Mathematical Model of the Reversing Two-Phase Induction Machine

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Abstract: - The paper presents a new high level mathematical model of the two-phase induction machine, called "in total fluxes". The equations of the model, which are very appropriate for the study of the machine operating under imposed voltage, including reversing operation, do not contain explicitly the rotor speed and the winding currents. The quantities used in the equations, which are the rotation angle and the total fluxes of the windings, allow the obtaining of the instantaneous electromagnetic torque in a similar expression with the well known and wide spread equation that contains winding currents. Finally, the movement equation (equilibrium of the torques) is valid both for dynamic regimes with important variations of the speed and stationary regime. The computer simulated results of the "in total fluxes" model for different regimes, and particularly for reversing operation, are more accurate in comparison with the results given by the traditional model called "in currents". These latter models contain equations which include the speed among variable quantities. The speed is supposed to be constant along infinitesimal time domains but this presumption is not exactly valid in practice for reversing operation.

Key-Words: - Two-phase induction machine, mathematical model with fluxes, simulation, reversing operation

1 Introduction

The two-phase induction machine (denoted with the acronym T-PIM in this paper) is the machine with minimal number of windings that operates on the basis of rotating magnetic field. It has two immobile windings on stator, 90 electrical degrees shifted in space. The other two equivalent windings, placed on rotor, are obviously in rotation. The position angle θ_R, between the homologous phase axes, reference *as* from stator and *ar* from rotor, is time dependent under a linear rule if the machine operates in a stationary regime [1,2,3]. This machine is used as motor (two-phase induction servomotor) in low power electric drives or as position or speed transducer (under generating duty) for the movement control of some rotating mechanical elements of positioning systems [4-16]. The two-phase induction motor with hollow-rotor made of non-magnetic metals (aluminum or alloy) and with high rotor resistance is frequently used for positioning. Due to its low inertia, this machine is proper for reversing rotation (bidirectional servomotor). This operation regime can be obtained by supplying the *control* phase winding with a voltage that has a different frequency from the *excitation* phase winding. The T-PIM is also frequently used as mathematical model *(αβ-αβ, ab-ab, ab-dq, DQ-dq)* for the study of the multiphase induction machines, mainly the three-phase ones, which operate as part of variable electric drives.

The study of this machine, operating under different regimes, uses 4 differential equations given by the II Kirchhoff theorem (voltage balance) and the torque equilibrium equation where both angular speed ω_R and its derivative $d\omega_R/dt$ are present. The second order movement equation can be replaced by two differential equations of the first order. Thus, the final system contains 6 differential equations of the first order and non-linear. The scientific literature generally uses as variable quantities the winding currents and the electromagnetic torque depends on them as well [2,3]. If the total fluxes produced by the windings are expressed in terms of currents and inductances, then the model becomes a hybrid one, denoted as *in total fluxes and currents*. A coordinate transformation can bring a linearization of the system. But these transformations bring new variable quantities which usually have a different frequency related to real ones. A new model is obtained, with two rotor windings and collinear stator and rotor axes, called _dq-dq_ *(DQ-dq)* [1,2,3]. The models mostly used in the scientific literature contain inside the circuit equations the rotor speed ω_R. The resolution of the system takes into consideration a maintaining constant of the speed or acceptance of a "slow variation". This is an important error source mainly

when the speed varies rapidly (start-up or reversal of the two-phase induction servomotor, for example) [4].

This paper presents a new model, called *in total fluxes*, which has simpler equations, the variables represent real quantities (no transformation is required) and the rotation speed, ω_R, is not present in an explicit form. The electric circuits have as independent variable the rotation angle, $\theta_R=var$ (its variation law is not necessary to be prior imposed). This model gives more accurate results for the study of variable speed regimes, the calculus time could be shorter, the precision of the unbalanced duty analysis is higher, and the implementation together with a position transducer and eventually with a rotor flux transducer is more effective.

2 The Two-Phase Induction Machine Model with Variable Quantities in Total Fluxes

The T-PIM is presented in Fig. 1a), the 4 windings with their characteristics in Fig. 1b) and the reduced rotor quantities [5] in Fig. 1c). The II Kirchhoff theorem gives 4 non-linear equations of the first order. The total fluxes depends both on time and rotation angle, θ_R, variable with time as well.
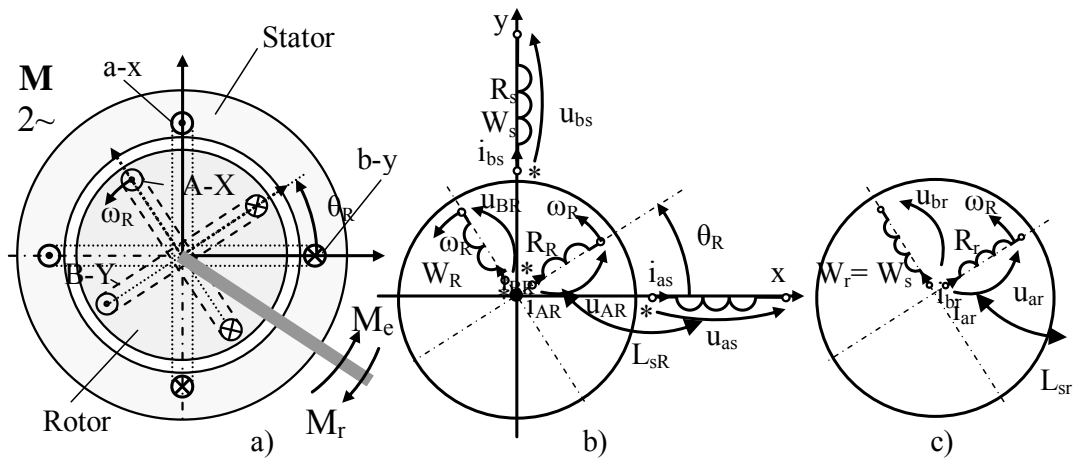

Fig.1. Two-phase induction machine: a) Physical model, b) Simplified depiction, c) Reduced rotor

If the total fluxes are expressed in terms of currents and inductances then a matrix equation can be obtained [1]:

$$
\begin{cases}
[u_{absr}]=[R_{s,r}][i_{absr}]+\dfrac{d[\psi_{absr}]}{dt}, \quad where \quad [\psi_{absr}]=[L_{abssrr}][i_{absr}], \\[4mm]
[L_{abssrr}]=L_{hs}\begin{bmatrix}
1+l_{\sigma s} & 0 & \cos\theta_R & -\sin\theta_R \\
0 & 1+l_{\sigma s} & \sin\theta_R & \cos\theta_R \\
\cos\theta_R & \sin\theta_R & 1+l_{\sigma r} & 0 \\
-\sin\theta_R & \cos\theta_R & 0 & 1+l_{\sigma r}
\end{bmatrix}, with:\dfrac{L_{\sigma s}}{L_{hs}}=l_{\sigma s},\dfrac{L_{\sigma r}}{L_{hs}}=l_{\sigma r}.
\end{cases} \tag{1}
$$

To obtain the currents, the left term is amplified with the reciprocal matrix:

$$[L_{abssrr}]^{-1}[\psi_{absr}]=[L_{abssrr}]^{-1}[L_{abssrr}][i_{absr}], \quad or \quad [i_{absr}]=[L_{abssrr}]^{-1}[\psi_{absr}] \tag{2}$$

The main issue is now to identify the *reciprocal matrix* [17]. This matrix is supposed to be similar to the direct matrix and by identification term by term and using the notations:

$$\Lambda_h=\frac{1}{L_{hs}}; \quad k=\frac{1}{l_{\sigma s}+l_{\sigma r}+l_{\sigma s}l_{\sigma r}}; \quad k_s=(1+l_{\sigma s})k; \quad k_r=(1+l_{\sigma r})k \tag{3}$$

one obtains:

$$[L_{abssrr}]^{-1} = \begin{bmatrix} k_r \Lambda_h & 0 & -k\Lambda_h \cos\theta_R & k\Lambda_h \sin\theta_R \\ 0 & k_r \Lambda_h & -k\Lambda_h \sin\theta_R & -k\Lambda_h \cos\theta_R \\ -k\Lambda_h \cos\theta_R & -k\Lambda_h \sin\theta_R & k_s \Lambda_h & 0 \\ k\Lambda_h \sin\theta_R & -k\Lambda_h \cos\theta_R & 0 & k_s \Lambda_h \end{bmatrix} \tag{4}$$

Then follows the product of the matrix $[R_{s,r}][L_{abssrr}]^{-1}[\psi_{absr}]$, and the voltage equation (1) using reduced quantities becomes after convenient grouping:

$$\frac{d\psi_{as}}{dt} + k_r \Lambda_h R_s \psi_{as} = u_{as} + k\Lambda_h R_s (\psi_{ar} \cos\theta_R - \psi_{br} \sin\theta_R), \tag{5-a}$$

$$\frac{d\psi_{bs}}{dt} + k_r \Lambda_h R_s \psi_{bs} = u_{bs} + k\Lambda_h R_s (\psi_{ar} \sin\theta_R + \psi_{br} \cos\theta_R), \tag{5-b}$$

$$\frac{d\psi_{ar}}{dt} + k_s \Lambda_h R_r \psi_{ar} = u_{ar} + k\Lambda_h R_r (\psi_{as} \cos\theta_R + \psi_{bs} \sin\theta_R), \tag{5-c}$$

$$\frac{d\psi_{br}}{dt} + k_s \Lambda_h R_r \psi_{br} = u_{br} + k\Lambda_h R_r (-\psi_{as} \sin\theta_R + \psi_{bs} \cos\theta_R), \tag{5-d}$$

The next step is the quantification of the electromagnetic torque which is possible by means of the stored energy principle or through the magnetic energy stored in the machine circuits [2,8,14] according to current values. The expression of the torque [3,4] together with (2) becomes:

$$M_e = \frac{p}{2}\left\{[i_{absr}]_t \cdot \frac{d[L_{abssrr}]}{d\theta_R}[i_{absr}]\right\} = \frac{p}{2}\left\{[\psi_{absr}]_t \cdot [L_{abssrr}]_t^{-1} \frac{d[L_{abssrr}]}{d\theta_R}[L_{abssrr}]^{-1} \cdot [\psi_{absr}]\right\} \tag{6}$$

The following equality is valid:

$$[L_{abssrr}]_t^{-1} \frac{d[L_{abssrr}]}{d\theta_R}[L_{abssrr}]^{-1} = -\frac{d[L_{abssrr}]^{-1}}{d\theta_R} \tag{7}$$

For validation, (7) is amplified to the left with $[L_{abssrr}] = [L_{abssrr}]_t$ and results:

$$[L_{abssrr}] \cdot [L_{abssrr}]^{-1} \frac{d[L_{abssrr}]}{d\theta_R}[L_{abssrr}]^{-1} = -[L_{abssrr}]\frac{d[L_{abssrr}]^{-1}}{d\theta_R} .$$

The first product is the unit matrix $[I]$, so: $\frac{d[L_{abssrr}]}{d\theta_R}[L_{abssrr}]^{-1} = -[L_{abssrr}]\frac{d[L_{abssrr}]^{-1}}{d\theta_R}$, an obvious expression

as a consequence of derivation rule of a constant product. Consequently, the active electromagnetic torque, given by (6), is:

$$M_e = -\frac{p}{2}[\psi_{absr}]_t \frac{d[L_{abssrr}]^{-1}}{d\theta_R}[\psi_{absr}] \tag{8}$$

a similar expression to the well known equation in currents. For the nonce, the torque depends with *total fluxes* alone. The calculus gives forwards:

$$M_e = pk\Lambda_h\left[-(\psi_{as}\psi_{ar} + \psi_{bs}\psi_{br})\sin\theta_R + (\psi_{bs}\psi_{ar} - \psi_{as}\psi_{br})\cos\theta_R\right] \tag{8'}$$

which proves that M_e depends on *total fluxes* and *rotor position angle*. The torque equation can be written as follows:

$$\frac{d\dot{\theta}_R}{dt} + \frac{k_z}{J}\dot{\theta}_R = \frac{p}{J}\left\{pk\Lambda_h\left[-(\psi_{as}\psi_{ar} + \psi_{bs}\psi_{br})\sin\theta_R + (\psi_{bs}\psi_{ar} - \psi_{as}\psi_{br})\cos\theta_R\right] - M_{st}\right\},$$
$$d\theta_R/dt = \dot{\theta}_R, \tag{9}$$

Now, the 6 equations system of the machine can be written by getting together the equation group (5-a÷5-d which has as variables the total fluxes and θ_R, but no speed) with the other 2 equations (9) which contain as variables only the total fluxes of the 4 windings, the speed and the position angle of the rotor.

3 Simulation Studies with the Model *in Total Fluxes* for a Reversing T-PIM

Taking into consideration (5-a) – (5-d) together with (9), one can deduce the 6 equation system under *operational calculus* form, obtained by means of the *Laplace transform*. The variables are represented by the *total fluxes* of the 4 windings, the *position angle* and the *speed* as a derivative of the angle:

$$\overline{\psi}_{as}(\overline{s} + k_r\Lambda_h R_s) = \overline{u}_{as} + k\Lambda_h R_s\left(\overline{\psi}_{ar}\cos\theta_R - \overline{\psi}_{br}\sin\theta_R\right), \tag{10-a}$$

$$\overline{\psi}_{bs}(\overline{s} + k_r\Lambda_h R_s) = \overline{u}_{bs} + k\Lambda_h R_s\left(\overline{\psi}_{ar}\sin\theta_R + \overline{\psi}_{br}\cos\theta_R\right), \tag{10-b}$$

$$\overline{\psi}_{ar}(\overline{s} + k_s\Lambda_h R_r) = \overline{u}_{ar} + k\Lambda_h R_r\left(\overline{\psi}_{as}\cos\theta_R + \overline{\psi}_{bs}\sin\theta_R\right), \tag{10-c}$$

$$\overline{\psi}_{br}(\overline{s} + k_s\Lambda_h R_r) = \overline{u}_{br} + k\Lambda_h R_r\left(-\overline{\psi}_{as}\sin\theta_R + \overline{\psi}_{bs}\cos\theta_R\right), \tag{10-d}$$

$$(\overline{s} + k_z/J)\dot{\theta}_R = (p/J)\{pk\Lambda_h[-(\overline{\psi}_{as}\overline{\psi}_{ar} + \overline{\psi}_{bs}\overline{\psi}_{br})\sin\theta_R + (\overline{\psi}_{bs}\overline{\psi}_{ar} - \overline{\psi}_{as}\overline{\psi}_{br})\cos\theta_R] - M_{st}\}, \tag{10-e}$$

$$\overline{s}\theta_R = \dot{\theta}_R. \tag{10-f}$$

where $\Lambda_h = 1/L_{hs}$; $\quad k = 1/(l_{\sigma s} + l_{\sigma r} + l_{\sigma s}l_{\sigma r})$; $\quad k_s = (1 + l_{\sigma s})k$; $\quad k_r = (1 + l_{\sigma r})k$; $l_{\sigma s} = L_{\sigma s}/L_{hs}$; $l_{\sigma r} = L_{\sigma r}/L_{hs}$.

The next step is a simulated analysis by using the proposed mathematical model for the study of the reversing operation of the two-phase induction servomotor. On the basis of the equations (10-a÷10.f), a block-diagram has been conceived, Fig. 2, with the following parameters:

$L_{ss} = L_{rr} = 0.1H;$ $\quad L_{hs} = 0.09H; L_{\sigma s} = 0.01H = L_{\sigma r};$
$R_s = 2\Omega; R_r = 20\Omega;$ $\quad J = 0.01; k_z = 0.02; p = 1;$
$U_{as\,max} = 600 = U_{bs\,max}; \omega_{as} = 314; \omega_{bs} = 307.72;$

It has to be pointed out that the *as* stator phase is connected to a supply source with a 50Hz frequency value and the *bs* stator phase is supplied with 49Hz frequency. According to theoretical arguments [4,8], the rotor must have a reversing movement corresponding to a frequency of 1Hz.

A part of the results obtained for the reversing operation regime show the variation of the applied voltages to the two windings: *as* – fig.3, and *bs* – fig.4. It is to be noticed the continuous alteration of the phase difference between the two voltages (the diagrams presents only the last 0.2 seconds of the simulation).

Fig. 5 presents the rotor speed variation: during the first half-second the machine rotates forwards and then backwards. The same phenomenon appears in Fig. 6 where the rotation angle, θ_R, is ascending during the first half-period, and then is descending towards initial value (close to zero), then this progression is repeating.

The time variation of the total rotor flux corresponding to *ar* phase is presented in Fig. 7 and for the stator *as* phase in Fig. 8. It is a fact that the total rotor flux has a frequency close to zero value when the rotor is close to synchronism and rises to 50 Hz for blocked rotor. The amplitude of this flux is about 2 Wb (except initial transients corresponding to supply source connection). The total stator flux has a 50 Hz frequency and amplitude of 2 Wb with some variations.

During the simulation, the active instantaneous torque oscillates around zero value, Fig. 9 (sometimes the torque is positive then negative). This repetitive sequence determines the reversing rotation of the rotor.

As regards the current variation of the *as* stator phase, a „beat" phenomenon is noticeable. Close to synchronism (quasi no-load operation), the current has a reduced amplitude, around 10 A, but for null speed (short-circuit operation) the amplitude rises up to 30 A, Fig. 10.

Useful information about the reversing operation of the two-phase induction servomotor are given by the mechanical ($\omega_R = f(M_e)$ – Fig. 11) and angular ($M_e = f(\delta)$ – Fig. 12) characteristics. One can see a variation of the speed in the range +310 rad/s to -310 rad/s and a variation of the torque between +60 Nm and -60 Nm with a symmetrical behavior for the two rotation senses.

Fig. 12 presents the angular characteristics where δ is the angle between the stator and rotor flux representative phase vectors. When the difference of phase between the two supply voltages is close to $\pi/2$ rad. and the rotor speed is close to synchronism then the electromagnetic torque is high, about 60 Nm and the internal angle is small since the load torque is small (close to zero). When the phase difference is zero then the torque is small and the internal angle is high, close to 1.57 rad ($\pi/2$). Again symmetry behavior as reversing servomotor is noticeable.

Fig. 2. Diagram-block of the two-phase induction oscilant servomotor
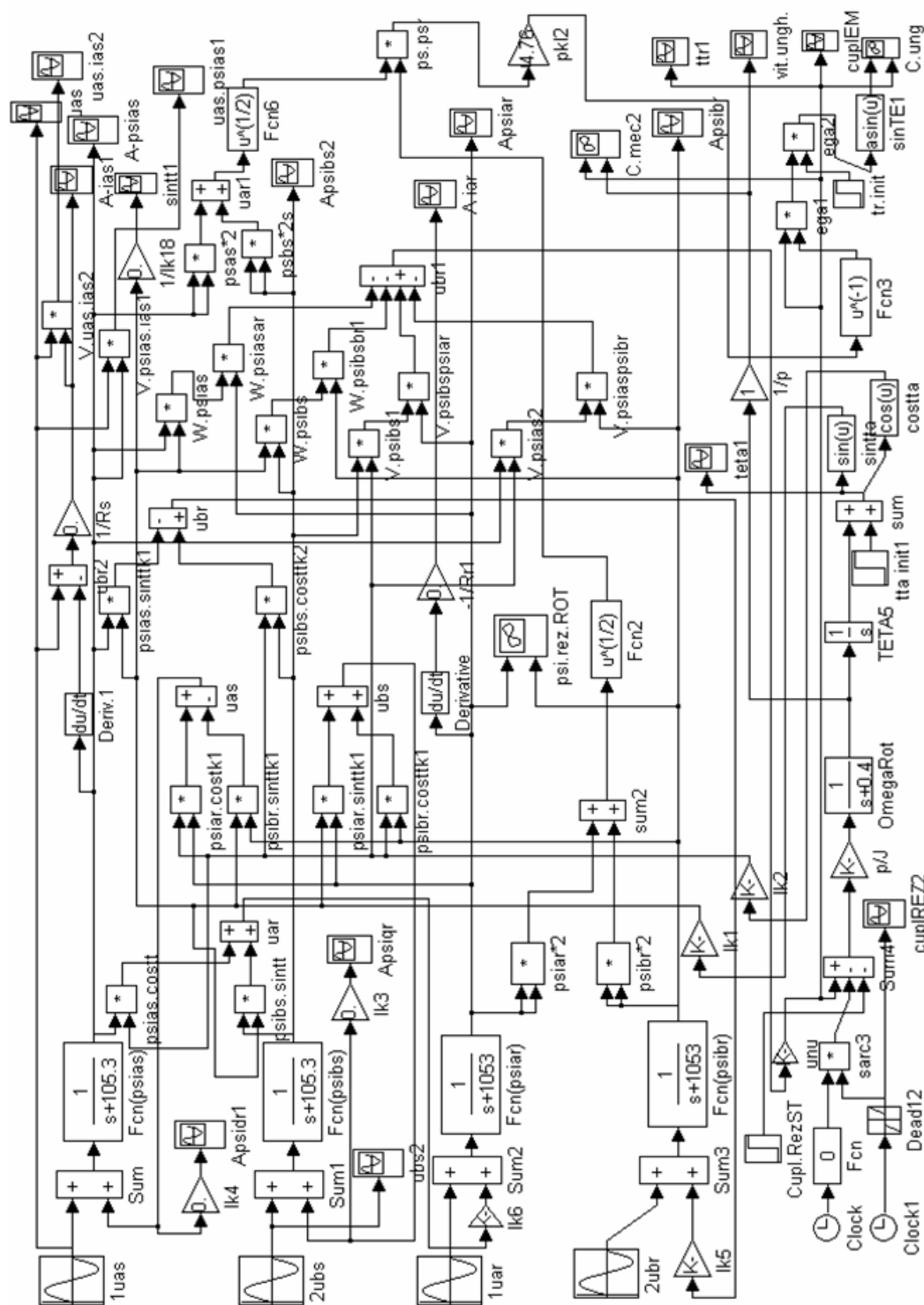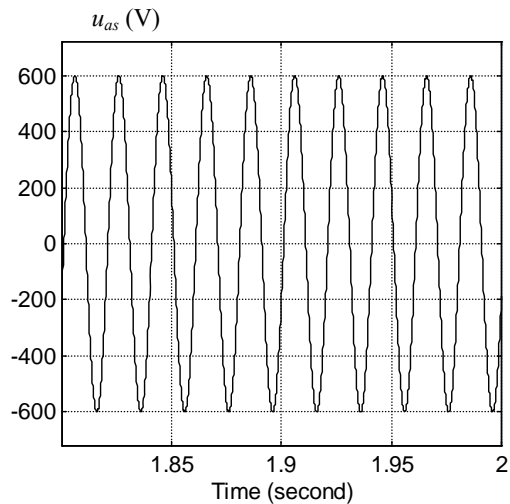
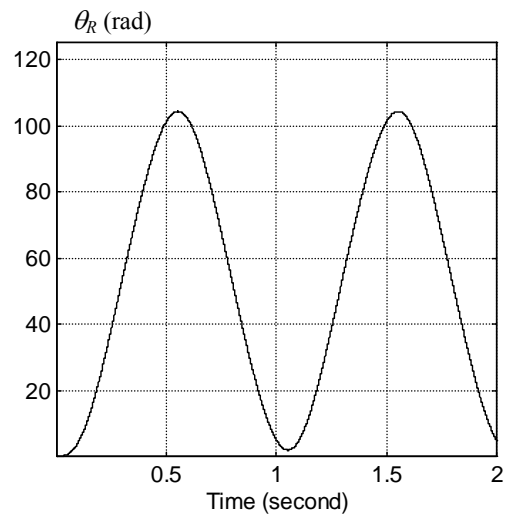Fig.3. Stator phase voltage, $u_{as}=f(t)$

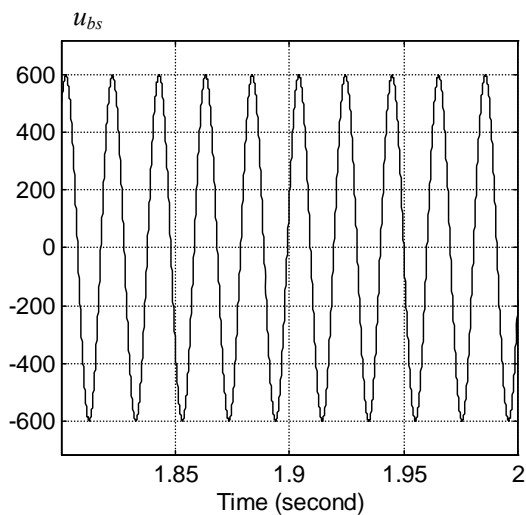Fig.6. Rotation angle, $\theta_R=f(t)$

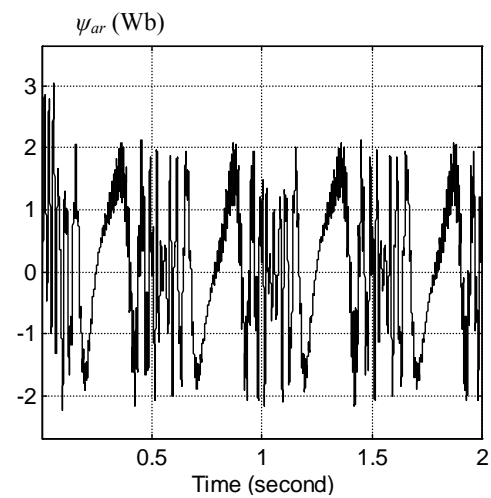Fig.4. Stator phase voltage, $u_{bs}=f(t)$

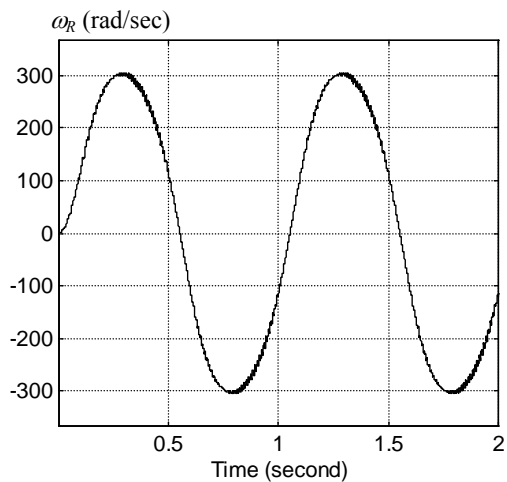Fig.7. Total rotor flux, $\psi_{ar}=f(t)$

Fig.5. Angular speed, $\omega_R=f(t)$

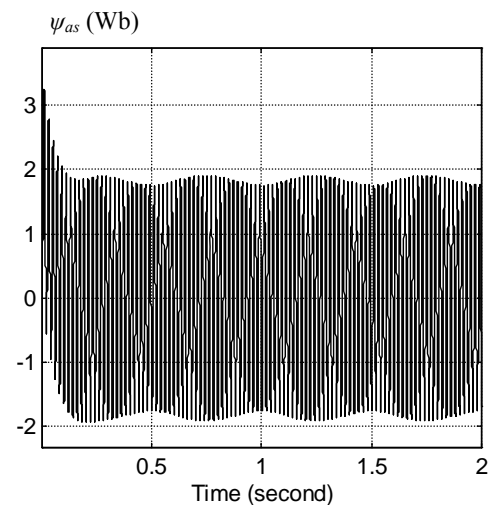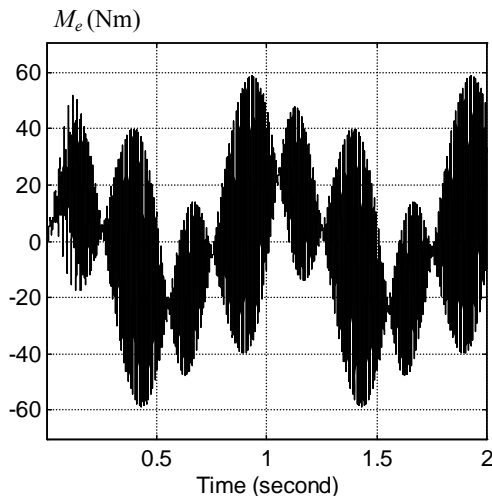Fig.8. Total stator flux, $\psi_{as}=f(t)$
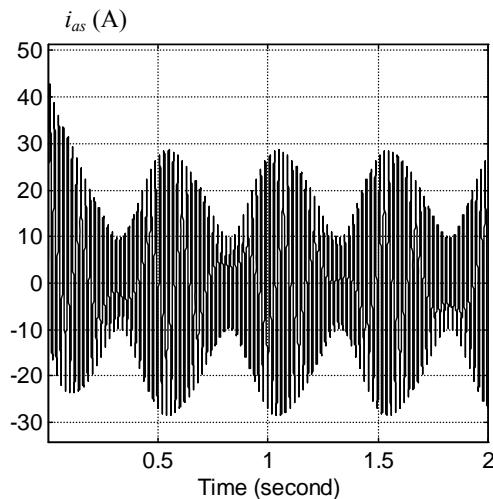
M_e (Nm)

Fig.9. Electromagnetic torque, $M_e=f(t)$

i_{as} (A)

Fig.10. Stator current, $i_{as}=f(t)$
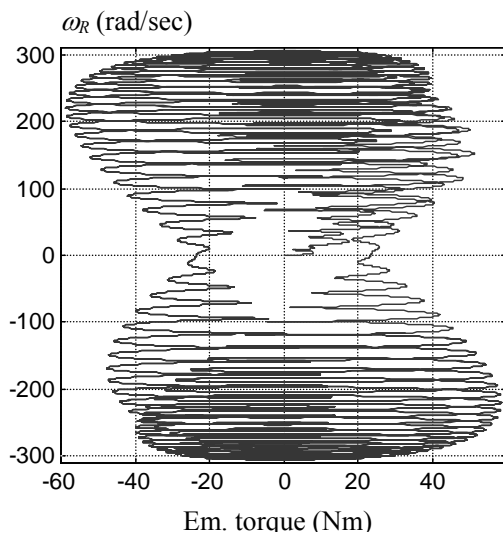
ω_R (rad/sec)

Fig.11. Mechanical characteristic, $\omega_R =f(M_e)$
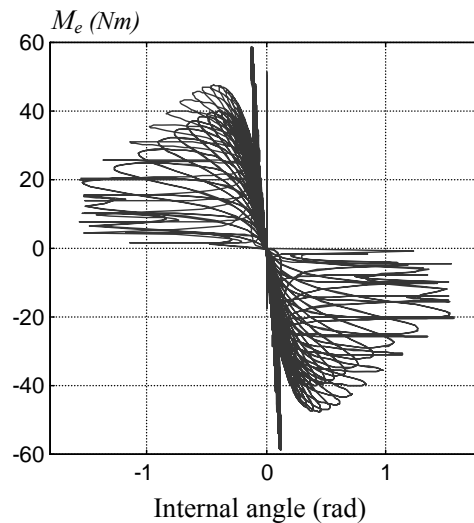
M_e (Nm)

Fig.12. Angular characteristic, $M_e=f(\delta)$

4 Conclusions

The model *in total fluxes* has simpler equations and diagram-block consequently. It allows a proper study of the reversing two-phase servomotor.

The variables (stator and rotor fluxes) represent real quantities and the rotation speed, ω_R, which varies within large limits in this operation regime, is not longer present in the equations in an explicit way. This fact determines a more precisely analysis on the basis of this model.

In comparison with the classical models, the proposed model uses as variables nothing but *total fluxes* and *rotation angle*. As consequence, there is more comfort in using it for the study of the unbalanced duties such as reversing rotation with frequencies between a few Hz to 10 Hz.

The model has a significant didactical importance and clear up that supply of the two-phase induction servomotor with different frequencies on the two windings determine a ''continuous'' phase control of the machine accompanied by the successive reversal of the electromagnetic torque sense. From analytically viewpoint, this is a validation of the experimental tests [8].

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