$,

Then

$$\begin{aligned} &\mu_A(x)-\eta \leq \\ &1-\iota_A(x)-\nu_A(x)-\eta \\ &\leq \varepsilon-\eta \leq 0 \end{aligned}$$

So

$$\begin{aligned} X &= 1-\iota_A(x)-0-(1-\iota_A(x)) \\ &= 0.0 \end{aligned}$$

If $\nu_A(x)+\varepsilon \leq 1-\iota_A(x)$,

Then

there are two subcases.

If $\mu_A(x)-\eta \leq 0$,

Then

$$X = 1-\iota_A(x)-0-(\nu_A(x)+\varepsilon) \geq 0$$

So

If $\mu_A(x)-\eta \geq 0$,

Then

$$\begin{aligned} X &= 1-\iota_A(x)-(\nu_A(x)+\varepsilon)-\mu_A(x)+\eta \\ &= 1-\iota_A(x)-\mu_A(x)-\mu_A(x)+\eta-\varepsilon \geq 0. \end{aligned}$$

Therefore, the first component of the second term is higher than the first component of the first term, while the inequality $\max(0,1-\nu_A(x)-\eta)-\max(0,\mu_A(x)-\eta) \geq 0$ is obvious. Therefore inclusion (38) is valid.
QED

In [Atanassov(2006)] there were shown cases in which some intuitionistic fuzzy (non-classical) negations do not satisfy De Morgan's laws. Now, by analogy with this result, we shall study the De Morgans' form of modal logic operators (see, e.g. [Feys(1965)]):

$$\begin{aligned} \Box A &= \neg\Diamond\neg A \\ \Diamond A &= \neg\Box\neg A \end{aligned}$$

In keeping with the objectives of this paper we will use the extended modal operators with a variety of negations.

$$\begin{aligned} \Box_\iota A &= \neg\Diamond_\iota\neg A \\ \Diamond_\iota A &= \neg\Box_\iota\neg A \end{aligned}$$

and will formulate the following assertion that is proved as above one.

Theorem 2: For every IFS A the following properties are valid:

1. $\neg_1\Box_\iota\neg_1 A = \Diamond_\iota A$
2. $\neg_1\Diamond_\iota\neg_1 A = \Box_\iota A$
3. $\neg_3\Box_\iota\neg_3 A = \Diamond_\iota A$
4. $\neg_4\Box_\iota\neg_4 A = \Diamond_\iota A$
5. $\neg_4\Diamond_\iota\neg_4 A \supset \Box_\iota A$
6. $\neg_7\Diamond_\iota\neg_7 A \subset \Box_\iota A$
7. $\neg_8\Diamond_\iota\neg_8 A = \Box_\iota A$
8. $\neg_9\Diamond_\iota\neg_9 A \subset \Box_\iota A$

Proof: Let us prove, for example (7). The remaining assertions can be proved similarly.
$\neg_8\Diamond_\iota\neg_8 A = \Box_\iota A$

$$\begin{aligned} A &= \langle x,\mu_A(x),\nu_A(x),\iota_A(x)\rangle \\ \neg_8 A &= \langle x,1-\iota_A(x)-\mu_A(x),\mu_A(x),\iota_A(x)\rangle \\ \Diamond_\iota\neg_8 A &= \langle x,1-\iota_A(x)-\mu_A(x),\mu_A(x),\iota_A(x)\rangle \\ \neg_8\Diamond_\iota\neg_8 A &= \langle x,1-\iota_A(x)-(1-\iota_A(x)-\mu_A(x)), \\ &\quad 1-\iota_A(x)-\mu_A(x),\iota_A(x)\rangle \\ &= \langle x,\mu_A(x)),1-\iota_A(x)-\mu_A(x),\iota_A(x)\rangle \\ \Box_\iota A &= \langle x,\mu_A(x)),1-\iota_A(x)-\mu_A(x),\iota_A(x)\rangle \end{aligned}$$

QED

3 Conclusion

A range of negations following the work of [Atanassov and Dimitrov(2007), Atanassova(2008), Atanassova(2007), Dimitrov(2008)] have been extended to incorporate contradictory evidence [Cubillo and Castiñeira(2005), Hinde and Patching(2007)]. An example proof of the properties, Theorem 1, has been presented and others may be proved similarly. Theorem 2 addresses some properties of modal operators in relation to the negations described, an example proof has been presented and the other properties may be proved similarly. In subsequent research the above properties for the case of extended intuitionistic fuzzy modal and topological operators will be studied.

References:

[Atanassov(1986)] Atanassov, K., 1986. Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20, 87–96.

[Atanassov(1999)] Atanassov, K., 1999. Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Physica-Verlag, Heidelberg.

[Atanassov(2006)] Atanassov, K., July 2006. On intuitionistic fuzzy negations and De Morgan's laws. In: Proc. of Eleventh International Conf. IPMU 2006. pp. 2355–2362.

[Atanassov and Dimitrov(2007)] Atanassov, K., Dimitrov, D., October 2007. On the negations over intuitionistic fuzzy sets. Notes on Intuitionistic Fuzzy Sets 13 (4), 2355–2362.

[Atanassov and Gargov(1989)] Atanassov, K., Gargov, G., 1989. Interval valued intuitionistic fuzzy sets. Fuzzy Sets and Systems 31, 343–349.

[Atanassova(2007)] Atanassova, L., December 2007. Modifications of an intuitionistic fuzzy implication. In: Proc. of the First Session of Section "Informatics" of Union of Bulgarian Scientists. pp. 2355–2362.

[Atanassova(2008)] Atanassova, L., 2008. On an intuitionistic fuzzy implication from Kleene-Dienes type. Advanced Studies in Contemporary Mathematics.

[Cubillo and Castiñeira(2005)] Cubillo, S., Castiñeira, E., February 2005. Measuring contradiction in fuzzy logic. International Journal of General Systems 34 (1), 39–59.

[Dimitrov(2008)] Dimitrov, D., October 2008. New intuitionistic fuzzy implications and their corresponding negations. In: Issues in Intuitionistic Fuzzy Sets and Generalized Nets. Vol. 6. Warsaw School of Information Technology.

[Feys(1965)] Feys, R., 1965. Modal Logics. Gauthier-Villars, Paris.

[Hinde and Patching(2007)] Hinde, C., Patching, R., 2007. Inconsistent Intuitionistic Fuzzy Sets. EXIT.