Definition of Geometry of Variable Radius Arch Dam with Degree of Polynomial in the 3D Finite Element Analysis

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Abstract: - The shape of an arch dam has of paramount importance in its ultimate behavior and eventually settles all design criteria. Variable curvature arch dams evolved to be economical in shape optimization studies. In Finite Element Analysis, the arch dam geometry is to be modeled very accurately. But the arch dam geometry may be varying in the three dimensions and with irregular boundaries depending on the design of profile according to site conditions; extrados intrados curves mainly in the horizontal and vertical directions need to be of higher order. Here in this paper, how effectively this type of complex arch dam geometry can be modeled by means of higher order hexahedral elements using Langrange polynomial of required degree in the respective directions is presented.

Keywords: Variable radius arch dams, Finite element analysis, Three dimensional mapping, Lagrange polynomials

1 Introduction
Arch dams can be analyzed by several methods of which Finite Element Method has become the most powerful, reliable and accurate these days as a theoretical prediction by using a mathematical model containing a set of differential equations will be more reliable than a physical model; experimental investigations, because the small-scale model may not always simulate all the features of the actual. For this purpose, it is suitable to go for a mathematical description of the physical model; each mathematical ordinary as well as partial differential equations express a certain conservative principle. Straight dams can be analyzed as a plain strain problem or by using basic three dimensional isoparametric elements. Though finite element method with two dimensional analysis using plane stress and plane strain as well as shell theory that actually approximates three-dimensional problem by two-dimensional one give good results for a thin arch dam, thick arch dam requires a rigorous three dimensional analysis. In the case of arch dams of variable curvature, a complex three dimensional solid continua of irregular geometry has to be analyzed.[1, 2] Each variable inside the continuum as well as its dependent variable must be in a balance among the various factors that influence the variable.

2 Problem Formulation
The actual solid continuum is to be discretised with the chosen basic elements ensuring intra element as well as inter element continuity and connectivity.[3] The arch dam discretisation are seen as an assemblage of horizontal arches and vertical cantilevers. The 16 node shell and general 3D solid elements of 8 and 21 nodes commonly provided for modeling the geometry are not sufficient to accurately define the shape of a thick variable curvature arch dam. This limitation in shape approximation as explained above is to be resolved by the help of higher order polynomial shape functions including more number of nodes at element level relevant to the curve in each direction to take care of the curvature. Here in this paper, a program has been developed to model the geometry of an arch dam using higher order polynomials of the relevant degree to suit with the curve in the respective directions are made use of. After arriving geometry accurately like this, the continuum can be discretised with the basic element chosen for carrying out the Finite Element Analysis.

2.1 Three dimensional mapping
The three-dimensional mesh with the required number of basic n- noded elements along the three directions is generated inside a cuboid of side 2 units and is mapped into the actual three-dimensional structure. This cube in the local coordinate system is to be mapped to the actual distorted continuum, knowing the global coordinates of the corresponding nodal points in the actual continuum with that of the basic element.[3] If the basic element chosen is an ‘n’ noded one, then, knowing the corresponding global values of the n points, the global coordinates of other points can be arrived with the help of shape functions by isoparametric mapping.
Any point in the local coordinate system is given by:

\[
x = \sum_{i=1}^{n} N_i x_i, \quad y = \sum_{i=1}^{n} N_i y_i, \quad z = \sum_{i=1}^{n} N_i z_i
\]

Thus, all the corresponding points of the continuum can be found out. The discretisation in the basic element can be mapped into the distorted continuum. Corresponding to each local coordinate, a single global Cartesian coordinate will be obtained. Violent distortion may be avoided to guard against non-uniqueness for which the degree of polynomial selected should match the curve.

### 2.1.1 Mesh Generation and Plotting

A mesh generator program developed from basic principles of isoparametric mapping in visual C++ with plotting in MatLab is used.\[4\] Here below Fig.2 shows the mesh plotted for the symmetric half of a USBR arch dam by inputting the global 20 nodes of the arch dam continuum.

### 3 Problem Solution

When the geometry modeling requires higher order, shape functions using Lagrange family of elements can be found out by the simple product of the shape functions in each direction at a point.\[5,6\] Thus, the shape function at the point \((i, j, k)\) is given by:

\[
N = N_{ijk} = N_{\xi} N_{\eta} N_{\zeta}
\]

where,

\[
N_{\xi_i} = \sum_{i=1}^{\text{deg}+1} \left( \xi - \xi_i \right) \quad \text{for } i \neq j
\]

\[
N_{\eta_j} = \sum_{j=1}^{\text{deg}+1} \left( \eta_j - \eta \right) \quad \text{for } i \neq j
\]

\[
N_{\zeta_k} = \sum_{k=1}^{\text{deg}+1} \left( \zeta_k - \zeta \right) \quad \text{for } i \neq j
\]

\(N_{\xi_i}, N_{\eta_j}, N_{\zeta_k}\) are the shape functions for the point \((i, j, k)\) in \(\xi, \eta, \zeta\) directions. The shape function components in each direction at a particular point can be found by the above relationship. The shape function at a point \( (\xi_i, \eta_j, \zeta_k) \) will be the product of the component shape function as above. A program for the shape functions of Lagrange family of elements is developed 'Visual C++'.

Arch dam geometry is arrived according to the guidelines for preliminary design of arch dams as per USBR recommendation. Mesh arrived with degree of polynomial 2 in the three directions for 27 noded elements are shown below.
2.1 Three dimensional mapping with Varying degree

While defining the arch dam geometry for a variable radius or curvature, being irregular and curved boundaries, the polynomial function selected for each direction should define the curve in the respective direction accurately. Simple isoparametric mapping cannot achieve this accurately for which higher order elements is useful for varying degree of approximation. In such cases of complex geometry the degree of curve in all the three directions may vary which can be accommodated using Langrange shape function as follows:

```c
//Langrange shape function of required degree in any direction (function call)

void shapeFunc(int degL, int degT, int degH, double pxi, double pet, double pze)
{
    ofstream fout("shape.out");
    double
    xxi[100],yyi[100],zzi[100],Nxii[100],Neti[100],Nzet[100];
    int i, j, k;
    for (i=2; i<=degL + 1; i++)
    { 
        xxi[i] = xxi[i-1] + 2.0/degL;
    }
    for (i=2; i<=degT + 1; i++)
    {
        yyi[i] = yyi[i-1] + 2.0/degT;
    }
    for (i=2; i<=degH + 1; i++)
    { 
        zzi[i] = zzi[i-1] + 2.0/degH;
    }
    for (i=1; i<=degL + 1; i++)
    {
        Nxii [i] = 1;
        for (j=1; j<=degL + 1; j++)
        {
            if (j != i)
            {
                Nxii [i] *= (pxi - xxi[j])/(xxi[i] - xxi[j]);
            }
        }
    }
    for (i=1; i<=degT + 1; i++)
    {
        Neti [i] = 1;
        for (j=1; j<=degT + 1; j++)
        {
            if (j != i)
            {
                Neti [i] *= (pet - yyi[j])/(yyi[i] - yyi[j]);
            }
        }
    }
    for (i=1; i<=degH + 1; i++)
    {
        Nzet [i] = 1;
        for (j=1; j<=degH + 1; j++)
        {
            if (j != i)
            {
                Nzet [i] *= (pze - zzi[j])/(zzi[i] - zzi[j]);
            }
        }
    }
    int node = 0;
    for (i=1; i<=degH + 1; i++)
    {
        for (j=1; j<=degT + 1; j++)
        {
            for (k=1; k<=degL + 1; k++)
            {
                node ++;
                N[node] = Nxii[k] * Neti[j] * Nzet[i];
            }
        }
    }
}
```

Fig.3. Discretisation with 27 noded brick element
A double curvature arch dam geometry derived by 80 nodal points with degree of polynomial 7 in length, 1 in thickness and 4 in height directions is discretised to 28 elements using 20 noded elements with the developed software is arrived like this. The mesh of the dam with node numbering is shown in below.

Fig.4. Double curvature arch dam- Geometry with 80 nodes-Discretisation with 20 noded brick element.

4 Conclusion
It is found that the method is capable of accommodating the effect of curvature accurately with required degree of curve in each direction. Since Lagrange shape functions of higher order are used for defining the geometry accurately, structures with complex geometry like double and multiple curvature arch dams can be effectively modeled and analysed with this software. This is developed using advanced programming technique in Visual C++ with verification by plotting in Matlab.

References: