A Generalization and a Quasi-Fractal Scheme of the Fibonacci Integer Sequence

J-B. CAZIER¹, C. MANDAKAS² and V. GEKAS²

¹ BioInformatics and BioStatistics Service, Cancer Research UK, WC2A 3PX, London, United Kingdom
² Department of Environmental Engineering, Technical University of Crete, Polytechniopolis, Chania 73100, Crete, GREECE

Abstract: Sequences of integers can produce irrational or asymmetric numbers. For example the golden section number, \( \phi \), can be obtained using the ratios of subsequent terms of the Fibonacci integer sequence. In the Academy of Plato there was research on the square roots of integer numbers. As Theon o Smyrnaiose tells us another famous irrational number, the square root of two, was produced as the common limit of convergence of two series of fractionals. We have shown that the numerators and denominators of those fractionals belong to the terms of certain integer sequence similar to the Fibonacci sequence, i.e. there are terms of a new, being a modified Fibonacci, integer sequence. We have also reported two new integer sequences in which the two ratios given by Archimedes as approximations of the square root of three. In this paper we furthermore generalize the method to give approximations of the square root of all the terms of the Fibonacci integer sequence i.e the square roots of 5, 8, 13, 21 etc. Thus a quasi fractal (self-similar scheme) appears. The practical interest of these numbers is that they appear in many systems of Physics, in general such as it is the golden tree and the bronchic tree of the respiratory system.

Keywords: fractals, square root of two, square root of three, golden section number, square root of Fibonacci sequence terms, integer sequences

1 Introduction

Irrational numbers can be produced by simple operation on sequence of integers. For example, the golden section number \( \phi \), can be obtained by convergence of its upper and lower bound limit from the Fibonacci integer sequence by alternating ratios of subsequent terms of the sequence [1]. As Plato put it “everywhere in the Universe the Rational is coupled with the Irrational” [Passas, 2]. The ancients Greeks were very keen on studying ratios and surfaces of simple geometric figures [Politeia, 3]. As Theon o Smyrnaiose tells us [4] another famous irrational number, the square root of two, was produced as the common limit of convergence of two series of fractionals. We have shown that the numerators and denominators of those fractionals belong to the terms of certain integer sequence similar to the Fibonacci sequence, i.e. there are terms of a new, being a modified Fibonacci, integer sequence. [5-
We have successfully constructed such an integer sequence giving accurate approximation of the square root of two alternating series, from higher than $\sqrt{2}$ and from lower than $\sqrt{2}$ values. We have also reported two new integer sequences in which the two ratios given by Archimedes [7] as approximations of the square root of three, appear [5, 8-9]. In this paper we furthermore generalize the method to give approximations of the square root of all the terms of the Fibonacci integer sequence i.e. the square roots of 5, 8, 13, 21 etc. Thus a quasi fractal (self-similar scheme) appears. The practical interest of these numbers is that they appear in many systems of Physics, in general [10]. In particular we deal with the application of the irrational numbers in the golden tree and in the bronchic tree of the respiratory system [5-6, 10].

2. The golden section number and the square roots of two and three.

2.1. The golden section number [1]

The Fibonacci sequence and its generating algorithm is well known:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

The base, arbitrarily but conveniently taken in any integer sequence, i.e. the two first terms, are 1, 1: 

$a_1 = 1$, for $n = 1$

$a_2 = 1$, for $n = 2$

The generation formula is a very simple one:

$a_n = a_{n-1} + a_{n-2}$, for $n > 2$

This sequence gives good approximations of the golden section number $\phi$, when the ratio of subsequent terms, $\frac{a_{2k}}{a_{2k+1}}$, are considered alternatively:

$$\frac{1}{1} < \frac{8}{5} < \frac{55}{34} < \frac{377}{233} < \phi < \frac{144}{89} < \frac{21}{13} < \frac{3}{2}$$

Two of the properties of the integer sequence are worth noting: Firstly by trying ourselves we observe that the limit reached depends on the generation formula and not on the base, i.e. irrespectively of the initial values, which in the Fibonacci case is \{1, 1\}, the subsequent-term ratios tend to $\phi$. Secondly there exists an index value, $n_e$ for which the $n_e^{th}$ term of the sequence is equal to the square root of the index (Figure 1). In the Fibonacci sequence, this occurs when the index value $n_e$ equals to 12; i.e. $\alpha_{12} = 144 = 12^2$

2.2. The square root of two [5, 8-9]

![Figure 1: Construction of the integer Fibonacci sequence. The 12th term is equal to 144, i.e. the square of 12. [9]](image-url)
A modification of the Fibonacci integer sequence can lead to the square root of two:

\[
1, 1, 2, 3, 5, 7, 12, 17, 29, 41, 70, 99, 169, 239, ...
\]

resembling the Fibonacci sequence, but created slightly differently with the following method.

The base is again \(\{1, 1\}\) but the generation formula, for \(n>2\), is

\[
\alpha_n = \alpha_{n-1} + \alpha_{n-2}, \quad \gamma \alpha n \geq 2 \text{ and } n=2k+1
\]

\[
\alpha_n = \alpha_{n-1} + \alpha_{n-3}, \quad \gamma \alpha n \geq 2 \text{ και } n=2k
\]

Here the progression of the sequence is “retarded” by adding a delay in the building of the next value. The same method of alternating ratios converges to the irrational square root of two:

\[
\begin{align*}
1 & < \frac{7}{5} < \frac{41}{29} < \frac{239}{169} < \sqrt{2} < \frac{99}{70} < \frac{17}{12} < \frac{3}{2}
\end{align*}
\]

Interestingly the above ratios are the same reported by Theon o Smyrnaios [2, 4]. He has obtained them using ‘analogies’, ratios but the result is the same as with the method of integer sequences. It is worth noting also that this ‘retarded Fibonacci’ sequence carries the same property as the usual Fibonacci, i.e. of having an index value which in this case is the 13, \(n_e = 13\), for the \(n_e\) term equals the square of \(n\): \(a_{13} = 169 = 13^2\).

Furthermore it appears that irrespectively of the initial values, applying the same generation formula the obtained formula converges to the same limit, i.e. the square root of two (Fig.2).

\[\text{Figure 2 Generation of the Integer Sequence converging to the square root of 2. The 13\textsuperscript{th} term equals the square of 13 (169)}\]

2.3. Extending the method to reach the square root of three [5, 8-9].

Archimedes left an approximation for the square root of three [7]:

\[
\frac{265}{153} < \sqrt{3} < \frac{1351}{780}
\]

The origin of the four integers constituting this approximation has puzzled ever since the modern reader[7]. We have reported [5, 8-9 the generation of two integer sequences that could lead to these four integers. The first sequence is the following:

\[
1, 1, 2, 3, 4, 7, 11, 15, 26, 41, 56, 97, 153, 209, 362, 571, 780, 1351, ...
\]

\[
\alpha_1 = \alpha_2 = 1, \quad \gamma \alpha n = 1, 2
\]

\[
\alpha_n = \alpha_{n-1} + \alpha_{n-2}, \quad \gamma \alpha n = 3k+1 \text{ και } n=3k
\]

\[
\alpha_n = \alpha_{n-1} + \alpha_{n-3}, \quad \gamma \alpha n = 3k+2
\]

The two out of the four integers, the numerator and the denominator of the one of the two ratios suggested by Archimedes, namely the integers...
780 and 1351, appear consecutively. A small modification of the initial base \{1, 1\} to \{2, 3\} with the same generation formula produces the following integer sequence:

\[ 2, 3, 5, 8, 11, 19, 30, 41, 71, 112, 153, 265, 418, \ldots \]

In the new sequence the other two out of the four Archimedes integers, namely the 153 and 265 appear also consecutively. The generating procedure for both is again following a 'retarded Fibonacci' sequence but now requiring three steps.

Easily it is observed that the ratios \( \frac{a_{3k+2}}{a_{3k+1}} \) tend to \( \sqrt{3} \), i.e., they are approximations of \( \sqrt{3} \). Again the limit value of the series is independent of the starting values \( \alpha_1 \) and \( \alpha_2 \). However there is no more an index value \( n_e \) of which the square is equal to \( n_e^2 \).

### 3. Generalization

Integer sequences following the same generating rules lead to a consistent irrational number, regardless of the base, e.g the first two terms. The integer sequences so far could be classified into three groups

A: those with no retardation step, following the simple \( a_n = a_{n-1} + a_{n-2} \) rule. They are associated with \( \phi \), the same as the Fibonacci well known sequence,

B: those retarding each other step, following the rule:

- \( a_n = a_{n-1} + a_{n-2} \) for \( n = 2k + 1 \)
- \( a_n = a_{n-1} + a_{n-3} \) else,

associated with the square root of 2,

C: those retarding each third step, i.e. following the rule

- \( a_n = a_{n-1} + a_{n-2} \) for \( n = 3k + 2 \)
- \( a_n = a_{n-1} + a_{n-3} \) for \( n = 3k + 1 \) and \( n = 3k \),

Those are associated with the square root of

Following this principle it is possible to create further sequences each 4th, 5th, 6th, n-th step. Remarkably these series seem to keep the property of converging to certain irrational values, namely the square roots of ratios of integers which are members, remarkably, of the Fibonacci sequence. Thus, a curious quasi fractal property is revealed for the Fibonacci sequence (Table 1)

<table>
<thead>
<tr>
<th>Generating formula</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n = a_{n-1} + a_{n-2} ) for ( n = 2k + 1 )</td>
<td>( \sqrt{3} )</td>
</tr>
<tr>
<td>( a_n = a_{n-1} + a_{n-3} ) else</td>
<td>( \sqrt{3} )</td>
</tr>
<tr>
<td>( a_n = a_{n-1} + a_{n-2} ) for ( n = 3k + 2 )</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>( a_n = a_{n-1} + a_{n-3} ) for ( n = 3k + 1 ) and ( n = 3k )</td>
<td>( \sqrt{2} )</td>
</tr>
</tbody>
</table>
\[ \alpha_n = \begin{cases} \alpha_{n-1} + \alpha_{n-3} & \text{for } n = 2k + 1 \\ \alpha_{n-1} + \alpha_{n-2} & \text{else} \end{cases} \]

\[ \alpha_n = \begin{cases} \alpha_{n-1} + \alpha_{n-3} & \text{for } n = 3k + 2 \\ \alpha_{n-1} + \alpha_{n-2} & \text{else} \end{cases} \]

\[ \alpha_n = \begin{cases} \alpha_{n-1} + \alpha_{n-3} & \text{for } n = 4k + 3 \\ \alpha_{n-1} + \alpha_{n-2} & \text{else} \end{cases} \]

\[ \alpha_n = \begin{cases} \alpha_{n-1} + \alpha_{n-3} & \text{for } n = 5k + 4 \\ \alpha_{n-1} + \alpha_{n-2} & \text{else} \end{cases} \]

\[ \alpha_n = \begin{cases} \alpha_{n-1} + \alpha_{n-3} & \text{for } n = 6k + 5 \\ \alpha_{n-1} + \alpha_{n-2} & \text{else} \end{cases} \]

4. Conclusion

In this paper we consider the Fibonacci sequence of integers, associated with the golden section number. The Fibonacci sequence modified generates two other sequences associated with the square roots of two and three. We have generalized the procedure in order to obtain the square roots of ratios of integers that are members of the Fibonacci sequence. Thus a quasi-fractal scheme appears and this adds another remarkable property to the golden section number and the Fibonacci sequence. Starting with a very simple base, such as the \{1, 1\} and other very simple ones, irrational numbers could be obtained verifying Plato who said that the rational and the irrational co-exist in the Universe.

References

[3] Plato “Politeia” 546c (in Greek)