Magnetic geometric dynamics around Tornado Trap

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Abstract: The contents of this paper include modeling and qualitative geometric analysis of magnetic Tornado trap. Section 1 gives a description of the Tornado magnetic trap, produced by two opposite currents flowing through a couple of concentric helical conductors, connected at the poles by two jumpers. Section 2 gives the components of the magnetic field around a Tornado configuration, using Biot-Savart-Laplace integrals. Section 3 presents magnetic morphology, simulations and critical points for the magnetic energy. Section 4 deals with linearizations of the magnetic dynamical systems around critical points of the magnetic energy.

Keywords: magnetic trap, magnetic geometric dynamics, energy of magnetic field, magnetic lines and surfaces, numerical simulation.

1 Description of the Tornado Trap

The magnetic traps are based on the idea that we must have a device capable to produce a magnetic field with a central point of minimum magnetic energy and nonzero magnetic intensity [1]-[14].

The idea of creating a closed magnetic system with a magnetic field increasing towards the periphery had appeared in 1936, but for the first time it was practically realized in 1969 [4].

The improvement of using traps with a closed structure of the magnetic field lines but simple mirror trap is that the time for increasing plasma confinement is greater. An example of such a closed magnetic structure is the Tornado magnetic structure.

A model of Tornado magnetic field, the Tornado-322 trap, was build in St-Petersburg Russia, and used for creating an ECR (Electron Cyclotron Resonance) heating source of MCI (Multicharged Ions) [2]. The Tornado magnetic field is produced by two opposite currents flowing through a couple of concentric helical conductors connected at the poles by two jumpers. Turns of the inner helix are placed between those of the outer helix in order to raise the magnetic field strength.

Let us consider a Tornado configuration on Fig.1, as a schematic diagram of a magnetic trap we study.

Fig.1 Schematic configuration of parabolic Tornado-trap.
We approximate the magnetic field associated to the Tornado trap with Biot-Savart-Laplace formula

\[ \overline{H}_a(M) = \int_{\gamma_a} \frac{\mathbf{J}_a \times P\mathbf{M}}{\left| P\mathbf{M} \right|} \, d\tau_p, \]

\( M \in \mathbb{R}^3 - \gamma_a, \mathbf{P} \in \gamma_a, \mathbf{J}_a \) is the conductor current density (like the tangent versor \( \dot{\gamma}_a \)) on \( \gamma_a \).

The effect of the total magnetic field around a configuration \( \Gamma = \biguplus_{a=1}^{p} \gamma_a \) depends on the geometry of the configuration \([4]-[14] \). The Biot-Savart-Laplace magnetic field obtained is irrotational (i.e rot(\( \overline{H}_a \)) = 0) and solenoidal (i.e div(\( \overline{H}_a \)) = 0).

### 2. Magnetic Field around Tornado Configuration

Let us consider the Tornado configuration presented as schematic diagram in Fig.1. The shape of the helices can be parabolic, spherical, elliptic, etc. We consider that the intensity of the electrical current through the wires is 1, so we neglect the contribution of the segments that connect the helices with the circles.

Under these hypotheses, the configuration shown in Fig.1 can be reduced to

\[ \Gamma = \{ e_1 \} \{ e_2 \} \{ C_1 \} \{ C_2 \}, \]

where the inner helix is \( e_2 \), the outer helix is \( e_1 \) and \( C_1, C_2 \) are the two circles situated at the poles of the Tornado configuration.

Let us find the components of total magnetic field when the inner helix \( e_2 \) and the outer helix \( e_1 \) have parabolic shapes, respecting the configuration described in Fig.1.

Explicitly, we accept for the helices \( e_1 \) and \( e_2 \) the following parameterizations:

\( e_1: x = (a^2 - b_1^2 t^2) \cos t, y = (a^2 - b_1^2 t^2) \sin t, \)

\( z = bt, t \in \left[ -\frac{a}{b}, \frac{a}{b} \right], \)

\( e_2: x = (a^2 - b_1^2 t^2) \cos t, y = -\left( a^2 - b_1^2 t^2 \right) \sin t, \)

\( z = -bt, t \in \left[ -\frac{a}{b}, \frac{a}{b} \right], \)

with \( a > 0, b > 0, b_1 > 0, b_1 > b \).

For helix \( e_1 \), the magnetic field \( H_a \) has the components (from Biot-Savart-Laplace integrals)

\[ H_{ye1} = \]

\[ b a^2 \int_{-\frac{a}{b}}^{\frac{a}{b}} \sin t \Phi_1(t, x, y, z) \, dt + b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ -2b^2 z \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt - b^2 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ z b^2 \int_{-\frac{a}{b}}^{\frac{a}{b}} t \sin t \Phi_1(t, x, y, z) \, dt + b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ -z b^2 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ + b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt - b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ + \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt, \]

\[ H_{ye2} = \]

\[ a^2 \int_{-\frac{a}{b}}^{\frac{a}{b}} \sin t \Phi_1(t, x, y, z) \, dt + \]

\[ -b^2 z \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ + a^2 b \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt - 2b^2 z \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ -z b^2 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt - \]

\[ + b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt - b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ -b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt - b^3 \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt + \]

\[ \int_{-\frac{a}{b}}^{\frac{a}{b}} \Phi_1(t, x, y, z) \, dt. \]
\[ H_{x_1} = -y \ a^2 \int_{a}^{b} \sin t \Phi_1(t,x,y,z) \, dt - x \ a^2 \int_{a}^{b} \cos t \Phi_1(t,x,y,z) \, dt + \]

\[ + 2b^2 \int_{a}^{b} t \sin t \Phi_1(t,x,y,z) \, dt - 2b^2 y \int_{a}^{b} t \cos t \Phi_1(t,x,y,z) \, dt + \]

\[ + x \ b^2 \int_{a}^{b} t^2 \cos t \Phi_1(t,x,y,z) \, dt + y \ b^2 \int_{a}^{b} t^2 \sin t \Phi_1(t,x,y,z) \, dt + \]

\[ + \int \left( a^2 - b^2 \ t^2 \right)^2 \Phi_1(t,x,y,z) \, dt, \]

where \( \Phi_1(t,x,y,z) = (x^2 + y^2 + z^2) - 2x(a^2 - b^2 t^2) \cos t - 2y(a^2 - b^2 t^2) \sin t - 2zt + b^2 t^2 + (a^2 - b^2 t^2)^2 \)^{3/2}. 

Likewise, magnetic field \( H_{c_2} \) has the components \( H_{x_2} = \)

\[ \int_{a}^{b} \frac{\sin t}{\Phi_2(t,x,y,z)} \, dt - z \ a^2 \int_{a}^{b} \frac{\cos t}{\Phi_2(t,x,y,z)} \, dt + \]

\[ 2b_1^2 \int_{a}^{b} \frac{t \sin t}{\Phi_2(t,x,y,z)} \, dt - b_1 \ a^2 \int_{a}^{b} \frac{t \cos t}{\Phi_2(t,x,y,z)} \, dt + \]

\[ + z \ b_1^2 \int_{a}^{b} \frac{t^2 \cos t}{\Phi_2(t,x,y,z)} \, dt + b_1^3 \int_{a}^{b} \frac{t^2 \sin t}{\Phi_2(t,x,y,z)} \, dt + \]

\[ + b_1^3 \int_{a}^{b} \frac{t^3 \cos t}{\Phi_2(t,x,y,z)} \, dt + b_1 \ y \int_{a}^{b} \frac{1}{\Phi_2(t,x,y,z)} \, dt, \]

where \( \Phi_2(t,x,y,z) = (x^2 + y^2 + z^2) - 2x(a^2 - b_1^2 t^2) \cos t + 2y(a^2 - b_1^2 t^2) \sin t + 2zt + b_1^2 t^2 + (a^2 - b_1^2 t^2)^2 \)^{3/2}. 

The circle \( C_1 : x^2 + y^2 = 1, \ z = a_1, \ a_1 > a, \) with parametric equations \( x = \cos t, \ y = -\sin t, \ z = a, \ t \in [0,2\pi], \) has the tangent versor \( v_{c_1} = (-\sin t, -\cos t, 0). \)
The magnetic field $H_{c_1}$ has the components

$$H_{x_{c_1}} = -(z - a_1) \int_0^{2\pi} \frac{\cos t}{\Psi_1(t,x,y,z)} dt,$$

$$H_{y_{c_1}} = (z - a_1) \int_0^{2\pi} \frac{\sin t}{\Psi_1(t,x,y,z)} dt,$$

$$H_{z_{c_1}} = x \int_0^{2\pi} \frac{\cos t}{\Psi_1(t,x,y,z)} dt - y \int_0^{2\pi} \frac{\sin t}{\Psi_1(t,x,y,z)} dt - \frac{1}{\Psi_1(t,x,y,z)} dt,$$

where $\Psi_1(t,x,y,z) = (x^2 + y^2 + z^2 - 2x \cos t + 2y \sin t - 2za_1 + a_1^2 + 1)^{1/2}$.

Similarly, $C_2 : x^2 + y^2 = 1, z = -a_1, a_1 > a$, gives the magnetic field $H_{c_2}$,

$$H_{x_{c_2}} = (z + a_1) \int_0^{2\pi} \frac{\cos t}{\Psi_2(t,x,y,z)} dt,$$

$$H_{y_{c_2}} = (z + a_1) \int_0^{2\pi} \frac{\sin t}{\Psi_2(t,x,y,z)} dt,$$

$$H_{z_{c_2}} = -x \int_0^{2\pi} \frac{\cos t}{\Psi_2(t,x,y,z)} dt - y \int_0^{2\pi} \frac{\sin t}{\Psi_2(t,x,y,z)} dt + \frac{1}{\Psi_2(t,x,y,z)} dt,$$

where $\Psi_2(t,x,y,z) = (x^2 + y^2 + z^2 - 2x \cos t - 2y \sin t + 2za_1 + a_1^2 + 1)^{1/2}$.

We approximate the total magnetic field around the configuration by

$$H = H_{c_1} + H_{c_2} + H_{c_1} + H_{c_2}.$$

According to the physical properties, the magnetic field is well represented by a pattern such as linearization, computing Biot-Savart-Laplace integral with a small number of intermediate points, etc.

Because the integrals are difficult to be computed (several elliptic functions are present), it is convenient to replace them by approximation via Simpson’s numerical method, for $n=7$. For numerical approximations and graphic simulations we can choose $a = 8\pi, b_1 = 1, b_1 = 1, a_1 = 9\pi$.

### 3 Tornado magnetic flow

The anti-Helmholtz configuration of the coils (the circles $C_1$ and $C_2$) is a pair of two current loops with opposite electric current. This is important because the magnetic field created by the two coils has a “zero point” in the middle between them; more importantly, this ensures the existence of a minimum energy state for the neutral atoms. The research team of the paper [2] added two spatial helices, $e_1$ and $e_2$, to create Tornado effect for multicharged ions.

The flow of the magnetic field $H = H_x i + H_y j + H_z k$ is described by the ODE’s system

$$\frac{dx}{dt} = H_x, \frac{dy}{dt} = H_y, \frac{dz}{dt} = H_z. \quad (1)$$

In case of the magnetic field studied, there are no equilibrium points, i.e., the algebraic system $H_x = 0, H_y = 0, H_z = 0$ has no solutions.

The oriented curves which are defined as solutions of the differential system (1) are called magnetic lines.

A MAPLE simulation of the magnetic field line fixed by the data $x(0) = 0.1, y(0) = 0.1, z(0) = 0.1$, is presented in Fig.2.

![Fig.2 Magnetic line](image-url)
The density of the energy of the magnetic field is \( f = \frac{1}{2}(H_x^2 + H_y^2 + H_z^2) \).

A simulation of a constant level set of density of energy,
\[
S = \{(x, y, z) \in [-100,100]^3 \mid f(x, y, z) = 0.01\},
\]
is presented in Fig.3 (non-connected surface, with a closed piece around the origin).

Fig.3 Constant level set of the density of the energy \( f \).

Critical point of the magnetic energy \( f \), (0, 0, 0), is a minimum point. This point it appears also as intersection of three surfaces (see Fig.4).

Fig.4 Critical point of the energy (intersection of three surfaces)

4 Magnetic dynamics linearization

The magnetic energy \( f = \frac{1}{2} \|H\|^2 \) associated to a Biot-Savart-Laplace vector field \( H \), produces the following magnetic dynamics
\[
\ddot{x} = -\frac{\partial f}{\partial x}, \quad x = (x_i)_{i=1,2,3}, \quad (2)
\]

For the nonlinear second order differential system (2), two kinds of linearizations are studied: one around an equilibrium point of the magnetic field \( H \), and another around a critical point of the total magnetic energy \( f \). The linearization around a critical point of the energy is not always the same with the one around an equilibrium point of the initial vector field [3].

Let’s consider that \( A \) is the Jacobian matrix of the magnetic field \( H \) at a fixed point.

**Proposition.**

1) If \( x_e \) is an equilibrium point of the first order ODEs system (1), then the linearization of system (1) around \( x_e \) is \( \ddot{x} = A(x_e)(x - x_e) \), and the linearization of system (2) around \( x_e \) is
\[
\ddot{x} = A(x_e)^T A(x_e)(x - x_e).
\]

2) Let \( x_e \) be a critical point of the magnetic energy. The linearization of the second order ODEs system (2) around \( x_e \) is \( \ddot{x} = B(x - x_e) \), where \( B = \left[A(x_e)^T A(x_e) + H(x_e) H Hess(H)(x_e)\right] \).

The magnetic field around a Tornado trap has no equilibrium point, but the critical point of the magnetic energy is \( x_c = (0,0,0) \).
The MAPLE linearization of the ODEs system (2) around critical point of the magnetic energy \( x_c \) is
\[
\begin{align*}
x_1 &= C_1 e^{-0.0263529t} + C_2 e^{0.0263529t} + C_3 + C_4 t + C_5 \sin(0.0687t) + C_6 \cos(0.0687t), \\
x_2 &= 693.391 e^{-0.0263529t} + 43010 C_5 \sin(0.0687t) + 40310 C_6 \cos(0.0687t) + 802.605309 C_3 + 802.605309 C_4 t, \\
x_3 &= 2425.318 C_5 e^{-0.0263529t} + 2425.318 C_6 e^{0.0263529t} + 28.183 C_5 \sin(0.0687t) + 28.183 C_6 \cos(0.0687t) + 822.1351569 C_3 + 822.1351569 C_4 t.
\end{align*}
\]
Conclusions

The Tornado magnetic trap is suitable for confinement of hot dense plasmas.

Our magnetic geometric dynamics confirms three phenomena: (1) the trapping force is proportional with the gradient of magnetic energy density; (2) the trajectories are suitable geodesics, (3) the movement is stable around the origin.

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