A Perception-Based Estimation of Uncertainty and its Application to Financial Portfolios

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Abstract: A risk-minimizing portfolio model under uncertainty with randomness and fuzziness is discussed. By a perception-based extension of estimations for fuzzy random variables, the risk-minimizing portfolio problem is developed. In the uncertainty model, the randomness and fuzziness are evaluated respectively by the probabilistic expectation and mean values with evaluation weights and \( \lambda \)-mean functions. The means, variances and the measurements of fuzziness for fuzzy numbers/fuzzy random variables are applied in the possibility case and the necessity case, and a risk estimation is derived from both random factors and fuzzy factors in the model. By quadratic programming approach, we derive a solution of the risk-minimizing portfolio problem. It is shown that the solution is a tangency portfolio. A numerical example is given to illustrate our idea.

Key–Words: Uncertainty modeling, perception-based estimation, fuzzy random variable, portfolio, risk-minimizing, possibility-necessity weight, pessimistic-optimistic index.

1 Introduction and notations

The portfolio is one of the most important tools for the asset management in finance. Portfolio models have been studied by many authors using mathematical programming on the basis of Markowitz’s ([8, 9]). When we deal with systems containing hidden information like actual financial markets, fuzzy logic works well since the markets contain the uncertain factors which are different from probabilistic essence and in which it is difficult to specify actual price values exactly ([4]). Fuzzy factors come from the lack of knowledge in the stock market, and it could be observed remarkably in the unreliable actions among markets, banks and investors when the financial crisis in October 2008. At that time, the amount of the loans, the bad debts and doubtful accounts in the related banks were fuzzy factors from the lack of knowledge to investors and the other banks. The unreliability in the market arises from the hidden information like the bad debts and doubtful accounts and it is related to the risk in the stock market. In this paper, randomness is applied to the uncertainty regarding the belief degree of frequency, and fuzziness is applied to imprecision of data because of a lack of knowledge regarding the current stock market. In this paper, we consider a risk-minimizing portfolio model under uncertainty of randomness and fuzziness.

This paper introduces the estimation from Yoshida [11], which has discussed a perception-based extension of estimations for fuzzy random variables from the viewpoint of Kruse and Meyer [6] and we represent a risk-minimizing portfolio with fuzzy random variables. Estimation of uncertain quantities is important in decision making. To represent uncertainty in a finance model, we use fuzzy random variables which have two kinds of uncertainties, i.e. randomness and fuzziness. Recently, Yoshida [10] introduced means, variances and measurements of fuzziness of fuzzy random variables using evaluation weights and \( \lambda \)-mean functions. In this paper, we estimate fuzzy numbers/fuzzy random variables by probabilistic expectations and these evaluations, which are characterized by a possibility-necessity weight for subjective estimation and a pessimistic-optimistic index for subjective decision. Especially we deal with evaluation weights derived from the possibility measure and the necessity measure for numerical computation in modeling.

In a portfolio model, we use triangle-type fuzzy numbers/fuzzy random variables for simplicity in numerical computation when we apply them to actual models. We discuss a risk-minimizing problem for the portfolio, where the risk is defined by both random factors and fuzzy factors in the portfolio model. By quadratic programming approach, we derive a solution of the risk-minimizing portfolio problem. We
show that the solution is a tangency portfolio. A numerical example is given to illustrate our idea.

In this paper, we consider a portfolio model with a bond and \( n \) stocks, where \( n \) is a positive integer. In the remainder of this section, we describe a bond price process and stock price processes. We deal with a model where an investor’s actions do not have any impact on the stock market, so-called small investors hypothesis ([9]). Let \( T \) be an expiration date and and let \( \mathbb{R} \) denote the set of all real numbers. Let \( (\Omega, \mathcal{M}, P) \) be a probability space, where \( \mathcal{M} \) is a \( \sigma \)-field of \( \Omega \) and \( P \) is a non-atom probability measure on \( \Omega \). Let a positive number \( r_i \) be an interest rate of a bond price at time \( t \) for \( t = 1, 2, \cdots, T \), and put a bond price process \( \{S_t^{B}\}_{t=0}^{T} \) by \( S_0^B = 1 \) and \( S_t^B := \prod_{i=1}^{t}(1 + r_i) \) for \( t = 1, 2, \cdots, T \) and := means that the left term is defined by the right term. Then, for an asset \( i = 1, 2, \cdot \cdot \cdot, n \), a stock price process \( \{S_t^{i}\}_{t=0}^{T} \) is given by rates of return \( R_t^{i} \) as follows. Set \( S_0^i := S_0^i(1 + R_0^i) \) for \( t = 1, 2, \cdots, T \), where \( \{R_t^i\}_{t=0}^{T} \) is assumed to be a uniform integrable sequence of real random variables with values in \([-1, \infty)\). Then we have In this paper, we present a portfolio model where stock price processes \( S_t^i \) take fuzzy values using fuzzy random variables, whose mathematical notations are introduced in Section 4.

2 Perception-based extension of estimations

First we introduce some notations of fuzzy numbers. Let \( \mathbb{R} \) denote the set of all real numbers. A fuzzy number is denoted by its membership function \( \bar{a} : \mathbb{R} \mapsto [0,1] \) which is normal, upper-semicontinuous, fuzzy convex and has a compact support ([5]). \( \mathcal{R} \) denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with their corresponding membership functions ([12]). The \( \alpha \)-cut of a fuzzy number \( \bar{a} \) is given by closed intervals \( \bar{a}_\alpha := \{x \in \mathbb{R} \mid \bar{a}(x) \geq \alpha\} \) \((\alpha \in (0,1])\) and \( a_\alpha := \text{cl}\{x \in \mathbb{R} \mid \bar{a}(x) > 0\} \), where \( \text{cl} \) denotes the closure of an interval. The \( \alpha \)-cut is written by closed intervals \( \bar{a}_\alpha = [\bar{a}_\alpha^-, \bar{a}_\alpha^+] \) \((\alpha \in [0,1])\). Hence we introduce a partial order \( \preceq \), so called the fuzzy max order, on fuzzy numbers \( \mathcal{R} \) ([5]):

For \( \bar{a}, \bar{b} \in \mathcal{R}, \bar{a} \preceq \bar{b} \) means that \( \bar{a}_\alpha \geq \bar{b}_\alpha \) and \( \bar{a}_\alpha \geq \bar{b}_\alpha \) for all \( \alpha \in [0,1] \). An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows ([12]): Let \( \bar{a}, \bar{b} \in \mathcal{R} \) and \( \zeta \geq 0 \). The addition \( \bar{a} \pm \bar{b} \) of \( \bar{a} \) and \( \bar{b} \) and the scalar multiplication \( \zeta \bar{a} \) of \( \zeta \) and \( \bar{a} \) are fuzzy numbers given respectively by their \( \alpha \)-cuts as follows: \( \bar{a} \pm \bar{b} = [\bar{a}_\alpha \pm \bar{b}_\alpha, \bar{a}_\alpha \pm \bar{b}_\alpha] \) and \( \zeta \bar{a} = [\zeta \bar{a}_\alpha, \zeta \bar{a}_\alpha] \) for \( \alpha \)-cuts \( \bar{a}_\alpha = [\bar{a}_\alpha, \bar{a}_\alpha] \) and \( \bar{b}_\alpha = [\bar{b}_\alpha, \bar{b}_\alpha] \).

Let \( \mathcal{X} \) be the set of all integrable real random variables on \((\Omega, P)\). A fuzzy-number-valued map \( \bar{X} : \Omega \mapsto \mathcal{X} \) is called a fuzzy random variable if the mappings \( \omega \mapsto \bar{X}_\omega^\alpha(\omega) \) are measurable for all \( \alpha \in [0,1] \), where \( \bar{X}_\alpha(\omega) := [\bar{X}_\alpha^-(\omega), \bar{X}_\alpha^+(\omega)] := \{x \in \mathbb{R} \mid \bar{X}(\omega)(x) \geq \alpha\} \) ([7]). A fuzzy random variable \( \bar{X} \) is said to be integrable if \( \omega \mapsto \bar{X}_\omega^\alpha(\omega) \) are integrable for all \( \alpha \in [0,1] \). The expectation of an integrable fuzzy random variable \( \bar{X} \) is a fuzzy number

\[
\bar{E}(\bar{X})(x) := \sup_{X \in \mathcal{X}} \inf_{\bar{X} : X(x) = x} \bar{X}(\omega)(X((\omega)), \quad (1)
\]

for \( x \in \mathbb{R} \), where \( \bar{X} \) is taken as the set of all integrable real random variables and \( E(X) := \int X dP \) (Kruse and Meyer [6]). Then, it is known that the expectation \( E(X) \) is a fuzzy number whose \( \alpha \)-cut is given by

\[
\bar{E}(\bar{X}_\alpha) = [E(\bar{X}_\alpha^-), E(\bar{X}_\alpha^+)]
\]

for \( \alpha \in [0,1] \). This extension is well-defined and has monotone, continuous and linear properties in Yoshida [11]. On the other hand, the variance does not satisfy the monotone properties in Yoshida [11]. In general, this set does not equal to the form of the interval (2) since the monotone properties does not hold. Thus the extension of variance is not well-defined. We can find some approaches regarding the variance in Carlsson and Fuller [2] and Yoshida [10]. In the next section, we introduce the variance with \( \lambda \)-mean functions and evaluation weights so that the extended variances are compatible to the extended means.

3 Mean, variance and the measurement of fuzziness

There are many researches for the estimation of fuzzy numbers ([1, 3]). In this paper, we use an evaluation method of fuzzy numbers/fuzzy random variables introduced by Yoshida [10] to estimate the rate of return (12) in the portfolio. Yoshida [10] studied an evaluation of fuzzy numbers by evaluation weights which are induced from fuzzy measures to evaluate a confidence degree that a fuzzy number takes values in an interval. With respect to fuzzy random variables, the randomness is evaluated by the probabilistic expectation and the fuzziness is estimated by evaluation weights and the following function. Let \( g^\lambda : \mathcal{I} \mapsto \mathbb{R} \) be a map such that

\[
g^\lambda([x, y]) := \lambda x + (1 - \lambda)y \quad (3)
\]

for \([x, y] \in \mathcal{I}\), where \( \lambda \) is a constant satisfying \( 0 \leq \lambda \leq 1 \) and \( \mathcal{I} \) denotes the set of all bounded closed intervals. This scalarization is used for the estimation

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of fuzzy numbers to give a mean value and a measurement of fuzziness in the possibility case and the mean \( \tilde{E}^N(\tilde{a}) \) in the necessity case are given as follows ([10]):

\[
\tilde{E}^P(\tilde{a}) := \int_0^1 g^\lambda(\tilde{a}_\alpha) w(\alpha) \, d\alpha, ~ (5)
\]

\[
\tilde{E}^N(\tilde{a}) := \int_0^1 g^\lambda(\tilde{a}_\alpha)(2 - 2\alpha) \, d\alpha. ~ (6)
\]

The mean \( \tilde{E}^\lambda \) has the following natural properties regarding the linearity and the monotonicity for the fuzzy max order.

**Lemma 1** ([10]). Let \( \lambda \in [0, 1] \). Let \( \tilde{E}^\lambda = \tilde{E}^P \) or \( \tilde{E}^\lambda = \tilde{E}^N \). For fuzzy numbers \( \tilde{a}, \tilde{b} \in \mathbb{R} \) and real numbers \( \theta, \zeta \), the following (i) – (iv) hold.

(i) \( \tilde{E}^\lambda(\tilde{a} + 1_{\{\theta\}}) = \tilde{E}^\lambda(\tilde{a}) + \theta \), where \( 1_{\{\cdot\}} \) is the characteristic function of a set.

(ii) \( \tilde{E}^\lambda(\zeta \tilde{a}) = \zeta \tilde{E}^\lambda(\tilde{a}) \) if \( \zeta \geq 0 \).

(iii) \( \tilde{E}^\lambda(\tilde{a} + \tilde{b}) = \tilde{E}^\lambda(\tilde{a}) + \tilde{E}^\lambda(\tilde{b}) \).

(iv) If \( \tilde{a} \succeq \tilde{b} \), then \( \tilde{E}^\lambda(\tilde{a}) \geq \tilde{E}^\lambda(\tilde{b}) \) holds.

Next we consider measurements regarding two kinds of uncertainty, i.e. fuzziness and randomness. Fuzziness is based on the imprecision of data and the variance is based on the randomness, and they are given as independent concepts in this paper. Therefore, they should be estimated in different ways. Yoshida [10] has studied a method to measure the size of fuzziness regarding fuzzy numbers. Let \( \tilde{a} \in \mathbb{R} \) be a fuzzy number. A measurement of fuzziness \( \tilde{F}(\tilde{a}) \) of the fuzzy number \( \tilde{a} \) is given as follows: Let \( \alpha \in [0, 1] \). For an interval \( \tilde{a}_\alpha = [\tilde{a}^-_\alpha, \tilde{a}^+_\alpha] \) as a number with fuzziness, let \( y \in \tilde{a}_\alpha \) be a real number without fuzziness, which is taken temporarily as a true value estimated for \( \tilde{a}_\alpha \). Then, a size of fuzziness should be given by the distance between \( y \) and \( \tilde{a}_\alpha \):

\[
\max\{\tilde{a}^-_\alpha - y, y - \tilde{a}^+_\alpha\}. \]

Therefore, the upper/lower measurements of fuzziness should be given by

\[
m^U(\tilde{a}_\alpha) := \frac{\tilde{a}^+_\alpha - \tilde{a}^-_\alpha}{2} \quad \text{and} \quad m^L(\tilde{a}_\alpha) := \frac{\tilde{a}^+_\alpha - \tilde{a}^-_\alpha}{2}. \]

The measurements of fuzziness are related to the imprecision of the data, and they should be defined without the subjective index \( \lambda \). Then, for \( m = m^U \) or \( m = m^L \), a measurement of fuzziness \( \tilde{F}(\tilde{a}) \) is given by

\[
\tilde{F}(\tilde{a}) := \int_0^1 m(\tilde{a}_\alpha) w(\alpha) \, d\alpha = \int_0^1 \tilde{F}(\tilde{a}_\alpha) w(\alpha) \, d\alpha, ~ (7)
\]

where \( \tilde{a}_\alpha = [\tilde{a}^-_\alpha, \tilde{a}^+_\alpha] \) is the \( \alpha \)-cut of the fuzzy number \( \tilde{a} \in \mathbb{R} \).

**Lemma 2.** Let a fuzzy number \( \tilde{a} \in \mathbb{R} \). Then, the measurement of fuzziness in the possibility case and the necessity case are given as follows.

\[
\tilde{F}^P(\tilde{a}) := \int_0^1 (\tilde{a}^+_\alpha - \tilde{a}^-_\alpha) \, d\alpha, \quad \tilde{F}^N(\tilde{a}) := \int_0^1 (\tilde{a}^+_\alpha - \tilde{a}^-_\alpha)(1 - \alpha) \, d\alpha.
\]

Now we have the following natural results about the possibility fuzziness measure \( \tilde{F}(\cdot) = \tilde{F}^P(\cdot) \) and the necessity fuzziness measure \( \tilde{F}(\cdot) = \tilde{F}^N(\cdot) \).

**Lemma 3.** Let \( \tilde{F} = \tilde{F}^P \) or \( \tilde{F} = \tilde{F}^N \). For fuzzy numbers \( \tilde{a}, \tilde{b} \in \mathbb{R} \) and real numbers \( \theta, \zeta \), the following (i) – (iv) hold.

(i) \( \tilde{F}(\tilde{a} + 1_{\{\theta\}}) = \tilde{F}(\tilde{a}) \).

(ii) \( \tilde{F}(\zeta \tilde{a}) = |\zeta| \tilde{F}(\tilde{a}) \).

(iii) \( \tilde{F}(\tilde{a} + \tilde{b}) = \tilde{F}(\tilde{a}) + \tilde{F}(\tilde{b}) \).

(iv) If \( \tilde{a} \succeq \tilde{b} \), then \( \tilde{F}(\tilde{a}) \geq \tilde{F}(\tilde{b}) \) holds.

Let \( \tilde{a} \in \mathbb{R} \) be a fuzzy number and let \( \nu \in [0, 1] \) be a parameter. For applications of the mean values and measurement of fuzziness in actual problems, we introduce a mean value and a measurement of fuzziness with a parameter \( \nu \):

\[
\tilde{E}^{\lambda, \nu}(\tilde{a}) := \nu \tilde{E}^P(\tilde{a}) + (1 - \nu)\tilde{E}^N(\tilde{a}), \quad (8)
\]

\[
\tilde{F}^{\nu}(\tilde{a}) := \nu \tilde{F}(\tilde{a}) + (1 - \nu)\tilde{F}(\tilde{a}). \quad (9)
\]
Then, \( \nu \) is called a possibility-necessity weight, and (8) and (9) are the mean value and the measurement of fuzziness with the possibility-necessity weight \( \nu \). We have the following theorem which is convenient for numerical calculations in applications.

**Lemma 4.** Let a fuzzy number \( \tilde{a} \in \mathcal{R} \) and \( \nu, \lambda \in [0,1] \). Then, the mean \( \tilde{E}^{\lambda, \nu}(\cdot) \) and the fuzziness measure \( \tilde{F}^{\nu}(\cdot) \) with the possibility-necessity weight \( \nu \) and the pessimistic-optimistic index \( \lambda \) are represented by

\[
\tilde{E}^{\lambda, \nu}(\tilde{a}) = \int_0^1 g^\lambda(\tilde{a}_\alpha) (\nu + 2(1-\nu)(1-\alpha)) \, d\alpha,
\]

\[
\tilde{F}^{\nu}(\tilde{a}) = \int_0^1 (\tilde{a}_+ - \tilde{a}_-)(\nu + (1-\nu)(1-\alpha)) \, d\alpha,
\]

where \( \lambda \)-mean function \( g^\lambda \) is given by \( g^\lambda(\tilde{a}_\alpha) = \lambda\tilde{a}_- + (1-\lambda)\tilde{a}_+ \).

Using evaluation weights, we give means, variances and covariances regarding fuzzy random variables. For the fuzzy random variable \( \tilde{X} \), the mean of the expectation \( E(\tilde{X}) \) is a real number \( E(E^\lambda(\tilde{X})) \) given by the following (10):

\[
E \left( \int_0^1 g^\lambda(\tilde{X}_\alpha) w(\alpha) \, d\alpha \right) / \int_0^1 w(\alpha) \, d\alpha.
\]

Then, from Lemma 1, we obtain the following results regarding fuzzy random variables.

**Lemma 5.** Let \( \lambda \in [0,1] \). For a fuzzy number \( \tilde{a} \in \mathcal{R} \), integrable fuzzy random variables \( \tilde{X}, \tilde{Y} \), and integrable real random variables \( Z \) and a nonnegative real number \( \zeta \), the following (i) – (vi) hold.

(i) \( E(\tilde{E}^\lambda(\tilde{X})) = \tilde{E}^\lambda(E(\tilde{X})) \).

(ii) \( E(\tilde{E}^\lambda(\tilde{a})) = \tilde{E}^\lambda(\tilde{a}) \) and \( E(\tilde{E}^\lambda(Z)) = E(Z) \).

(iii) \( E(\tilde{E}^\lambda(\tilde{X} + \tilde{a})) = E(\tilde{E}^\lambda(\tilde{X})) + \tilde{E}^\lambda(\tilde{a}) \) and \( E(\tilde{E}^\lambda(\tilde{X} + Z)) = E(\tilde{E}^\lambda(\tilde{X})) + E(Z) \).

(iv) \( E(\tilde{E}^\lambda(\zeta \tilde{X})) = \zeta E(\tilde{E}^\lambda(\tilde{X})) \).

(v) \( E(\tilde{E}^\lambda(\tilde{X} + \tilde{Y})) = E(\tilde{E}^\lambda(\tilde{X})) + E(\tilde{E}^\lambda(\tilde{Y})) \).

(vi) If \( \tilde{X} \geq \tilde{Y} \) almost surely, then \( E(\tilde{E}^\lambda(\tilde{X})) \geq E(\tilde{E}^\lambda(\tilde{Y})) \) holds.

Next we introduce variances and covariances of fuzzy random variables from the viewpoint of \( \lambda \)-mean functions and evaluation weights. From the results in [10], for fuzzy random variables \( \tilde{X} \) and \( \tilde{Y} \), we define variances \( V^\lambda(\tilde{X}) \) and covariances \( Cov^\lambda(\tilde{X}, \tilde{Y}) \) by the following (19) and (12):

\[
E \left( \int_0^1 G^\lambda_{\tilde{X}_\alpha} G^\lambda_{\tilde{Y}_\alpha} w(\alpha) \, d\alpha \right) / \int_0^1 w(\alpha) \, d\alpha.
\]

For \( \lambda, \gamma \in [0,1] \), where we put \( G^\lambda_{\tilde{X}_\alpha} := g^\lambda(\tilde{X}_\alpha) - E(g^\lambda(\tilde{X}_\alpha)) \), \( G^\gamma_{\tilde{Y}_\alpha} := g^\gamma(\tilde{Y}_\alpha) - E(g^\gamma(\tilde{Y}_\alpha)) \). We can find other approaches in Carlsson and Fuller [2] which discuss the variance of fuzzy numbers by possibility theory. Hence we obtain the following natural properties about the variance \( V^\lambda(\cdot) \) and covariance \( Cov^\lambda,\gamma(\cdot, \cdot) \).

**Lemma 6** ([10]). Let \( \lambda, \gamma \in [0,1] \). For fuzzy numbers \( \tilde{a}, \tilde{b} \in \mathcal{R} \), integrable fuzzy random variables \( \tilde{X}, \tilde{Y} \), integrable real random variables \( X, Y \) and a nonnegative real number \( \zeta \), the following (i) – (vi) hold.

(i) \( V^\lambda(\tilde{a}) = 0 \) and \( V^\lambda(\tilde{X}) = V(X) \), where \( V(\cdot) \) is the variance of real random variables.

(ii) \( V^\lambda(\tilde{X} + \tilde{a}) = V^\lambda(\tilde{X}) \).

(iii) \( V^\lambda(\zeta \tilde{X}) = \zeta^2 V^\lambda(\tilde{X}) \).

(iv) \( Cov^\lambda,\gamma(\tilde{X}, \tilde{a}) = Cov^\lambda,\gamma(\tilde{a}, \tilde{X}) = 0 \) and \( Cov^\lambda,\gamma(\tilde{X}, \tilde{Y}) = Cov(X, Y) \), where \( Cov(\cdot, \cdot) \) is the covariance of real random variables.

(v) \( Cov^\lambda,\gamma(\tilde{X} + \tilde{a}, \tilde{Y} + \tilde{b}) = Cov^\lambda,\gamma(\tilde{X}, \tilde{Y}) \).

### 4 A portfolio model

Now we deal with a case when the rate of return \( \{R^t_i\}_{t=1}^T \) has some imprecision. We define a rate of return process with imprecision \( \{R^t_i\}_{t=1}^T \) by a sequence of triangle-type fuzzy random variables by the sum of fuzzy numbers:

\[
\tilde{R}^t_i(\omega) := 1_{\{R^t_i(\omega)\}}(\cdot) + \tilde{a}^t_i(\cdot)
\]

for \( \omega \in \Omega \), where \( 1_{\{\cdot\}} \) denotes the characteristic function of a singleton and \( \tilde{a}^t_i(\cdot) \) is a triangle-type fuzzy number

\[
\tilde{a}^t_i(x) = \begin{cases} 
0 & \text{if } x < -c^t_i \\
\frac{x + \bar{c}^t_i - c^t_i}{\bar{c}^t_i} & \text{if } -c^t_i \leq x < 0 \\
\frac{x - \underline{c}^t_i}{\bar{c}^t_i - \underline{c}^t_i} & \text{if } 0 \leq x < c^t_i \\
0 & \text{if } x \geq c^t_i 
\end{cases}
\]

with a positive number \( c^t_i \), which is called a fuzzy factor for asset \( i \) at time \( t \). For assets \( i = 1, 2, \cdots, n \), we define stock price processes \( \{S^t_i\}_{t=0}^T \) by the rates of return with imprecision \( \tilde{R}^t_i \) as follows: \( \tilde{S}^0_i := S^0_i \) is a constant and \( \tilde{S}^t_i := \tilde{S}^0_i \prod_{s=1}^t (1 + R^s_i) \) for \( t = 1, 2, \cdots, T \). Hence, we deal with a portfolio with trading strategies given by portfolio weight vectors \( w = (w^1, w^2, \cdots, w^n) \) such that \( w^1 + w^2 + \cdots + w^n = 1 \).
1 and \( w^i \geq 0 (i = 1, 2, \ldots, n) \). For the trading strategy \( w = (w^1, w^2, \ldots, w^n) \), the rate of return with imprecision for the portfolio is given by

\[
\tilde{R}_t := w^1 \tilde{R}_t^1 + w^2 \tilde{R}_t^2 + \cdots + w^n \tilde{R}_t^n. \tag{12}
\]

This paper discusses a risk-minimizing model regarding (12) under guarantee of rewards, whose mathematical notations are introduced in Section 5.

5 A risk-minimizing model

In this section, we discuss a risk-minimizing problem for (12) using the method in Section 3. Let the mean, variance and covariance of the fuzzy random variables \( \tilde{R}_t^i \) by \( \tilde{\mu}_t^i := E(\tilde{R}_t^i) \), \( (\sigma_t^i)^2 := V(\tilde{R}_t^i) \) and \( \tilde{\sigma}_t^{ij} := Cov(\tilde{R}_t^i, \tilde{R}_t^j) \) for \( \lambda \in [0, 1] \) and \( i, j = 1, 2, \ldots, n \). From Lemmas 3.5 and 3.6 and (6), we obtain the following results regarding the rates of returns \( \tilde{R}_t^i \): Then \( \tilde{\mu}_t^i = \mu_t^i + \tilde{E}(\tilde{a}^i) \), \( (\sigma_t^i)^2 = (\tilde{\sigma}_t^i)^2 \) and \( \tilde{\sigma}_t^{ij} = \tilde{\sigma}_t^{ij} \), where we put the mean, variance and covariance of the real random variables \( R_t^i \) by \( \mu_t^i := E(R_t^i) \), \( (\sigma_t^i)^2 := V(R_t^i) \) and \( \sigma_t^{ij} := Cov(R_t^i, R_t^j) \). For the trading strategy \( w = (w^1, w^2, \ldots, w^n) \) such that \( w^1 + w^2 + \cdots + w^n = 1 \), from Lemmas 3.5 and 3.6 the expectation \( \tilde{\mu}_t \) and variance \( (\tilde{\sigma}_t)^2 \) regarding the rate of return with imprecision for the portfolio \( \tilde{R}_t = w^1 \tilde{R}_t^1 + w^2 \tilde{R}_t^2 + \cdots + w^n \tilde{R}_t^n \) in (12) is given as follows. \( \tilde{\mu}_t := \sum_{i=1}^n w^i \tilde{\mu}_t^i = \sum_{i=1}^n w^i (\mu_t^i + \tilde{E}(\tilde{a}^i)) \) and \( (\tilde{\sigma}_t)^2 := \sum_{i=1}^n \sum_{j=1}^n w^i \tilde{w}^j (\tilde{\sigma}_t^{ij}) \), where \( \tilde{\sigma}_t^i := (\sigma_t^i)^2 \). Hence, to guarantee the lower bound regarding the expectation \( \tilde{\mu}_t \) of the rate of return for the portfolio we estimate \( \tilde{\mu}_t \) taking the index \( \lambda \) pessimistic (\( \lambda = 1 \)) and the necessity mean \( \tilde{E}(\tilde{a}^i) = 1 - \alpha (\alpha \in [0, 1]) \): \( \tilde{\mu}_t^i = \sum_{i=1}^n w^i (\mu_t^i + \tilde{E}(\tilde{a}^i)) \) and \( (\tilde{\sigma}_t)^2 := \sum_{i=1}^n \sum_{j=1}^n w^i \tilde{w}^j (\tilde{\sigma}_t^{ij}) \) for \( \lambda \in [0, 1] \), where \( \tilde{a}_t^i = [\tilde{a}_t^{i-}, \tilde{a}_t^{i+}] = [-\tilde{c}_t^i(1 - \alpha), \tilde{c}_t^i(1 - \alpha)] \) from (7). Therefore, \( \sum_{i=1}^n w^i (\mu_t^i - \tilde{\sigma}_t^i) \) is the lower bound regarding the expectation \( \tilde{\mu}_t \) of the rate of return for the portfolio. Next, in this model, we deal with the risk derived from the uncertainty which consists of randomness and fuzziness. Since these factors are independent, we define the risk of the portfolio by the combination of the variance \( (\tilde{\sigma}_t)^2 \) and the measurement of fuzziness \( E(\tilde{R}_t) \). Then, for positive constants \( \kappa_1 \) and \( \kappa_2 \), its upper bound is given by the possibility case \( w^P(\alpha) = 1 (\alpha \in [0, 1]) \): \( \kappa_1 (\tilde{\sigma}_t)^2 + \kappa_2 E(\tilde{R}_t) \leq \kappa_1 \sum_{i=1}^n \sum_{j=1}^n w^i \tilde{w}^j (\tilde{\sigma}_t^{ij}) + \kappa_2 \tilde{\sigma}_t^{ij} \), where \( \kappa := \frac{\kappa_2}{\kappa_1} > 0 \). Thus for simplicity, taking \( \kappa_1 = 1 \),

\[
\tilde{\rho} := \sum_{i=1}^n \sum_{j=1}^n w^i \tilde{w}^j (\tilde{\sigma}_t^{ij}) + \kappa_2 \tilde{\sigma}_t^{ij} \tag{13}
\]
is the upper bound of the risk of randomness and fuzziness.

First, we deal with a risk-minimizing model without a bond. For a given constant \( \gamma \) which means the expected rate of return to be guaranteed for the portfolio, we discuss the following problem with allowance for short selling trading strategies.

Risk-minimizing problem (RM1): Minimize the risk \( \tilde{\rho} := \sum_{i=1}^n \sum_{j=1}^n w^i \tilde{w}^j (\tilde{\sigma}_t^{ij} + \kappa_2 \tilde{\sigma}_t^{ij}) \) with trading strategies \( w = (w^1, w^2, \ldots, w^n) \) \((w^1 + w^2 + \cdots + w^n = 1)\) under the condition \( \sum_{i=1}^n w^i (\tilde{\mu}_t^i - \frac{2}{3} \tilde{\sigma}_t^i) = \gamma \).

**Theorem 1.** The solution of problem (RM1) is

\[
w = \xi \Sigma^{-1} 1 + \eta \Sigma^{-1} \tilde{\mu} \tag{14}
\]

and then the risk is

\[
\tilde{\rho} = \frac{\Lambda \gamma^2 - 2B \gamma + C}{\Delta}, \tag{15}
\]

where \( \tilde{\mu}_t^i := \mu_t^i - \frac{2}{3} \tilde{\sigma}_t^i \) and \( \tilde{\sigma}_t^{ij} := \sigma_t^{ij} + \kappa_2 \tilde{\sigma}_t^{ij} \) for \( i, j = 1, 2, \ldots, n \), we put \( \tilde{\mu} := [\mu_t^1, \mu_t^2, \ldots, \mu_t^n]^T, 1 := [1 \cdots 1]^T, \Sigma := [\tilde{\sigma}_t^{ij}], \xi := \frac{C - B \eta}{\Delta}, \eta := \frac{\Lambda \gamma - B}{\Delta}, A := \Gamma \Sigma^{-1} 1, B := \Gamma \Sigma^{-1} \tilde{\mu}, C := \tilde{\mu}^T \Sigma^{-1} \tilde{\mu}, \Delta := AC - B^2 \).

The solution \( w^* := w \) in Theorem 1 is called a minimal risk solution. The set of minimal risk portfolios \( \mathcal{E} := \{ (\tilde{\rho}, \tilde{\mu}) | \tilde{\rho} = \Lambda (\tilde{\mu}^2 - 2B\tilde{\mu} + C) \} \) and \( \tilde{\mu} \geq \frac{B}{\Lambda} \) is also called the efficient frontier.

Next, we discuss a risk-minimizing model with a bond. Let \( \mu_t^i = r_i \) and \( \tilde{\sigma}_t^i = 0 \) for the bond. For a given constant \( \gamma \), we discuss the following problem with allowance for short selling trading strategies.

Risk-minimizing problem (RM2): Minimize the risk \( \tilde{\rho} := \sum_{i=1}^n \sum_{j=1}^n w^i \tilde{w}^j (\tilde{\sigma}_t^{ij} + \kappa_2 \tilde{\sigma}_t^{ij}) \) with trading strategies \( w = (w^0, w^1, w^2, \ldots, w^n) \) \((w^0 + w^1 + w^2 + \cdots + w^n = 1)\) under the condition \( \sum_{i=0}^n w^i (\tilde{\mu}_t^i - \frac{2}{3} \tilde{\sigma}_t^i) = \gamma \).

**Theorem 2.** The solution of problem (RM2) is

\[
w^* = \tilde{\Sigma}^{-1} (\tilde{\mu} - r_1), \tag{16}
\]

and then the corresponding risk is

\[
\tilde{\rho} = \frac{(\gamma - r_1)^2}{A(r_1)^2 - 2Br_1 + C}, \tag{17}
\]

where \( \zeta := \frac{\gamma - r_1}{A(r_1)^2 - 2Br_1 + C} \).
From Theorems 4.1 and 4.2, we obtain the following results for the Problem (RM2) without allowance for short selling trading strategies i.e., \( w^j \geq 0 (i = 0, 1, 2, \cdots, n) \).

**Theorem 3.** Assume \( \Sigma^{-1}(\bar{\mu} - r_1) \geq 0 \) and \( \Sigma^{-1}(\bar{\mu} - r_1) \neq 0 \). Then there exists only one solution \((\bar{\rho}^{**}, \bar{\mu}^{**})\) of Problem (RM2) in the efficient frontier \( \mathcal{E} \) of the minimal risk portfolios with the risk-free asset, which is given by

\[
(\bar{\rho}^{**}, \bar{\mu}^{**}) = \left( \frac{A(r_1)^2 - 2B r_1 + C}{AB - B^2}, \frac{B r_1 - C}{AB - B^2} \right).
\]

This solution is called a tangency portfolio. Then, the corresponding trading strategy \( w^{**} := (w^0, w^1, w^2) \) is given as follows and satisfies \( w^{**} \geq 0 \). Hence \( w^0 = \frac{\zeta}{B-AB} \) and \( w^n = 1 - \left( (w^n)^\zeta \right)^1 \), with \( \zeta := \frac{A(r_1)^2 - 2B r_1 + C}{AB - B^2} \).

The solution \((\bar{\rho}^{**}, \bar{\mu}^{**})\) is called a tangency portfolio, and we can easily check that \((\bar{\rho}^{**}, \bar{\mu}^{**})\) maximizes the Sharpe ratio \( \bar{\mu}^{**} / \sqrt{\bar{\rho}^{**}} \) with respect to portfolios in the efficient frontier \( \mathcal{E} \). Tangency portfolios are widely used in the financial market, and it is known that the Sharpe ratio is a measure of risk-adjusted performance of a trading strategy in portfolio theory (9). For the tangency portfolio (18), it is easily checked that the Sharpe ratio is calculated by

\[
\frac{\bar{\mu}^{**} - r_1}{\sqrt{\bar{\rho}^{**}}} = \sqrt{A(r_1)^2 - 2B r_1 + C}.
\]

Finally we give a numerical example. Let \( n = 3 \) and \( \kappa = 0.5 \). Let the interest rate of the bond \( r_1 = 0.04 \). Take a mean and variance-covariance matrix of rate of return and fuzzy factors as follows. \( \mu^1 = 0.05, \mu^2 = 0.08, \mu^3 = 0.06, \sigma^1 = 0.02, \sigma^2 = 0.04, \sigma^3 = 0.02, \sigma^1_{11} = 0.40, \sigma^2_{22} = 0.20, \sigma^3_{33} = 0.30, \sigma^1_{12} = \sigma^2_{21} = 0.03, \sigma^3_{31} = 0.02, \sigma^3_{23} = 0.30 \). From Theorem 3 we obtain the tangency portfolio \((\bar{\rho}^{**}, \bar{\mu}^{**}) = (0.130019, 0.05344)\) with the trading strategy \( w^{**} = (w^0, w^1, w^2, w^3) \) = (0, 0.09438, 0.42699, 0.47895). Hence \( \bar{\mu}^{**} = 0.053448 \) is the mean of the rate of return and \( \bar{\rho}^{**} = 0.095007 \) is the risk given by the variance for the portfolio. For the tangency portfolio \((\bar{\rho}^{**}, \bar{\mu}^{**})\), from (19), we can easily calculate that the Sharpe ratio is \( \bar{\mu}^{**} / \bar{\rho}^{**} = 0.0372946 \). In the case without fuzziness, we also obtain the tangency portfolio \((\bar{\rho}^{**}, \bar{\mu}^{**}) = (0.095007, 0.073349)\) with the trading strategy \( w^{**} = (w^0, w^1, w^2, w^3) \) = (0, 0.00518, 0.67003, 0.32478). Hence \( \bar{\mu}^{**} = 0.073349 \) is the mean of the rate of return and \( \bar{\rho}^{**} = 0.095007 \) is the risk given by the variance for the portfolio.

**References:**


