Abstract: - By considering Fibonacci sequences as impulsive response of a dynamical system a family of prototype low pass digital filters is introduced. Realizability properties, bandwidth and rise time are also included.

Key-Words: - Digital Filters, Fibonacci sequences, Bandwidth, Rise time.

1 Introduction

There is a large amount of literature on Fibonacci numbers and their generalizations. Biographical information about Fibonacci can be found at the MacTutor History of Mathematics Archive [1]. A compendium of information about the Fibonacci numbers to art, architecture, and music can be found at [2] (Fibonacci Numbers and The Golden Section in Art, Architecture and Music).

A quarterly journal since 1963 is dedicated to researches related to Fibonacci numbers and related questions: The website for the Fibonacci Quarterly can be found at [4] (Fibonacci Quarterly Home Page). Also useful are the websites of wikipedia [5] and mathworld [6].

A large variety of generalizations is available including Fibonacci polynomials, tribonacci numbers, k-nacci or multinacci numbers, [7] [8], and Fibonacci tiles [9].

Many properties of Fibonacci and related sequences are discussed in: “Matrix methods for Fibonacci and related sequences” by R. C. Johnson, November 31, 2007 [10].

Note that the Fibonacci sequence described by:

\[ F_{k+2} = F_{k+1} + F_k, \quad F_0 = 0, \quad F_1 = 1 \]  

(1)

may be considered as the impulsive response of the basic Fibonacci dynamic system:

\[ x(k+1) = Ax(k) + bu(k) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \]  

(2)

\[ y(k) = C \cdot x(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(k) \]

Indeed starting with \( x(0) = [0 \ 1]^T \), and \( u(k)=0 \), the first sequence terms are:

\[
\begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 \\ 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 \end{bmatrix}
\]

Of course with \( x(0) = [2 \ 1]^T \) we obtain the Lucas sequence.

The transfer function of the system (2) is:

\[ F(z) = \frac{1}{z^2 - z - 1} \]  

(3)

By computing the eigenvalues and the eigenvectors of the dynamic matrix \( A \) we have:

\[ A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = U \cdot \begin{bmatrix} \Phi & 0 \\ 0 & -\Phi \end{bmatrix} \cdot U^{-1} \]

\[ U = \begin{bmatrix} 1 & 1 \\ \Phi & -\Phi \end{bmatrix} \]  

(4)

\[ \Phi = 1 + \phi = \frac{1 + \sqrt{5}}{2} \]

\[ \phi = 0.6180339884... \]

The matrix \( A \) is symmetric, then its eigenvectors are orthogonal.

The irrational number \( \phi \) is known as the golden section.

The generic power of \( A \) is:
A^N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^N = \\
\begin{bmatrix} 1 & 1 \\ \Phi - \phi \end{bmatrix} \begin{bmatrix} \Phi^N & 0 \\ 0 & (-\phi)^N \end{bmatrix} \begin{bmatrix} \phi & 1 \\ 1 & \Phi + \phi \end{bmatrix} (5)

Therefore starting with x(0) = [0 1]^T the N-th sequence term is:

\[
x(N) = \frac{1}{1+2\phi} \begin{bmatrix} 1 & 1 \\ \Phi - \phi \end{bmatrix} \begin{bmatrix} \Phi^N & 0 \\ 0 & (-\phi)^N \end{bmatrix} \begin{bmatrix} \phi & 1 \\ 1 & \Phi + \phi \end{bmatrix} x(0)
\]

corresponding to the well known Binet formula.

In the following sections we consider some generalizations of this basic Fibonacci dynamical system and some possible implementations as digital filters.

2 Fibonacci Dynamic Systems

A generalization of the Fibonacci system (2) (3) is possible by considering a state space of dimension n>2 as follows:

\[
x(k+1) = \alpha \begin{bmatrix} 0 & 1 \\ 1 & \mathbf{1}_{n-1} \end{bmatrix} \cdot x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(k) \\
y(k) = c \cdot x(k) + \mathbf{d} \cdot u(k)
\]

For n=2, \alpha=1, C=(1,0), d=0 we obtain the system (2). With n>2, \alpha=1, C=(1,0,...,0), d=0 we obtain trinacci, tetranacci, n-acci sequences.

Note that these systems are positive, i.e. are realizable by using components only with positive values, and accordingly to Perron-Frobenius Theorem [11] have only one positive eigenvalue of greatest modulus \(\lambda_{\text{max}}\). The characteristic polynomial is

\[
a(z) = \det \begin{bmatrix} z\mathbf{1} - \alpha \begin{bmatrix} 0 & 1 \\ 1 & \mathbf{1}_{n-1} \end{bmatrix} \end{bmatrix} = z^n - \alpha \cdot z^{n-1} - \alpha^2 \cdot z^{n-2} \ldots - \alpha^{n-1}
\]

and:

\[
1 \leq \lambda_{\text{max}} < 2 \quad n \in [1, \infty).
\]

Indeed by rewriting the polynomial \(a(w)\), \(w=z/\alpha, \alpha>0, \text{as}:

\[
a(w) = w^n - w^{n-1} - w^{n-2} - \ldots - w - 1 = \frac{w^n - w^{n-1} - 1}{w-1}
\]

and applying a secant procedure the root of maximum modulus, for any dimension n, i.e. the maximum eigenvalue, may be evaluated as:

\[
\frac{\lambda_{\text{max}}}{\alpha} \approx 2 - \Phi - 1 = 2 - \frac{\phi}{\Phi - 1} = 2 \frac{\Phi - 1}{\Phi^{n-1}}
\]

The eigenvector related to \(\lambda_{\text{max}}\) is:

\[
u = [1, \lambda_{\text{max}}, \lambda_{\text{max}}^2, \ldots, \lambda_{\text{max}}^{n-1}]^T (10)
\]

To get stable filters, starting from basic Fibonacci systems, a simple choice is:

\[
0 < \alpha < 0.5 \quad \forall n
\]

3 Low Pass Fibonacci Digital Filters

A low pass digital filter of “n” order based on Fibonacci generalized systems in state space form is:

\[
x(k+1) = \alpha \begin{bmatrix} 0 & \mathbf{1}_{n-1} \\ 1 & \mathbf{1}_{n-1} \end{bmatrix} \cdot x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(k) (12a)
\]

\[
y(k) = \gamma \cdot [c_0, c_1, c_2, \ldots, c_{n-1}] \cdot x(k) (12b)
\]

It is easy to evaluate the transfer function due to the nearly controllable canonical form of the state space representation:

\[
G(z) = \frac{\alpha \cdot c_0 + \alpha^2 \cdot c_1 z^{-1} + \ldots + \alpha^{n-1} c_{n-1} z^{-n+1}}{z^n - \alpha \cdot z^{n-1} - \ldots - \alpha^{n-1} \cdot z^0} (13)
\]

Then the Fibonacci filter is identified by:

- Its order “n”
- The parameters “\(\alpha\)” and “\(\gamma\)”
- The vector \(c=[c_0, c_1, \ldots, c_{n-1}]\)

The parameters \(\alpha\) and \(\gamma\) can be established as follows:

- The Fibonacci filter, in state space form, is a positive system, then the dynamic matrix has a real positive dominant eigenvalue \(\lambda_F\) (the well known Frobenius eigenvalue) which is a function of the order “n”, but always included in the real interval \([1,2]\). The parameter “\(\alpha\)” can be evaluated according to the choice of the
real positive dominant pole \( \lambda_D \) of the Fibonacci filter:

\[
\alpha = \frac{\lambda_D}{\lambda_F}
\]

- “\( \gamma \)” is usually chosen to normalize at “1” the Bode gain of the filter; that is:

\[
\gamma = \frac{\alpha^0 - \alpha^1 - \alpha^2 - \ldots - \alpha^n}{\alpha^{n-1} c_0 + \alpha^{n-2} c_1 + \ldots + \alpha^0 c_{n-1}}
\]

(14)

- The output vector \( e \) may be chosen in such a way to place a suitable number “\( m \)” of zeros in “-1” to ensure a good “low pass” characteristic of the Fibonacci filter. Moreover the number “\( m \)” of these zeros establishes the maximum phase lag of the filter:

\[
\phi(\pi) = n \cdot \pi - m \cdot \frac{\pi}{2}
\]

(15)

The components of the vector “\( e \)” are:

\[
\begin{align*}
c_i &= \begin{cases} 
\frac{m}{\alpha^{n-1-i}} & i = 0, 1, 2, \ldots, m \\
0 & i = m + 1, m + 2, \ldots, n - 1
\end{cases}
\end{align*}
\]

(16)

With the previous considerations the transfer function of a generalized Fibonacci filter is:

\[
G(z) = \frac{2 - \alpha^{n+1}}{1 - \alpha} \cdot \frac{(z+1)^m 2^m}{z^n - \alpha \cdot z^{n-1} - \ldots - \alpha^n \cdot z^0}
\]

(17)

\[\text{with } \alpha = \frac{\lambda_D}{\lambda_F}\]

To clarify the characteristics of the Fibonacci filters, let us examine the frequency response of a filter of the second order with one zero in “-1” for different values of the dominant pole (\( \lambda_D \)= 0.2, 0.4, 0.6, 0.8). The results are shown in Fig.1.

Some considerations are in order:

- The magnitude response is strongly dependent on the dominant eigenvalue of the filter: greater is the dominant eigenvalue (i.e. slower the dynamic dominant mode) lesser the filter bandwidth.

- The phase lag of the filter is almost independent by the dominant eigenvalue. A greater phase lag corresponds to a greater dominant eigenvalue.

The increment of the zero number placed in “-1”, with the limit of the filter order, can modify the frequency response of the filter and its dynamic behaviour. Fig. 2 reports the frequency response of the second order filters for dominant poles \( \lambda_D = \) 0.2, 0.4, 0.6, 0.8, as in the previous figure, but with two zeros in “-1”.

With the increment of the zeros in “-1” the magnitude of the frequency response substantially does not change while the phase lag decreases of \( \pi/2 \) at the maximum value of the frequency: 0.5 Hz·sec.

It is also interesting to examine the frequency response of filters with the same dominant pole but with different order. The following figure reports magnitude and phase of filters, by increasing the order but with dominant pole fixed to 0.8 and a number of zeros in “-1” equal to the filter order.
Fig. 2. Frequency Response of Second Order Fibonacci Filters

It appears that the magnitude of the frequency response is weakly dependent on the order of the filter, while the phase lag increases with the order “n”. This happens also for filters with different dominant eigenvalue. Then:

- The bandwidth of the filters is largely unaffected by the order “n”
- The phase lag is minimum for a filter of the second order

Concluding it is sufficient to consider filters of the second order and the filter choice can be limited to the choice of its dominant eigenvalue which establishes the filter bandwidth.

Fig. 3. Frequency Response of Fibonacci Filters with Fixed Dominant Pole

Figure 4. reports the bandwidth of a second order filter for different values of the dominant pole, with

\[ p = \lambda_{\text{max}} = 2 \frac{\alpha}{\Phi + 1} \]  

For \( \alpha = 0 \) we get \( B_3 = 0.18202833 \ldots \), the bandwidth of the second order binomial filter. Approximately we have, with the due care for the definition of rise time in the discrete case, :

\[ B_3 = 0.18 \cdot \left(1 - \frac{2\alpha}{1 + \Phi}\right) \]  

Concluding it is sufficient to consider filters of the second order and the filter choice can be limited to the choice of its dominant eigenvalue which establishes the filter bandwidth.
Finally the following table reports the rise time, the bandwidth and their product for a proper filter (one zero in “-1”). The results are in good agreement with the formula (18).

<table>
<thead>
<tr>
<th>Dom. pole</th>
<th>Bandwidth</th>
<th>Rise time</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.201</td>
<td>1.74</td>
<td>0.350</td>
</tr>
<tr>
<td>0.4</td>
<td>0.139</td>
<td>2.67</td>
<td>0.371</td>
</tr>
<tr>
<td>0.6</td>
<td>0.081</td>
<td>4.47</td>
<td>0.362</td>
</tr>
<tr>
<td>0.8</td>
<td>0.035</td>
<td>9.85</td>
<td>0.345</td>
</tr>
</tbody>
</table>

Tab. 1. Bandwidth and rise time relationship.

4 Conclusions

A family of prototype digital low pass filters has been introduced as a generalization of a basic Fibonacci dynamic system. Realizability properties are presented; bandwidth and rise time as function of the dominant pole are reported. Extensions to band pass or high pass filters are possible by using standard procedures of filter design [12], [13].

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