ON THE EROSION PROCESS PREDICTION
OF THE DUCTILE MATERIALS

VIOREL-PUIU PAUN\textsuperscript{1}, CONSTANTIN UDRISTE\textsuperscript{2}, CONSTANTIN PATRASCOIU\textsuperscript{3}

\textsuperscript{1} Physics Department I
Faculty of Applied Sciences
“Politehnica” University of Bucharest
313 Splaiul Independentei, Bucharest, 060042
ROMANIA

\textsuperscript{2} Mathematics Department I
Faculty of Applied Sciences
“Politehnica” University of Bucharest
313 Splaiul Independentei, Bucharest, 060042
ROMANIA

\textsuperscript{3} Faculty of Engineering and Management of the Technological Systems
Drobeta Turnu Severin, University of Craiova
ROMANIA

Abstract: - The occurrence of cavitation erosion damage is prevalent in high speed engines and bearing systems operating under high clearance conditions. The cavitation erosion prediction for the hydraulic machines components is very important in the engineering research. The present paper studies this process in the terms of the volume loss curve of erosion cavitations progress. An analytical model for the prediction of cavitation erosion of ductile materials is proposed.

Key-Words: - erosion, cavitation, volume loss curve, Bessel function, ductile material

1 Introduction
The cavitation erosion in hydraulic systems is an old problem, but the damage mechanism that culminates in material loss was not known with certainty until recently. An investigation is described that aimed at clarifying the damage mechanism in cavitation erosion and applying that knowledge to make hydraulic equipment more resistant to cavitation.

The cavitation phenomena in physics is the process of change in state (phase) of the matter (e.g. water) from liquid to vapour due to pressure drop of the surrounding domain. This pressure drop usually happens when the liquid is in rapid motion (flow). The typical example of such phenomena is found in turbomachinery and hydrodynamics. In both cases this constitutes a limitation for the performances of the device that rotates the fluid because of the drawbacks of cavitation inception amongst which are the vibrations, the erosion, the acoustic inconveniences and the performance degradation.

Virtually the cavitation damage is caused by the repeated nucleation, growth and collapse of bubbles against a metal surface in a liquid. The cavitation process is a form of erosion-corrosion which occurs on surfaces such as propellers, hydrofoils, pipelines, valves, engines, pump components and impellers that undergo large changes in liquid pressure.
The surface of the bearing exhibits areas where material has been removed or eroded during operation. The depth of these features is often limited to the immediate surface of the material, however if the bearing system is allowed to operate under these conditions for prolonged periods then the damage can penetrate deep into the bearing material, in extreme cases through to the steel backing.

The cavitation erosion results from rapid fluctuations in oil film pressure. When the pressure in an isolated area of the oil film drops below the bulk vapor pressure of the oil, a small, vapor-filled cavity is formed. This cavity then travels to an area of higher pressure where it collapses with the surrounding oil impinging on the adjacent bearing material. This action can eventually erode the bearing surface, as shown in the technical literature.

In the last decade, a lot of papers have been dedicated to the cavitation erosion study [1-3]. Roughly speaking, the techniques of cavitation erosion prediction can be classified into three main categories:

1. empirical correlations with material properties or with electrochemical or noise measurements;
2. simulation techniques using special test devices to reproduce a given aggressiveness in an accelerated way;
3. analytical methods.

One of the most common analytical methods is the physico-mathematical modeling [4] which provides the so-called particular analytical models.

2 Problem Formulation

The main objective of the analytical model is to quantitatively predict the cavitation erosion. In respect to experimental observations the volume loss curve of ductile materials is usually described by the time function

\[ V(t) = A \cdot U(k, I_t) \]  \hspace{1cm} (1)

\( A \) is the eroded surface area and \( U \) represents the erosion progress function depending on the applied phenomenological model [5, 6]. In the general definition of function \( U \), \( k \) is a set of real parameters (practically 3 parameters are sufficient) determined by fitting the erosion curve to the experimental data and \( I \) is the measure of cavitations intensity. In the phenomenological models \( t \) is the cumulative exposure duration [6].

The typical erosion curves for unit eroded surface area, i.e., the graphs of \( v(t) \) and \( V(t) \) functions, are given in Fig.1 and Fig. 2. The function \( v(t) \) is the first order derivative with respect to time of the volume loss function, \( V(t) \), defined in equation (1).

![Fig. 1 Theoretical curve of volume loss rate](image1)

![Fig. 2 Theoretical curve of volume loss](image2)
(incubation period), Ac (acceleration period), Dc (deceleration period) and Ss (steady state erosion period). We remark that, during the incubation period, the volume loss $V$ and the volume loss rate $v$ are null. At the end of incubation period, the volume loss rate curve $v(t)$, can have a point of discontinuity (can be nonsmooth at the end point of incubation period).

Let $(t_0, \tan \theta)$ be the end point of incubation period. If $\theta\neq 0$ is the angle between the time axis and tangent to the volume loss curve at $(t_0,0)$, as in Fig. 2, then we have a discontinuity point for the volume loss rate curve $v(t)$, like in Fig. 1.

During the incubation period, the volume loss and the volume loss rate curves are null. To simplify, we choose the time interval $[-\epsilon, 0]$ as incubation period, and we study the volume loss, and the volume loss rate curves for $t\geq 0$ ($t_0$ is the end point of incubation period).

Then, the most accurately experimentally data for very long time suggested that the volume loss rate curve $v(t)$ can look as in Fig. 3.

![Fig. 3 Experimental curve of volume loss rate](image)

Based on realistic physical considerations and experimental data, we accept that the volume loss rate curve $v(t)=\frac{dV}{dt}(t)$ is the solution of the differential equation

$$t^2 \frac{d^2v}{dt^2} + t \frac{dv}{dt} + (\beta^2 t^2 - 1)v = \alpha(\beta^2 t^2 - 1)$$

(2)

where $\alpha \geq 0, \beta > 0$ are real constants, depending of eroded material.

Setting in equation (2)

$$v(t) = \alpha + \varphi(\beta t)$$

(3)

after an elementary calculus, we find the following differential equation

$$t^2 \beta^2 \frac{d^2\varphi}{dt^2} + t\beta \frac{d\varphi}{dt}(\beta t) + (\beta^2 t^2 - 1)\varphi(\beta t) = 0$$

(4)

Replacing $\beta t$ by $x$, we obtain that $\varphi$ is solution of differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx}(x) + (x^2 - 1)y = 0$$

(5)

a particular case (for $n=1$) of Bessel's differential equation [7]

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx}(x) + (x^2 - n^2)y = 0$$

(6)

Recall that the general solution of Bessel's equations (6) is

$$y(x) = C_1 J_n(x) + C_2 Y_n(x)$$

(7)

where $J_n(x)$ is the Bessel function of the first kind, and $Y_n(x)$ is the Bessel function of the second kind (also known as the Weber function), [7]. Since $Y_n(x)$ is divergent at $x = 0$, the associated coefficient $C_2$ is forced to be zero to obtain a mechanically meaningful result (there is no source or sink at $x=0$), [8]. For this reason the general solution is $y(x)=C_1 J_n(x)$.

Additionally, we can say that the Bessel function of the first kind $J_n(x)$ satisfies

$$ \frac{d}{dx} [x^n J_n(x)] = x^{n-1} J_{n-1}(x)$$

(8)

and, for any integer $n$

$$J_{-n}(x) = (-1)^n J_n(x)$$

(9)

Without restraining the generality of the presentation for $J_n(x)$, the asymptotic forms can be alternatively used. For small $x$, i.e., fixed $n$ and $x\rightarrow 0$,
\[ J_n(x) = \frac{1}{2^n n!} x^n \]  \hspace{1cm} (10)

and for large \( x \), i.e., fixed \( n \) and \( x \gg n \),

\[ J_n(x) = \frac{2}{\sqrt{\pi x}} \cos \left[ x - (2n + 1) \frac{\pi}{4} \right] \]  \hspace{1cm} (11)

In this moment, we can go back to the differential equations (5) and (4). They admit the solutions

\[ y(x) = C_1 J_1(x) \]  \hspace{1cm} (12)

respectively

\[ \varphi(\beta t) = C_1 J_1(\beta t) \]  \hspace{1cm} (13)

each of them written in the corresponding variables. The mathematical purpose being now reached, we can return to the functions with physical significance, previously defined. Therefore, using relation (13) we find the solution of the differential equation (2)

\[ v(t) = \alpha + C_1 J_1(\beta t) \]  \hspace{1cm} (14)

in fact the volume loss rate curves expression.

The volume loss curve is obtained by a simple integral

\[ V(t) = \int_0^t (\alpha + C_1 J_1(\beta s)) ds = \int_0^t \left[ d \left[ J_0(\beta s) \right] \right] ds = \int_0^t \frac{d}{ds} \left[ J_0(\beta s) \right] ds = \frac{1}{\beta} J_0(\beta s) \]  \hspace{1cm} (15)

because \( J_0(0) = 1 \). Now, the formula (15) becomes

\[ V(t) = \alpha t - \frac{C_1}{\beta} J_0(\beta t) + \frac{1}{\beta} \]  \hspace{1cm} (20)

Imposing the boundary condition \( V(0) = 0 \), meaning that

\[ 0 = V(0) = -\frac{C_1}{\beta} J_0(0) + \frac{1}{\beta} \]  \hspace{1cm} (21)

and because \( J_0(0) = 1 \), we obtain that \( C_1 = 1 \).

The final result is

\[ v(t) = \alpha + \frac{1}{\beta} J_1(\beta t) \]  \hspace{1cm} (22)

and the volume loss function, for unit eroded surface area, becomes

\[ V(t) = \alpha t + \frac{1}{\beta} \left[ 1 - J_0(\beta t) \right] \]  \hspace{1cm} (23)

If \( A \) is the eroded surface area, we can write

\[ V(t) = A \alpha t + \frac{A}{\beta} \left[ 1 - J_0(\beta t) \right] \]  \hspace{1cm} (24)

3 Problem Solution

Further on, we present the phenomenological support, respectively the both macroscopic and microstructural study on a known ductile material, which led us to the developing of the analytical model proposed. The material “45”, conform to its original denomination, is a carbon steel of high quality, hardened and tempered. This alloy contains C(0.43), Mn(0.63), Si(0.26), P(0.030), S(0.033) and Fe in rest. In the images below, the scanning electron microscopy of the cavitated surface [9], at three different magnificences, for the material entitled “45”, is presented.
The mean depth of erosion values were calculated while the maximum depth of erosion measured under the microscope on the cross-sections through the centre of the specimens.

The cavitation damage is generally very shallow. The tested specimens of material entitled “45” evidently show a plastic deformation as shear bands at the cavitated edge compared to the centre of the specimens, as in Fig. 6. These were photographed at higher magnification. No scaling effect could be seen [9]. This microscopic proof of the material behaviour is conforming to the ductile material general typology [10].

The mathematical constants $\alpha$ and $\beta$ have a correct physical interpretation in the analytical model above developed. The asymptotic behaviour appreciation of the volume loss rate is relevant. Continuing the Bessel functions theory, we can compute the limits of $J_1(x)$

$$\lim_{x \to \infty} J_1(x) = 0, \quad \lim_{x \to 0^+} J_1(x) = 0,$$

(25)

to finally obtaining

$$\lim_{t \to \infty} v(t) = \lim_{t \to \infty} \left[ \alpha + \frac{1}{\beta} J_1(\beta t) \right] = \alpha$$

(26)

and
From the equalities (26) and (27) it follows that the constant \( \alpha \) has two physical meanings. Conform to equation (26) \( \alpha \) is the end value of the volume loss rate. The equation (27) affirms that \( \alpha \) is the right derivative of \( V(t) \) function (volume loss curve) at the point \( t_0 \).

Considering that the geometrical tangent to the volume loss curve has a \( \theta \) angle with the time axis, we have the relation

\[
\frac{dV}{dt}(t_0) = v(t_0) = \tan \theta
\]  
(28)

Thus, the end value of the volume loss rate must be equal to \( \tan \theta \). If \( \tan \theta \neq 0 \), then the volume loss rate curve is discontinuous at \( t_0 = 0 \). The constant \( \beta \) is a scale factor. Other authors claim that the \( \alpha \) and \( \beta \) coefficients are in connection with the material properties (work hardening ability and resistance to cavitation, respectively) [6]. Nevertheless in both cases these constants (coefficients, parameters) should be determined a posteriori.

In the present analytical model, the real parameters \( \alpha \) and \( \beta \) will be obtained by fitting the erosion curve to the experimental data.

In this moment, taking into account the asymptotic behaviour, we can explicit the two Bessel functions

\[
Y_0(x) = \frac{2}{\pi x} \cos \left( x - \frac{\pi}{4} \right)
\]  
(29)

and

\[
Y_1(x) = \frac{2}{\pi x} \cos \left( x - \frac{3\pi}{4} \right)
\]  
(30)

necessary to write the ultimate two expressions of the volume loss rate curve and the volume loss curve. Due to the fact that the erosion process happens in a very large time interval (days, weeks), we can say that the asymptotic behaviour is naturally reached very quickly. Therefore, we can now present the volume loss rate curve

\[
v(t) = \alpha t + \frac{1}{\beta} \sqrt{\frac{2}{\pi \beta t}} \cos \left( \beta t - \frac{\pi}{4} \right)
\]  
(31)

and respectively, the volume loss curve,

\[
V(t) = \alpha t + \frac{1}{\beta} \left[ 1 - \sqrt{\frac{2}{\pi \beta t}} \cos \left( \beta t - \frac{\pi}{4} \right) \right]
\]  
(32)

as functions of time, according to the initial assumed program.

Because we didn’t have some real experimental data at our disposal necessary to a correct fitting of the theoretical curves in order to obtain the optimum values for the parameters \( \alpha \) and \( \beta \), we nevertheless tried to find these parameters in a heuristic manner. We will present the functions graphical representation (equations (31) and (32)) for several values of the parameters \( \alpha \) and \( \beta \), considered in accordance with the chosen ductile material, in the sense of certifiably reproducing their known experimental curves.

![v(t) graph for \( \alpha = 1000 \) and \( \beta = 100 \)](image)

![v(t) graph for \( \alpha = 100 \) and \( \beta = 10 \)](image)
4 Conclusion

An analytical model, at macroscopic level, for the erosion behaviour prediction of ductile materials in the specific conditions is proposed.

The volume loss and the volume loss rate curves, in the cavitation erosion process, are correctly represented by the Bessel function of the first kind, $J_0(x)$ and $J_1(x)$, but univocally depend only on two real parameters $\alpha$ and $\beta$.

For each simulation the real parameters $\alpha$ and $\beta$ can be determined by fitting the erosion curves to the experimental data of the studied ductile materials.

Although the present model is fully predictive, we must be aware that several assumptions or shortcuts were necessary to complete the modeling. Among the most critical ones, we can mention the influence of the deformation rate which was ignored.

In addition, the material was simply characterized by classical engineering tests whereas the natural solicitations in cavitation erosion have a complex character which cannot be surprised in a single type test. As a future work, we should also develop mechanical tests capable to reproduce until identification the real cavitation erosion process.
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