Observations concerning the dynamics for k-order nonlinear discrete determinist exchange rate models

MIRELA-CATRINEL VOICU
Faculty of Economic Sciences
West University of Timișoara
ROMANIA

Abstract: - In this paper we present some results for a particular type of k-order exchange rate models. The relation between the cycles of period \( p \) (\( p \geq 1 \)) is investigated. Given their nonlinear nature, these systems can display a more complex evolution (like chaotic behavior). We present such examples of behavior, using numerical simulations. The algorithms implementation is made using VBA (Visual Basic for Applications) program in Excel, and the images in the figures in this paper are made using Mathematica.

Key-Words: - nonlinear system, period-p cycle, chaos.

1 Introduction

According to [2], a general equation modeling the exchange rate evolution is given by:

\[ S = X_t E_t(S_{t+1}) \]  

(1)

In the above equation, \( S_t \) is the exchange rate at the moment \( t \); \( X_t \) describes the exogenous variables that drive the exchange rate at the moment \( t \); \( E_t(S_{t+1}) \) is the expectation held at the moment \( t \) in the market about the exchange rate at the moment \( t+1 \); \( b \) is the discount factor that speculators use to discount the future expected exchange rate (\( 0 < b < 1 \)).

This model allows us to take into account two components for forecasting: a forecast made by the chartists \( E_ch(S_{t+1}) \) and a forecast made by the fundamentalists \( E_f(S_{t+1}) \):

\[ E_t(S_{t+1})/S_{t-1} = (E_ch(S_{t+1})/S_{t-1})^m \cdot (E_f(S_{t+1})/S_{t-1})^{1-m}, \]

(2)

where \( m_t \) is the weight given by the chartists and \( 1-m_t \) is the weight given by the fundamentalists at the moment \( t \).

The fundamentalists assume the existence of an equilibrium exchange rate \( S^* \). If at the moment \( t-1 \) the exchange rate \( S_{t-1} \) is above, respectively below, the equilibrium rate \( S^* \), the fundamentalists expect the future exchange rate \( S_{t+1} \) to go down, respectively increase, with the speed \( \alpha \). More precisely, if they observe a deviation today, then their forecasts is the following:

\[ E_f(S_{t+1}) = \left( \frac{S^*}{S_{t-1}} \right)^\alpha, \quad \alpha > 0. \]  

(3)

The chartists use the past values of the exchange rate to detect patterns that they extrapolate in the future. An equation which gives a general description of the different models used by chartists is the following:

\[ E_ch(S_{t+1})/S_{t-1} = f(S_{t-1}, ..., S_{t-N}). \]  

(4)

According to [2], it is possible to specify such a rule, in general terms, as follows:

\[ E_ch(S_{t+1})/S_{t-1} = \left( \frac{S_{t-1}}{S_{t-2}} \right)^{C_1} \left( \frac{S_{t-2}}{S_{t-3}} \right)^{C_2} \cdots \left( \frac{S_{t-N+1}}{S_{t-N}} \right)^{C_{N-1}}. \]  

(5)

The exact nature of this rule is determined by the exponents \( C_i \). These can be positive, negative, or zero. The weight \( m_t \), in equation (2), given by chartists is

\[ m_t = \frac{1}{1 + \beta(S_{t-1} - S^*)}, \quad \beta > 0. \]  

(6)

The parameter \( \beta \) measures the precision degree of the fundamentalists' estimation. When the exchange rate is in the neighbourhood of the equilibrium rate, chartists' behavior dominates. When the exchange rate differs from the fundamental rate, then the expectation will be dominated by the fundamentalists.

In this paper we consider the case \( X_t = 1 \) (which means that \( S^* = 1 \)) and for chartists we consider the expectation:

\[ E_ch(S_{t+1})/S_{t-1} = \left( \frac{S_{t-1}}{S_{t-k}} \right)^c, \quad c > 1, \quad k \geq 2, \quad k \in \mathbb{N}. \]  

(7)

In equation (2) we will use the expectations given by equations (3) and (7). In equation (1) we will use the expectations given by equation (2). In this way, we obtain
the following difference equation:

\[ S_i = S_{i-1} \left( \frac{(2+\alpha)b}{1+\beta(e^{\alpha-1})} \right) + (1-\alpha)S_i \left( \frac{-2b}{1+\beta(e^{\alpha-1})} \right) \]  

(8)

If we denote \( s_i = \ln S_i \), then equation (8) can be written in the form:

\[ s_i = \left( \frac{(2+\alpha)b}{1+\beta(e^{\alpha-1})} \right) s_{i-1} + \left( \frac{-2b}{1+\beta(e^{\alpha-1})} \right) s_i \]  

(9)

with \( s_i \in \mathbb{R} \) and \( t \in \mathbb{Z} \). We can rewrite equation (9) in the following vectorial form:

\[ (s_{i+2}, s_{i+3}, \ldots, s_{i+k+1}) = F(s_{i+1}, s_i, s_{i+k+1}) \]  

(10)

where

\[ F: \mathbb{R}^k \rightarrow \mathbb{R}^k, \quad F(x_1, x_2, \ldots, x_k) = (F_1(x_1, x_2, \ldots, x_k), \ldots, F_k(x_1, x_2, \ldots, x_k)) \]

is defined in the following way:

\[ F_i(x_1, x_2, \ldots, x_k) = \phi(x_1)x_i + \phi(x_k)x_i, \]

\[ \phi(x) = \frac{(2+\alpha)b}{1+\beta(e^{\alpha-1})}x_i + \frac{-2b}{1+\beta(e^{\alpha-1})}x_i \]

In Sections 2 and 3 we will present some analytical results for system (10) and in Section 4 we will present some numerical simulations.

2 Analytical results

2.1. Steady-state existence, unicity and stability

Proposition 1. In the case in which \( c > 1, b \in (0, 1), \alpha > 0 \) and \( \beta > 0 \), the system (10) has a unique fixed point and this point is \( (0, 0, \ldots, 0) \in \mathbb{R}^k \).

Proposition 2. For \( c > 1, \alpha \in (0, 1), \beta > 0 \) and \( b \in \left( 0, \frac{1}{2c+1} \right) \), and any initial condition of system (10), the limit \( p \) is 0. This implies that the fixed point \( (0, 0, \ldots, 0) \in \mathbb{R}^k \) is globally attractive.

Proposition 3. The fixed point is stable for \( b \in (0, y(c,k)) \), where \( \frac{1}{2c+1} \leq y(c,k) < 1 \).

2.2. Period-\( p \) cycles

We recall that, a period-\( p \) point \( x \in \mathbb{R}^k \) of system (10) is a point for which \( F^p(x) = x \) and \( F^i(x) \neq x, \quad \forall i = 1, p - 1 \).

Observation 1. If \( k < p \) and system (10) has a period-\( p \) cycle, then the period -\( p \) cycle has following form:

\[
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_p \\
 x_1 \\
 \end{pmatrix}
\quad F
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_p \\
 x_1 \\
 \end{pmatrix}
\quad F
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_p \\
 x_1 \\
 \end{pmatrix}
\quad \cdots
\quad F
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_p \\
 x_1 \\
 \end{pmatrix}
\quad F
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_p \\
 x_1 \\
 \end{pmatrix}
\]  

(11)

where \( x_i = x_{pi}, \forall i = 1, p, \) and this means that the period-\( p \) cycle has the form:

\[
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_p \\
 x_1 \\
 \end{pmatrix}
\]  

(12)

Observation 2. Using equation (9), from the relation (12) we can observe that a period-\( p \) cycle is given by \( p \) consecutive points (from \( R \)):

\[
\begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_p \\
 x_1 \\
 \end{pmatrix}
\]  

(13)

Observation 3. If \( k \geq p \) and system (10) has a period-\( p \) cycle, then the period-\( p \) cycle verifies the following relation:

\[
\begin{pmatrix}
 x_1 \\
 x_p \\
 x_{pi} \\
 \vdots \\
 x_1 \\
 \end{pmatrix}
\]  

(14)

where \( x_i = x_{pi}, \forall i = 1, p, \) and this means that the cycle has the following form:

\[
\begin{pmatrix}
 x_1 \\
 x_p \\
 x_{pi} \\
 \vdots \\
 x_1 \\
 \end{pmatrix}
\]  

(15)

where \( k = i + jp, \quad j \geq 1, \quad i = 0, p - 1. \)
Now, we provide a periodical-$p$ cycle classification in respect to the systems' orders.

**Proposition 4.** Let $p$ and $i$ be fixed values where $p \geq 1$ and $2 \leq i \leq p+1$. If $j_0 \geq 0$ exists so that the system

$$s_i = [(c + \alpha)m + b + (1 - \alpha)b]s_{i-1} - cmbs_{i-k}$$

(where $k = i + j_0p$) has a cycle of period $p$, then the system

$$s_i = [(c + \alpha)m + b + (1 - \alpha)b]s_{i-1} - cmbs_{i-k}$$

(where $k = i + j_0p$, $j \geq 0$) has a cycle of period $p$.

Proof. If $j_0 = 0$, $k_i < p$ (this means that $k_0 = i$) and the system (15) has a period-$p$ cycle, then the cycle has the form presented in the following relation:

$$\begin{align*}
(x_1) \rightarrow (x_2) \rightarrow \cdots \rightarrow (x_{p-1}) \rightarrow (x_p) \rightarrow \cdots \rightarrow (x_{p-k}) \rightarrow (x_{p-k+1}) \rightarrow (x_{p-k+2}) \rightarrow \cdots \rightarrow (x_{p-k+i}) \rightarrow (x_{p-k+i+1}) \rightarrow \cdots \rightarrow (x_{p-k+i+j-1}) \rightarrow (x_{p-k+i+j})
\end{align*}$$

We now consider that $j_0 = 0$. For the system of $k$-order ($k = i + p$), we consider the followings vectors (from $R^{i+p}$):

$$\begin{align*}
(x_1) \rightarrow (x_2) \rightarrow \cdots \rightarrow (x_{p-1}) \rightarrow (x_p) \rightarrow \cdots \rightarrow (x_{p-k}) \rightarrow (x_{p-k+1}) \rightarrow (x_{p-k+2}) \rightarrow \cdots \rightarrow (x_{p-k+i}) \rightarrow (x_{p-k+i+1}) \rightarrow \cdots \rightarrow (x_{p-k+i+j-1}) \rightarrow (x_{p-k+i+j})
\end{align*}$$

and we can observe (from the system (15)) that these vectors also form a periodical-$p$ cycle for the system (16).

We denote now $C_i = \{k \mid k \geq 2 \text{ and } k \text{ modulo } p = i\}$, which represents the class of the number $k$ for which the remainder ($k \text{ modulo } p$) after numerical division of $k$ by $p$ returns the value $i$. For a fixed $p$ ($p \geq 1$) value, we can observe that we have $p$ different classes ($2 \leq i \leq p+1$).

We consider now that $p \geq 1$ and we want to present a classification of period-$p$ cycles in respect to the $k$-order of the system (10).

**Proposition 5.** We now fix the period $p$ ($p \geq 1$). In respect to order $k$ ($k \geq 2$) of the system (10), we find $p$ qualitatively different possibilities for the classification of periodical-$p$ cycles (corresponding to the $p$ class $C_i$, where $2 \leq i \leq p+1$).

Proof. For the system of second order, a periodical-$p$ cycle has the form

$$\begin{align*}
(x_1) \rightarrow (x_2) \rightarrow \cdots \rightarrow (x_{p-1}) \rightarrow (x_p) \rightarrow \cdots \rightarrow (x_{p-k}) \rightarrow (x_{p-k+1}) \rightarrow (x_{p-k+2}) \rightarrow \cdots \rightarrow (x_{p-k+i}) \rightarrow (x_{p-k+i+1}) \rightarrow \cdots \rightarrow (x_{p-k+i+j-1}) \rightarrow (x_{p-k+i+j})
\end{align*}$$

and the vector $(x_1, \ldots, x_p)$ is a solution for the system:

$$\begin{align*}
x_1 &= \phi(x_1)x_2 + \phi(x_2)x_1 \\
\vdots \\
x_i &= \phi(x_{i-1})x_{i+1} + \phi(x_{i+1})x_{i-1} \\
x_{i+j} &= \phi(x_i)x_{i+j} + \phi(x_{i+j})x_i \\
x_j &= \phi(x_i)x_{i+j} + \phi(x_{i+j})x_i
\end{align*}$$

For the system of third order, a periodical-$p$ cycle has the form
We can observe that the relations (20)-(23) describe different systems, which generally have different solutions. Also, using the Proposition 4, we have completed the proof.

Our result can now lead to a generalization for a nonlinear discrete dynamical system:

**Proposition 6.** If the system of \( k \) order

\[
x_i = \phi(x_{i-1}, \ldots, x_{i-k})
\]

(where \( k = \max \{i_1, \ldots, i_k\} \)) has a period \( -p \) cycle (consisting, as above, of \( p \) consecutive points from \( R \)), then the system

\[
x_i = \phi(x_{i-j}, \ldots, x_{i-1+p-j})
\]

\( j_1, \ldots, j_k \geq 0 \)

of the order \( k_i \) where \( k_i = \max \{j_1, \ldots, j_k, p + j_k\} \) has a period-\( p \) cycle formed with the same \( p \) consecutive points from \( R \), \( \forall j_1, \ldots, j_k \geq 0 \).

### 3 Numerical simulations

We now recall some notions which will be used in this section. We say that a set \( A \) is an attracting set with the fundamental neighbourhood \( U \), if it verifies the following properties (see [5]):

1. **attractivity:** for every open set \( V \supset A \), \( F^t U \subset V \) for all sufficiently large \( t \).
2. **invariance:** \( F^t(A) = A \), for all \( t \).
3. **\( A \) is minimal:** there is no proper subset of \( A \) that satisfies conditions 1 and 2.

The basin of attraction is the set of initial points \( x \) so that \( F^t(x) \) is close to \( A \) when \( t \to \infty \).

It is possible to classify the different attractors: attracting fixed point, attracting \( n \)-cycle, quasiperiodic attractor and strange attractor. An attractor, as an experimental object, gives a global description of the asymptotic behavior of a dynamical system.

When a deterministic mechanism presents complex behavior with intermittence, we can conclude that the series evinces chaos under certain conditions. The sensitive dependence on initial conditions is one of the most essential aspects in identifying the chaos. We recall that the sensitive dependence on initial conditions means that two trajectories starting very close together will rapidly diverge from each other.

The strange attractor is associated with a chaotic state of time evolution and is characterized by the sensitive dependence on initial conditions.

A measure of the average rate of exponential divergence exhibited by a chaotic system is given by the Lyapunov exponents of the system; the positivity of one of these exponents can suggest the presence of chaos.

The Lyapunov exponents \( \lambda_1, \lambda_2, \ldots, \lambda_4 \) are given by
where $J\left(F(s_0, s_1, \ldots, s_{t+k})\right)$ represents the Jacobian matrix of the function $F$. For a period-$p$ point the Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_k$ are given by

\[ \left\{ e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_k t} \right\} = \lim_{n \to \infty} \left\{ \text{eigenvalues of} \left( \prod_{s=0}^{t} J(F(s_0, s_1, \ldots, s_{t+k})) \right)^{\frac{1}{t}} \right\} \]

We recall now that for an attracting period-$p$ cycle the Lyapunov exponents are negative; in case of a bifurcation point, at least one Lyapunov exponent is zero; for a limit cycle one Lyapunov exponent is zero and the others are negative and for a chaotic behavior the highest Lyapunov exponent is positive while the sum of all Lyapunov exponents is negative.

In order to compute the Lyapunov exponents, when system (10) displays a chaotic behavior, we use the method proposed in [1], based on the Householder QR factorization and the implementation method proposed in [8].

Fig. 1 Chaotic attractors in the case $b = 0.95$, $\alpha = 2$, $c = 2$, $k = 2$ and $(s_0, s_1, s_2) = (0.02, -0.02, s_i)$, the space $(s_i, s_{i+1})$

We have made many numerical simulations and we have found many situations in which the system displays chaotic behavior. In order to illustrate these, now, we give some examples. The implementation of the algorithms is made using VBA (Visual Basic for Applications) program in Excel, and the images from the
figures are made using Mathematica.

In Figure 1 we present the case \( b = 0.95, \alpha = 2, c = 2, k = 2 \) and \( s_0 = 0.02, s_1 = -0.02, \) the space \((s_1, s_{i+1})\).

In Figure 2 we present the case \( b = 0.95, \alpha = 2, c = 2, k = 3 \) and \((s_0, s_1, s_2) = (0.02, -0.02, s_2)\), the space \((s_i, s_{i+1})\).

In Figure 2, we have

\[
s_i = \left(\frac{(2 + \alpha)b}{1 + \beta (e^c - 1)}\right) s_i + \left(\frac{-2b}{1 + \beta (e^c - 1)}\right) s_0.
\]

In Figure 3 we fix: \( \alpha = 2, b = 0.95, \beta = 600, c = 2 \) and the initial condition: \((s_0, ..., s_{k+1}) \in R^k, s_0 = 0.02\), where \( s_i = -0.02, s_j = \varphi(s_{i-1}), s_{i+1} + \varphi(s_{i-1}), s_{i+2} \) for \( i \in 2, k - 1 \) and \( s_j = \varphi(s_{i-1}) s_{i+1} + \varphi(s_{i-1}), s_{i+2} \) for \( i \in 2, k - 1 \). We make \( k \) variable, where \( k \) represents the order of the system for each particular case.

From Figures 1, 2 and 3 we can observe a similarity between the images of attractors for each particular order of the systems.

**4 Conclusion**

This study leads to the conclusion that there are similarities between the dynamics of the studied systems. These results are interesting from a mathematical viewpoint. But also, these results lead to economic interpretations.

We have used our implementation methods for detecting the dynamics of a nonlinear system, presented in [4]. These methods present the way in which, in a very short time, we can obtain a very large number of observations with the help of the computer which, at the same time, is conclusive for the obtained results.

**References:**


