A Probability Model for Multiple-Source Interference

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Abstract: Two simple and novel methods are presented to analytically calculate the probability of interference caused by multiple sources that are scattered randomly over an area. We use the free space model and the two-ray propagation model to calculate the field strengths. Our model does not assume equal transmitting power for every source. The power of a transmitter is regarded as a random variable. Our first model deals with the case that the nearest source has the biggest impact on the victim. In our second model we deal with the case that the contribution of many remote sources may result in a cumulative contribution.

Key–Words: free-space model, two-ray model, interference probability, FM transmitter

1 Introduction

The prediction of radio interference is an important facet of spectrum management. As radio spectrum is a limited resource, and capacity demands are increasing, the radio spectrum has to be used optimally. Sharing of frequency bands by different kinds of radio systems has become a necessary measure to save frequency space. However, when several systems, are operating in each others spatial vicinity and are using frequency bands close to one another, the compatibility between those systems has to be studied.

Law makers, spectrum managers and manufacturers can use several methods to study scenarios of the use of future appliances and the consequences in terms of interference. Often they have to deal with complex scenarios where traditional analytical methods cannot offer any satisfactory solution. In those cases Monte Carlo simulation programs, such as for example SEAMCAT [1] can be used to study a variety of compatibility scenarios, see e.g. [2], [3]. However, in some cases also (simple) analytical models can give a satisfying impression of the compatibility of two radio systems. Moreover, an analytical model can provide more understanding of the underlying mechanisms of interference phenomena than a simulation. This also makes it useful for educational purposes.

In this paper we develop two analytical models to calculate the power densities and field strengths caused by multiple sources distributed in the neighborhood of a victim. In the first model we consider the situation in which the largest contribution of interference is caused by the nearest source. This may be the case when the radiated power of the sources is low, when the density of sources is low or when propagation conditions in the area are poor.

In general the interference effect is not limited to the immediate environment of the victim receiver. The interference detectable by the victim receiver might not be a problem when a source is remote. However, if there are many sources, each of them will make its own contribution to the interference. A receiver will thus pick up a large number of interfering sources, in total having the potential to cause unacceptable interference. For this case, we developed a simple model to calculate analytically the probability density function of the total received power from multiple interferers.

2 The Model

We assume that in the area around the victim a number of radio sources that transmit on the same frequency as the one that is received by the victim are present. We introduce the stochastic variable \(Q_G\) which denotes the number of sources in an area \(G\). The expected number of sources per square units in size is called \(\rho\).

In our models \(Q_G\) is the result of a Poisson-process (see [4]). This means that

\[
P(Q_G \leq q) = \sum_{k=0}^{q} \frac{e^{-\mu} \mu^k}{k!}
\]  

(1)

1. \(Q_G\) is Poisson distributed, i.e. the probability that there are equal or less than \(q\) sources in region \(G\) is
the area of a surface
stant in time. However, we consider that several val-
with respect to a probability density function (prob-
 according to some discrete distribution, i.e.

Here, and in the remainder of this paper for
any function \( M() \), \( \mathbb{E} M(z) := \int_Q M(z)f_Z(z)dz \)
(\( \sum_i f_Z(i) \)) denotes the expectation of \( M(z) \)
with respect to a probability density function (probability
mass function) \( f_Z(.) \) of \( Z \), and \( \|S\| \) denotes
the area of a surface \( S \).

The power \( Y \) of a source is assumed to be con-
stant in time. However, we consider that several values
\( Y_i, 1 \leq i \leq m < \infty \) of \( Y \) may occur for different
sources according to some discrete distribution, i.e.

\[
\mathbb{P}(Y = y_i) = \begin{cases} p_1 & \text{if } i = 1 \\ p_2 & \text{if } i = 2 \\ \vdots & \vdots \\ p_m & \text{if } i = m \end{cases} \tag{2}
\]

with \( \sum_i p_i = 1 \). In other words the transmitted
power of a randomly picked source has one of the con-
stant values \( y_1, y_2, \ldots, y_m \) according to the probability
mass function (2).

We call \( O \) the position of the victim. Around \( O \)
we distinguish between four different kinds of areas
that have different properties (see Figure 1). These ar-
eas have simple circular shapes. The disk shaped area
with center \( O \) and radius \( R_0 \) is assumed to be free of
interfering sources due to practical reasons. The value
of \( R_0 > 0 \) depends on the situation that is treated.
For example, the victim may be an FM receiver inside
a car, making it highly improbable that an interfer-
ring micro FM transmitter can come closer than, let us
say, 1 meter (see Section 5). The annulus with max-
imum radius \( R_1 \) around this disk, is called the free
space propagation zone \( A \). Annulus \( B \) with radii \( R_1 \)
and \( R_2 \), is called the two-ray propagation zone. The
received power of the signals from sources in these
areas are calculated with

\[
P_r(R) = \begin{cases} \frac{P_t G_t G_r A^2}{(4\pi R^2)} & \text{if } R_0 < R \leq R_1 \\
\frac{P_t G_t G_r h_t^2 h_r^2}{R^4} & \text{if } R_1 < R \leq R_2 \\
0 & \text{otherwise} \tag{3}
\end{cases}
\]

With \( P_r(R) \) the power that is received by the victim,
\( P_t(R) \) the signal power of the transmitter. \( G_t \) and \( G_r \)
are the antenna gains of the transmitter and the re-
ceiver respectively. Also, \( \lambda \) is the wavelength of
the transmitted radio frequency, and \( h_t \) and \( h_r \) are the an-
tenna heights.

The crossover distance \( R_1 \) is given by

\[
R_1 = \frac{4\pi h_t h_r}{\lambda} \tag{4}
\]

which follows from (3). The value of \( R_2 \) is the radio
horizon distance. We assume that radio signals from
sources positioned at a greater distance than \( R_2 \) are
below the receivers sensitivity threshold.

In some practical cases, the density of sources is
low or the radiated power is small. It is then not neces-
sary to take remote sources into account when calcu-
ating the total received power of a victim. For those
situations, we provide a simplified model which fo-
cusses on the contribution of the nearest source only.
In the following we will refer to this approach as
model I.

3 Results Model I

In this model we are only interested in the power that
is received from the nearest source in the vicinity of
the victim. Assuming that the nearest source is con-
siderably close to the victim, we apply the free-space
model on its propagation path.

Lemma 1 Under Poisson arrivals with intensity \( \rho \) of
random points on a plane, take \( R > R_0 \) the distance
from a reference point \( O \) to the nearest point of Pois-
on arrivals. Then \( X = R^2 \) has a shifted exponential
probability density function with parameter \( \rho \pi \), or

\[
f_X(x) = \rho \pi e^{-\rho \pi(R^2-R_0^2)}1_{[R_0^2,\infty)}(x) \tag{5}
\]

Proof: Define \( R \) the distance between \( O \) and
the nearest Poisson arrival point. Then, it read-
ily follows \( \mathbb{P}(R > r) = e^{-\rho \pi(r^2-R_0^2)}|\rho \pi(r^2 -
Theorem 2 Suppose $Y$ has a probability density function as defined in (2) and $X = R^2$ has density (5). Then $H = cY/X$, with $c > 0$ a constant, has the contaminated probability density function

$$f_H(t) = \sum_{i=1}^{m} p_i \frac{cY_i e^{-\rho(cy_i/R_a^2)}}{t^2} \left[ 0, \frac{cy_i}{R_a^2} \right] (t)$$

with

$$EH = c\rho\pi e^{\rho R_a^2} \Gamma(0, \rho\pi R_a^2) \mathbb{E}Y$$

$$EH^2 = c^2 \rho^2 \left( \frac{1}{R_a^6} - \frac{EH}{c\mathbb{E}Y} \right)$$

where $\Gamma(a, x) := \int_{x}^{\infty} z^{a-1} e^{-z} dz$ denotes the incomplete gamma function.

Proof: First, we observe that for a particular $i$ we have

$$P\left( \frac{cy_i}{X} < t \right) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1 - P(X < cy_i/t), & \text{if } 0 < t \leq \frac{cy_i}{R_a^2} \\ 1, & \text{if } t > \frac{cy_i}{R_a^2} \end{cases}$$

So, for $0 \leq t < cy_i/R_a^2$ we have $P(H < t) = \sum_{i=1}^{m} p_i (1 - P(X < cy_i/t) = \sum_{i=1}^{m} e^{-\rho(cy_i/t - R_a^2)} p_i$. The density of $H$ follows by taking the derivative. Further, we have

$$EH = \sum_{i=1}^{m} p_i \int_{cy_i}^{\infty} \rho\pi cy_i e^{-\rho \left( \frac{cy_i}{t} - R_a^2 \right)} \frac{dt}{t} = \sum_{i=1}^{m} cY_i p_i \rho \pi e^{\rho R_a^2} \int_{\rho R_a^2}^{\infty} u^{-2} \exp(-u) du,$$

where we applied the substitution $u = cy_i/\rho R_a^2$. Also, with partial integration it follows readily

$$EH^2 = \sum_{i=1}^{m} p_i \int_{cy_i}^{\infty} \rho\pi cy_i e^{-\rho \left( \frac{cy_i}{t} - R_a^2 \right)} \frac{dt}{t} = \sum_{i=1}^{m} (cY_i)^2 p_i \rho^2 \pi^2 e^{\rho R_a^2} \int_{\rho R_a^2}^{\infty} u^{-2} \exp(-u) du = \left( \frac{\rho}{R_a^6} - (\rho \pi)^2 e^{\rho R_a^2} \Gamma(1, \rho \pi R_a^2) \right)^2 \mathbb{E}Y^2.$$

Filling in the result for $EH$ finishes the proof. \hfill \Box

Combining the result of Theorem 2 with the free-space propagation formula (first part of (3)), i.e. choosing $c = G_i G_r \lambda^2/(4\pi)^2$, gives the probability density function of the received cumulated power flux density radiated from radio sources that are scattered in the vicinity of the victim according to a Poisson process with intensity $\rho$.

4 Results Model II

Lemma 3 Let $Q$ be Poisson distributed with parameter $\mu$ on an annulus $G(O, R_a, R_b)$ with midpoint $O$ and radii $R_a$ and $R_b$, with $0 < R_a < R_b$. The density of $X = D^2 = (x^2 + y^2)^{\eta/2}$, $\eta \geq 2$ of every point $(x, y)$ from the Poisson process on $G$ is

$$f_{X_q}(x) = \frac{2}{\eta(R_b^2 - R_a^2)} x^{\eta-2} \mathbb{E}X_{q, R_a, R_b}(x).$$

Proof: The distribution of $D$ is equal to $D \sim \mathcal{P}(D \leq d) = \pi d^2/(\pi R_b^2 - \pi R_a^2) = d^2/(R_b^2 - R_a^2)$. So, its density is $f_D(d) = 2d/(R_b^2 - R_a^2)$. The density of $X_q = u(D) = D^\eta$ becomes $f_{X_q}(x) = f_D(u^{-1}(x)) | \frac{d}{dx} u^{-1}(x) | = f_D(x^{1/\eta}) | \frac{d}{dx} x^{1/\eta} | = \frac{2x^{1/\eta}}{\eta(R_b^2 - R_a^2)} x^{\eta-2}.$

Lemma 4 The density of $H_q = cY/X_q$ (with $c > 0$ a constant) is equal to

$$f_{H_q}(t) = \sum_{i=1}^{m} \frac{2}{\eta R_a^6} (cY_i)^{\frac{1}{2}} p_i t \left( \frac{cy_i}{R_a^2} \right) \left[ 0, \frac{cy_i}{R_a^2} \right] (t)$$

and

$$EH_2 = \frac{c \ln(R_b^2/R_a^2)}{R_b^2 - R_a^2} \mathbb{E}Y$$

$$EH_4 = \frac{c \mathbb{E}Y}{(R_a R_b)^2}$$

$$EH_6 = \frac{c^2}{3} \mathbb{E}Y^2 \frac{R_a^{-6} - R_b^{-6}}{R_b^2 - R_a^2}$$

Proof: First, note that for a particular $i$ we have

$$P(cy_i/X_q < t) = P(X_q > cy_i/t), cy_i/R_a^2 < t \leq cy_i/R_b^2 \Rightarrow \sum_{i=1}^{m} p_i P(X_q > cy_i/t) = \sum_{i=1}^{m} cY_i p_i \rho e^{\rho cy_i/R_a^2} \int_{cy_i}^{R_a^2} u^{-2} \exp(-u) du.$$
Figure 2: Illustration of the proof of lemma 4 with \( m = 3 \).

when \( k \neq \frac{1}{2} \). Taking \( k = 1 \) and \( k = 2 \) concludes the proof. \( \square \)

4.1 Contribution of Annulus A

Assume in annulus A there are \( Q_A \sim \text{Pois}(\mu_A) \) sources present. In order to calculate the total amount of field strength received by the victim in \( O \) we have to calculate the sum \( S_A \) of the contributions of all the sources in A.

\[
S_A = \begin{cases} 
0 & \text{if } Q_A = 0 \\
\sum_{i=1}^{Q_A} H_{A,i} & \text{if } Q_A > 0
\end{cases} \quad (18)
\]

The density of \( S_A \) is compound Poisson [5] and it is difficult to find an expression for it. However, in many practical cases \( R_1 \) is relatively small compared to the density of sources, so in those cases we can neglect the probability of presence of sources in A. As an approximation we choose to calculate the expected value of \( S_A \) and use it as a constant contribution in our further calculations. By Wald’s equation [6] we simply derive

\[
\mathbb{E}S_A = \mu_A \mathbb{E}H_{A,1} \quad (19)
\]

where we used the independence of \( H_{A,1} \) and \( Q_A \).

4.2 Contribution of Annulus B

For \( S_B \) we have the similar expression

\[
S_B = \begin{cases} 
0 & \text{if } Q_B = 0 \\
\sum_{i=1}^{Q_B} H_{B,i} & \text{if } Q_B > 0
\end{cases} \quad (20)
\]

Generally \( |B| \gg |A| \) and so, it will contain the largest portion of sources in the neighborhood of the victim. Assuming that the expected number of sources in B is large, we apply a result of Robbins [7].

Lemma 5 (Robbins) Let us suppose that \( U_1, U_2, \ldots, U_n, \ldots \) are independent and identically distributed random variables with mean \( \mathbb{E}U_1 \) and variance \( \text{Var}(U_1) \). Let us put \( V_n = U_1 + U_2 + \ldots + U_n \). Let further \( N_\lambda \) denote a non-negative integer-valued random variable independent of the \( U_i \). Define \( \sigma^2 = \mathbb{E}N_\lambda \text{Var}(U_1) + \text{Var}(N_\lambda)(\mathbb{E}U_1)^2 \). If \( \lambda \to \infty \) implies

1. \( \sigma^2 \to \infty \)

2. \( \sqrt{\text{Var}(N_\lambda)} = o(\sigma^2) \)

3. \( N_\lambda \to \mathcal{N}(\mathbb{E}N_\lambda, \text{Var}(N_\lambda)) \)

Then \( V_{N_\lambda} = \sum_{i=1}^{N_\lambda} U_i \) is asymptotically normal, or \( V_{N_\lambda} \to \mathcal{N}(\mathbb{E}U_1 \mathbb{E}N_\lambda, \sigma^2) \)

Recall that in our model the integer-valued variable is \( \text{Pois}(\mu_B) \) distributed with \( \mu_B = \rho \|B\| \) and the properties of the \( U_i \) are as mentioned in lemma 4. In order to check the conditions, first notice that \( \sigma^2 = \mu_B \mathbb{E}H_{B,1}^2 \). Now, let \( \mu_B \to \infty \), then the first two conditions follow immediately. Condition 3 is a well known property of the Poisson distribution. So, now we can state that \( S_B \sim \mathcal{N}(\mu_B \mathbb{E}H_{B,1}, \mu_B \mathbb{E}H_{B,1}^2) \), or

\[
f_{S_B}(z) = \frac{1}{\sqrt{2\pi \mu_B \mathbb{E}H_{B,1}^2}} e^{-\frac{(z-\mu_B \mathbb{E}H_{B,1})^2}{2\mu_B \mathbb{E}H_{B,1}^2}} \quad (21)
\]

4.3 Total Received Power

The victim receives signals from sources out of both annuli A and B. We are interested in the total incoming power flux density in \( O \). So, we have to add (19) and (21) to get the probability density function

\[
f_{S_{\text{tot}}}(z) = ES_A + f_{S_B}(z) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(z-\alpha)^2}{2\sigma^2}} \quad (22)
\]

with \( \alpha = \mu_A \mathbb{E}H_{A,1} + \mu_B \mathbb{E}H_{B,1} \), and \( \sigma^2 = \mu_B \mathbb{E}H_{B,1}^2 \). Combining this with the propagation formulas given in (3) leads to the following result.

Corollary 6 A victim surrounded by a \( \text{Pois}(\mu) \) number of interfering sources with a distribution of the power \( Y \) as given in (2) receives a total power flux density of \( P_r \) which has the Normal probability density function

\[
f_{P_r}(v) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(v-\alpha)^2}{2\sigma^2}} \quad (24)
\]
where
\[
\alpha = \pi \left[ c_1 \ln \left( \frac{R_1^2}{R_0^2} \right) + c_2 \frac{R_2^2 - R_1^2}{(R_1 R_2)^2} \right] E Y \rho \quad (25)
\]
\[
\sigma^2 = \frac{1}{3} \pi c_2^2 (R_1^{-6} - R_2^{-6}) E Y^2 \rho \quad (26)
\]
with propagation constants \(c_1 = G_t G_r \lambda^2 / (4\pi)^2\) and \(c_2 = G_t G_r h_i^2 h_r^2\).

The total field strength \(E_{\text{tot}}\) generated by sources located in area \(A\) and \(B\) can be calculated using the power sum law, i.e. by taking the square root of the sum of all power flux densities
\[
E_{\text{tot}} = \sqrt[Q_A]{\sum_{i=1}^{Q_A} H_{A,i}^2} + \sqrt[Q_B]{\sum_{i=1}^{Q_B} H_{B,i}^2} \quad (27)
\]
So, we have to calculate the square root of the density function (24).

**Lemma 7** Let \(U \sim N(\mathbb{E}U, \text{Var}(U))\) and \(V = U^1/2\). For \(\mathbb{E}U \gg \text{Var}(U)\) the density of \(V\) can be approximated by
\[
f_V(v) = 2v \frac{e^{-\frac{(-v^2 - v \mathbb{E}U)^2}{2 \text{Var}(U)}} + e^{-\frac{(-v^2 + v \mathbb{E}U)^2}{2 \text{Var}(U)}}}{\sqrt{2 \pi \text{Var}(U)}} 1_{[0,\infty)}(v) \quad (28)
\]
**Proof:** Because the Normal distribution is defined on \(\mathbb{R}\), it is generally impossible to calculate the distribution of the square root of a normally distributed random variable. However, in our case the probability of a negative value for \(U\) is very low as it is the sum of positive valued random variables. In other words, the difference between \(U\) and its absolute value \(|U|\) is expected to be small. So, our approach is to calculate the probability density function of \(|U|^{1/2}\).

The transformation of \(U\) into \(V = |U|\) results in the folded Normal distribution [8].
\[
f_V(v) = \frac{e^{-\frac{(-v^2 - v \mathbb{E}U)^2}{2 \text{Var}(U)}} + e^{-\frac{(-v^2 + v \mathbb{E}U)^2}{2 \text{Var}(U)}}}{\sqrt{2 \pi \text{Var}(U)}} 1_{[0,\infty)}(v) \quad (29)
\]
Applying \(W = h(V) = V^{1/2}\) gives us \(f_W(w) = f_V(h^{-1}(w)) \left| \frac{dh^{-1}(w)}{dw} \right| = f_V(w^2) |2w|\).

**Corollary 8** A victim surrounded by a Pois(\(\mu\)) number of interfering sources with a power distribution as given in (2) receives a total signal strength of \(E_{\text{tot}} = \sqrt{S_{\text{tot}}}\) with the following distribution
\[
f_V(v) = 2v \frac{e^{-\frac{(-v^2 - \alpha)^2}{2 \sigma^2}} + e^{-\frac{(-v^2 + \alpha)^2}{2 \sigma^2}}}{\sigma \sqrt{2 \pi}} 1_{[0,\infty)}(v) \quad (30)
\]
with \(\alpha\) and \(\sigma\) as defined in (25) and (26) respectively.

### 5 Example: Low Power FM Transmitter Interference

A low power FM transmitter is a device that can be connected to a portable audio device, such as an mp3 or CD player. The sound is broadcasted through the transmitter on an FM frequency which can be picked up by appliances such as car or portable radios. The range of most devices on the market varies from 10 to 20 meters, and they can broadcast on FM frequencies from 88.0 to 108.0 MHz.

Despite its low power the unlicensed use of FM transmitters is not yet allowed in some European countries. In the Netherlands the use of FM transmitters is only recently allowed under the condition that no interference with the existing use of the FM band is caused. The prediction of the level of interference is important for radio communication authorities.

In this subsection we aim to calculate the cumulative distribution function of the power flux density received by an FM radio, caused by FM transmitters broadcasting on the same frequency. We regard the FM radio as a victim centered in the origin \(O\). Assume FM transmitters are randomly distributed over a large area around \(O\). Only a part of these transmitters will use the same frequency as the victim, with some constant density \(\rho\) transmitters / km\(^2\) = \(\rho 10^{-6}\) transmitters / m\(^2\).

Now, as an example let us assume that there are three FM transmitters on the market with power \(y_1 = 40\) nW, \(y_2 = 50\) nW, and \(y_3 = 60\) nW. Assume that they occur according to the rate \(1:2:1\). In other words, the transmitted power probability of a randomly picked FM transmitter is \(\text{Pr}(Y = y_1) = 0.5\), and \(\text{Pr}(Y = y_1) = \text{Pr}(Y = y_3) = 0.25\). It follows readily that \(\mathbb{E}Y = 50\) and \(\text{Var}(Y) = 2550\). Further, assume that since FM transmitters are mostly used in cars, the probability of the occurrence of an FM transmitter closer than 1 meter to the victim is zero, i.e. \(R_0 = 1\) m. We also need to find \(R_1\) and \(R_2\). We use the free space propagation model to calculate the impact of the closest FM transmitters. The signals from the remaining FM transmitters that are located between \(R_1\) and the line of sight horizon \(R_2\) will be treated with the two-ray model. Recall the cross-over distance (4) between both propagation models \(R_1\) is given by \(R_1 = (4\pi h_i h_r)/\lambda\). In this case we may assume \(h_i = h_r = 1\) m. If we take 100 MHz as the transmitted frequency, the value of the wave length becomes \(\lambda \approx 3.10^9/100.10^9 = 3\) m, so that \(R_1 \approx 4\) m. If we assume \(G_t = G_r = 1\) we find also \(c_1 = 0.0570\) and \(c_2 = 1\). The line of sight horizon is \(R_2 = 4124\sqrt{h_i} = 4124\) m [9].

For small values of \(\rho\) we may assume that the nearest source has the biggest impact on the victim.
and apply the results of Model I in section 3. Theorem 2 gives the probability density function of the received power from the nearest source. For higher values of \( \rho \) we can apply corollary 6 and find that the total received power by a victim is approximately normally distributed with expected value \( \alpha = 34.6\rho \) and variance \( \sigma^2 = 0.652\rho \). In figure 3 cumulative distribution functions of the total received power flux density are depicted for several values of \( \rho \).

![PFD Cumulative Distribution Functions](image)

Figure 3: Cumulative distribution functions of the received power flux density by a victim for several values of source density \( \rho \). For \( \rho = 500 \text{ sources/km}^2 \) we used the result of Model I. For \( \rho = 1000 \), \( 2000 \) and \( 4000 \) we applied the result of Model II.

6 Conclusion

In this article we regarded the situation of a victim receiver surrounded by transmitting radio sources. The sources are randomly distributed over the area and have a randomly chosen transmitting power. We have proposed two models to calculate the cumulative powers received by the victim. Analytical results are obtained with the use of some results of probability theory. We illustrated the results by applying them on a scenario of interfering FM transmitters.

Our models are based on some assumptions and simplifications of the reality. Therefore the calculation results should be considered as indications of a worst case situation. On the other hand, the models simplicity makes it easy to understand and to adapt them. Moreover, the analytical results derived in this paper are simple functions which makes it possible to obtain results quickly.

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