Analytical results and numerical simulations for k-order nonlinear discrete determinist exchange rate models

MIRELA-CATRINEL VOICU
Faculty of Economic Sciences
West University of Timișoara
ROMANIA

Abstract: In this paper we present some qualitative results for a particular type of k-order exchange rate models. These results concern the existence of the fixed point, its stability and its attraction domain and the existence of the period-two cycles. Given the nonlinear nature, these systems can display a more complex evolution (chaotic behavior, period doubling-bifurcation, limit cycle and period-p cycle with p > 2). In the last section, we present such examples of behavior, using numerical simulations. The algorithms implementation is made using VBA (Visual Basic for Applications) program in Excel, and the images of the figures in this paper are made using Mathematica.

Key-Words: nonlinear system, fixed point, period-p cycle, attractors, numerical simulations.

1 Introduction

According to [2] a general equation modeling the exchange rate evolution is given by:

\[ S_t = X_t E_c(S_{t+1}) \]  
(1)

In the above equation, \( S_t \) is the exchange rate at the moment \( t \); \( X_t \) describes the exogenous variables that drive the exchange rate at the moment \( t \); \( E_c(S_{t+1}) \) is the expectation held at the moment \( t \) in the market about the exchange rate at the moment \( t+1 \); \( b \) is the discount factor that speculators use to discount the future expected exchange rate (0 < b < 1).

This model allows us to take into account two components for forecasting: a forecast made by the chartists \( E_c(S_{t+1}) \) and a forecast made by the fundamentalists \( E_f(S_{t+1}) \):

\[ E_c(S_{t+1})/S_{t-1} = (E_c(S_{t+1})/S_{t-1})^{m_c} \]  
(2)

where \( m_c \) is the weight given by the chartists and \( 1 - m_c \) is the weight given by the fundamentalists at the moment \( t \).

The fundamentalists assume the existence of an equilibrium exchange rate \( S^* \). If at the moment \( t-1 \) the exchange rate \( S_{t-1} \) is above, respectively below, the equilibrium rate \( S^* \), the fundamentalists expect the future exchange rate \( S_{t+1} \) to go down, respectively increase, with the speed \( \alpha \). More precisely, if they observe a deviation today, then their forecasts is the following:

\[ E_f(S_{t+1})/S_{t-1} = \left( \frac{S^*}{S_{t-1}} \right)^\alpha, \quad \alpha > 0. \]  
(3)

The chartists use the past values of the exchange rate to detect patterns that they extrapolate in the future. An equation which gives a general description of the different models used by chartists is the following:

\[ E_c(S_{t+1})/S_{t-1} = f(S_{t-1}, ..., S_{t-N}). \]  
(4)

According to [2] it is possible to specify such a rule, in general terms, as follows:

\[ E_c(S_{t+1})/S_{t-1} = c_1 \left( \frac{S_{t+1}}{S_{t+2}} \right) c_2 \left( \frac{S_{t+2}}{S_{t+3}} \right) ... \left( \frac{S_{t+N}}{S_{t+N+1}} \right)^{c_{N+1}}. \]  
(5)

The exact nature of this rule is determined by the coefficients \( c_i \). These can be positive, negative, or zero. The weight \( m_c \), in equation (2), given by chartists is

\[ m_c = \frac{1}{1 + \beta(S_{t-1} - S^*)^2}, \quad \beta > 0. \]  
(6)

The parameter \( \beta \) measures the precision degree of the fundamentalists’ estimation. When the exchange rate is in the neighbourhood of the equilibrium rate, chartists’ behavior dominates. When the exchange rate differs from the fundamental rate, then the expectation will be dominated by the fundamentalists.

In this paper we consider the case \( X_t = 1 \) (which means that \( S^* = 1 \)) and for chartists we consider the expectation:

\[ E_c(S_{t+1})/S_{t-1} = \left( \frac{S_{t+1}}{S_{t+k}} \right)^c, \quad c > 1, \quad k \geq 2, \quad k \in \mathbb{N}. \]  
(7)

In equation (2) we will use the expectations given by
In equation (1) we will use the expectations given by equation (2). In this way, we obtain the following difference equation:

\[ S_t = \left[ \frac{(2+\alpha \beta)(1+\alpha \beta)}{1+\beta(e^{\alpha \beta}-1)} \right] \left[ \frac{-2b}{1+\beta(e^{\alpha \beta}-1)} \right] S_{t-k} \]

If we denote \( S_t = \ln S_t \), then equation (8) can be written in the form:

\[ S_t = \left( 1+\alpha \beta \right) e^{\alpha \beta} S_{t-1} + \left( 1-\alpha \beta \right) S_{t-1} \]

with \( S_t \in R \) and \( t \in Z \). We can rewrite equation (9) in the following vectorial form:

\[ \left( S_{t+1}, ..., S_{t+k+1} \right) = F(S_{t+1}, S_t, S_{t+k}) \]

where \( F : R^k \rightarrow R^k \), \( F(x_1, ..., x_k) = \left( F_1(x_1, ..., x_k), ..., F_k(x_1, ..., x_k) \right) \), is defined in the following way:

\[ F_i(x_1, ..., x_k) = \varphi(x_i) x_i + \psi(x_i) x_i, \quad \varphi(x) = \frac{(2+\alpha \beta)}{1+\beta(e^{\alpha \beta}-1)} + (1-\alpha \beta) \beta \quad \text{and} \quad \psi(x) = \frac{-2b}{1+\beta(e^{\alpha \beta}-1)} \frac{(1-\alpha \beta) \beta}{1} \].

In Sections 2 and 3 we will present some analytical results for system (10) and in Section 4 we will present some numerical simulations.

2 Fixed point. Existence, unicity, stability and attraction domain

2.1. Steady-state existence, unicity and stability

A fixed point for system (10) is a point \( \left( x^*, ..., x^* \right) \in R^k \) for which \( \left( x^*, ..., x^* \right) = F(x^*, ..., x^*) \).

We recall that a fixed point \( \left( x^*, ..., x^* \right) \) is stable if, for any sufficiently small neighbourhood \( U \supseteq \left( x^*, ..., x^* \right) \) there is a neighbourhood \( V_U(\alpha, b, \beta, c) \supseteq \left( x^*, ..., x^* \right) \) so that \( F^t(x_1, ..., x_k) \in U \) for every point \( (x_1, ..., x_k) \in V_U(\alpha, b, \beta) \) and all \( t > 0 \), where \( F^t = F \circ F^{t-1} \).

If there is a neighborhood \( V_U(\alpha, b, \beta, c) \supseteq \left( x^*, ..., x^* \right) \) so that \( F^t(x_1, ..., x_k) \rightarrow \left( x^*, ..., x^* \right) \), when \( t \rightarrow \infty \), for every point \( (x_1, ..., x_k) \in V_U(\alpha, b, \beta) \), then the fixed point is asymptotically stable (attracting fixed point).

Proposition 1. In the case in which \( c > 1, \ b \in \ (0, 1), \ \alpha > 0 \ \text{and} \ \beta > 0 \) system (10) has a unique fixed point and this point is \( (0, ..., 0) \in R^k \).

Observation 1. The Jacobian matrix of function \( F \)

(defined in relation (10)) for \( (0, ..., 0) \in R^k \), is the matrix

\[
\begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
-cb & 0 & \cdots & (c+1)b
\end{pmatrix}
\]

and it has the determinant

\((-1)^k \lambda^k + (-1)^{k-1}(1+c)b \lambda^{k-1} + (-1)^k bc\). Calculating the eigenvalues of the Jacobian Matrix, we can establish when the fixed point is stable or unstable.

We do not give a mathematical solution for this problem, but we can use the computer, like in the examples from Table 1, where we make \( c=2 \):

<table>
<thead>
<tr>
<th>System order</th>
<th>The fixed point is stable for</th>
<th>The fixed point is unstable for</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k=2 )</td>
<td>( b \in (0, 0.5) )</td>
<td>( b \in (0.5, 1) )</td>
</tr>
<tr>
<td>( k=3 )</td>
<td>( b \in (0, 0.3165) )</td>
<td>( b \in (0.3165, 1) )</td>
</tr>
<tr>
<td>( k=4 )</td>
<td>( b \in (0, 0.2654) )</td>
<td>( b \in (0.2654, 1) )</td>
</tr>
<tr>
<td>( k=5 )</td>
<td>( b \in (0, 0.2428) )</td>
<td>( b \in (0.2428, 1) )</td>
</tr>
<tr>
<td>( k=6 )</td>
<td>( b \in (0, 0.2306) )</td>
<td>( b \in (0.2306, 1) )</td>
</tr>
<tr>
<td>( k=10 )</td>
<td>( b \in (0, 0.2119) )</td>
<td>( b \in (0.2119, 1) )</td>
</tr>
</tbody>
</table>

Table 1: Fixed point stablility

2.2. Attraction domain for the fixed point

In the case in which the fixed point is stable, it is important to study its attraction domain. In order to make such a study, now, we give the following result:

Proposition 2. Under the assumption:

\( c > 1, \ b \in \left( 0, \frac{1}{2c+1} \right), \ \alpha \in (0, 1), \ \beta > 0, \ s_t \in R, \ t \in Z \), the following relations are verified:

**c.1.** For \( S_{t+1}, S_{t+k} < 0 \):

- **c.1.1.** if \( S_{t+1} \in \left( -\infty, \frac{(c+1)+(1-\alpha)\beta(e^{\alpha \beta}-1)^2}{c} \right) \) then
  \(-S_{t+1} > S_{t+k+1} > 0 > S_{t+k} > S_{t+i} \)

- **c.1.2.** if \( S_{t+1} = \frac{(c+1)+(1-\alpha)\beta(e^{\alpha \beta}-1)^2}{c} \) then
  \( S_{t+k} = 0 > S_{t+k} > S_{t+i} \)

- **c.1.3.** if
  \( S_{t+1} = \left\{ \frac{(c+1)+(1-\alpha)\beta(e^{\alpha \beta}-1)^2}{c} S_{t+k}, \frac{(c+1)b+(1-\alpha)\beta(e^{\alpha \beta}-1)^2}{1+cb+\beta(e^{\alpha \beta}-1)} S_{t+k} \right\} \) then
  \( 0 > S_{t+k+1} > S_{t+k} \quad 0 > S_{t+k+1} > S_{t+i} \)
c.1.4. if \( s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb + \beta(x^{t+1} - 1)} \) then 
\( 0 > s_{t+1} = s_{t+1} > s_{t+1} \)
c.1.5. if \( s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb + \beta(x^{t+1} - 1)} x_{t+1}, 0 \) then 
\( 0 > s_{t+1} > s_{t+1} > s_{t+1} \)
c.2. For \( s_{t+1}, s_{t+1} > 0 \) :
c.2.1. if \( s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb + \beta(x^{t+1} - 1)} s_{t+1} \) then 
\( s_{t+1} = s_{t+1} > s_{t+1} > 0 \)
c.2.2. if \( s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb + \beta(x^{t+1} - 1)} s_{t+1} \) then 
\( s_{t+1} = s_{t+1} \)
c.2.3. if \( s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb + \beta(x^{t+1} - 1)} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > 0, s_{t+1} > s_{t+1} > 0 \)
c.2.4. if \( s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta(x^{t+1} - 1)^2}{c} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > s_{t+1} \)
c.2.5. if \( s_{t+1} = \frac{(c+1)b + (1-\alpha)b\beta(x^{t+1} - 1)^2}{c} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > 0 \)
c.3. For \( s_{t+1} < 0, s_{t+1} > 0 \) :
c.3.1. if \( s_{t+1} = \frac{-(c+1)b - 1 + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > 0 \)
c.3.2. if \( s_{t+1} = \frac{(c+1)b - 1 + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > 0 \)
c.3.3. if \( s_{t+1} = \frac{(c+1)b - 1 + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > 0 \)
c.4. For \( s_{t+1} > 0, s_{t+1} < 0 \) :
c.4.1. if \( s_{t+1} = \frac{(c+1)b - 1 + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > 0 \)
c.4.2. if \( s_{t+1} = \frac{(c+1)b - 1 + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb} s_{t+1} \) then 
\( s_{t+1} > s_{t+1} > 0 \)

then \( s_{t+1} > 0 > s_{t+1} = s_{t+1} \)
c.4.3. if \( s_{t+1} = \frac{(c+1)b - 1 + (1-\alpha)b\beta(x^{t+1} - 1)^2}{1 + cb} s_{t+1} \) then 
\( s_{t+1} > 0 > s_{t+1} > s_{t+1} > 0 \)
c.5. if \( s_{t+1} < 0 \) and \( s_{t+1} = 0 \) then \( 0 < s_{t+1} < s_{t+1} \)
c.6. if \( s_{t+1} > 0 \) and \( s_{t+1} = 0 \) then \( s_{t+1} = s_{t+1} < 0 \)
c.7. if \( s_{t+1} = 0 \) and \( s_{t+1} < 0 \) then \( s_{t+1} < s_{t+1} < 0 \)
c.8. if \( s_{t+1} = 0 \) and \( s_{t+1} > 0 \) then \( 0 < s_{t+1} < s_{t+1} \)

Remark 1. For \( c > 1, \alpha \in (0,1), \beta > 0 \) and \( b \in \left(0, \frac{1}{2c+1}\right) \), we find that \( |s_{t+1}| \leq \max|s_{t+1}| |s_{t+1}|, \forall s_{t} \in R, t \in Z \). This relation implies that 
\( \max|s_{t+1}| |s_{t+1}|, \forall s_{t} \in R, t \in Z \).

For \( c = 1, \alpha = 1, \beta = 0 \) and \( b \in \left(0, \frac{1}{2c+1}\right) \), we find that \( |s_{t+1}| \leq 1 |s_{t+1}|, \forall s_{t} \in R, t \in Z \).

We define: \( s_{t+1}^* = \frac{s_{t+1}^*}{s_{t+1}^*} \) for \( j \in Z, i = 1, \ldots, k \), then the sequence \( s_{t+1}^* \) is monotonously decreasing and positive. This means that the sequence \( s_{t+1}^* \) is convergent. If \( p = \lim_{t \to \infty} s_{t+1}^* \), then:

1. \( \lim_{t \to \infty} s_{t+1} = p \) or
2. \( \lim_{t \to \infty} s_{t+1}^* = -p \) or
3. \( \lim s_{t+1} = t + jk \) for \( t + jk \in T \), \( \lim s_{t+1} = t + jk \) for \( t + jk \in T \), where \( T \cap T = \emptyset \), \( T \cup T = Z \) and \( T \) or \( T \) are infinite.

Using Proposition 2 and Remark 1, we provide the following result:

Proposition 3. For \( c > 1, \alpha \in (0,1), \beta > 0 \) and \( b \in \left(0, \frac{1}{2c+1}\right) \) and any initial condition of system (10), the limit \( p \) is 0. This implies that the fixed point \( (0,0) \in R^k \) is globally attractive.

Proposition 4. The fixed point is stable for \( b \in \left(0, y(c,k)\right) \), where \( y(c,k) \) is

3 Period-two cycles for system (10)

A period-2 point of system (10) is a solution of the
equation \( (s_1, \ldots, s_k) = F^2(s_1, \ldots, s_k) \) where \((s_1, \ldots, s_k) \neq F(s_1, \ldots, s_k)\). The relations
\( (s_2, \ldots, s_{k+1}) = F(s_1, s_2, \ldots, s_k) \), \( (s_3, \ldots, s_{k+2}) = F(s_2, \ldots, s_{k+1}) \) and
\( (s_4, \ldots, s_{k+3}) = F(s_3, \ldots, s_{k+2}) \) imply that
1) for \( k \), an odd number, a period – two cycle has the form: \( (s_2, s_3, \ldots, s_{k+1}, s_1, s_2, s_3, \ldots, s_k) \), where the vectors are from \( R^k \), 
\( (s_2, s_3, \ldots, s_k, s_1) = F(s_2, s_3, \ldots, s_k, s_1) \) and 
\( (s_3, s_4, \ldots, s_k, s_1) = F(s_3, s_4, \ldots, s_k, s_1) \).
2) for \( k \), an even number, a period – two cycle has the form: 
\( (s_2, s_3, \ldots, s_k, s_1) \), where the vectors are from \( R^k \), 
\( (s_2, s_3, \ldots, s_k, s_1) = F(s_2, s_3, \ldots, s_k, s_1) \) and 
\( (s_3, s_4, \ldots, s_k, s_1) = F(s_3, s_4, \ldots, s_k, s_1) \).

3.1. The case when \( k \) is an even number
Now we consider that \( k \) is an even number and we get the following proposition:

Proposition 5. If \( c > 1 \) and \( b \in (0,1) \), if
\[
\alpha \in \left(0, \frac{1+c}{b}\right] \quad \text{or} \quad \alpha \in \left(1 + \frac{1+c}{b}, \infty\right]
\]
and
\[
\beta \in \left(0, \frac{(c+1)b^2(\alpha-1)+1+cb}{(\alpha-1)^2b^2-1}\right],
\]
then system (10) has no cycles of period two.

If \( \alpha \in \left(1 + \frac{1+c}{b}, \infty\right) \) and \( \beta \in \left(\frac{(c+1)b^2(\alpha-1)+1+cb}{(\alpha-1)^2b^2-1}, \infty\right) \)
then system (10) has an unique cycle of period two. This cycle is \( (s_2, s_3, \ldots, s_k, s_1) \) where \( s_1 \) and \( s_2 \) are solutions of the equation:
\[
\phi\left(\frac{\phi(x)}{1-\psi(x)}\right) - \frac{\phi(x)}{1-\psi(x)} = 1,
\]
which means that \( s_1 \) and \( s_2 \) are solutions of equation:
\[
\frac{(c+1)b^2(\alpha-1)+1+cb}{(\alpha-1)^2b^2-1} = 1.
\]

The numbers \( s_1 \) and \( s_2 \) verify the relation \( s_1s_2 < 0 \). Let \( s_1 > 0 \) be the positive number.

If \( \beta \in \left(\frac{(c+1)b^2(\alpha-1)+1+cb}{(\alpha-1)^2b^2-1}, \frac{(1+c)b^2+1}{(\alpha-1)^2b^2-1}\right] \), then we find that \( s_1 > \ln \left(1 + \frac{1+(c+1)b}{\sqrt{\beta[(\alpha-1)b-1]}^2}\right) \) and
\[
s_2 < -\ln \left(1 + \frac{1+(c+1)b}{\sqrt{\beta[(\alpha-1)b-1]}^2}\right) .
\]

If \( \beta \in \left(\frac{(2c+1)b^2+1}{(\alpha-1)b^2-1}, \infty\right) \), then we find that
\[
s_1 \in \left(\ln 1 + \frac{1+(c+1)b}{\sqrt{\beta[(\alpha-1)b-1]}}, -\ln 1 - \frac{1+(c+1)b}{\sqrt{\beta[(\alpha-1)b-1]}^2}\right) \]
and
\[
s_2 \in \left(\ln 1 - \frac{1+(c+1)b}{\sqrt{\beta[(\alpha-1)b-1]}}, -\ln 1 + \frac{1+(c+1)b}{\sqrt{\beta[(\alpha-1)b-1]}^2}\right) .
\]

From Propositions 4 and 5 we get the following result:

Proposition 6. If \( c > 1 \), \( b \in \left(0, \frac{1}{c+1}\right) \), \( \alpha \in \left(1 + \frac{1+c}{b}, \infty\right) \)
and \( \beta \in \left(\frac{(c+1)b^2(\alpha-1)+1+cb}{(\alpha-1)^2b^2-1}, \infty\right) \), then the fixed point \( (0,\ldots,0) \) in \( R^k \) of the system is locally attractive.

3.2. The case when \( k \) is an odd number
Now, we consider that \( k \) is an odd number and we get the following proposition:

Proposition 7. For \( c > 1 \) and \( b \in (0,1) \), if
\[
\alpha \in \left(0, 1 + \frac{1}{b}\right] \quad \text{or} \quad \alpha \in \left(1 + \frac{1}{b}, \infty\right)
\]
and
\[
\beta \in \left(0, \frac{b^2(\alpha-1)+1}{(\alpha-1)^2b^2-1}\right],
\]
then system (10) has no cycles of period two.

If \( \alpha \in \left(1 + \frac{1}{b}, \infty\right) \) and \( \beta \in \left(\frac{b^2(\alpha-1)+1}{(\alpha-1)^2b^2-1}, \infty\right) \), then system (10) has an unique period-2 cycle. This cycle is \( (s_2, s_3, \ldots, s_k, s_1) \) where \( s_1 \) and \( s_2 \) are solutions of the equation:
\[
(\phi(\phi(x) + \psi(x))x) + \psi((\phi(x) + \psi(x))x)(\phi(x) + \psi(x)) = 1,
\]
which means that \( s_1 \) and \( s_2 \) are solutions of equation:
\[
\frac{ab}{1+b(\phi(x) + \psi(x))x^2} + (1-\alpha)b\left(\frac{ab}{1+b(\phi(x) + \psi(x))x^2}\right) = 1,
\]
Numbers \( s_1 \) and \( s_2 \) verify the relation \( s_1s_2 < 0 \). Let
$s_1 > 0$ be the positive number.

If $\beta \in \left[ b^2(\alpha - 1) + 1 \right] \frac{b+1}{[(\alpha - 1)^2 b^2 - 1] [(\alpha - 1)b - 1]}$ then

$s_1 > \ln \left( 1 + \frac{1}{\sqrt{\beta} [(\alpha - 1)b - 1]} \right)$ and $s_2 < -\ln \left( 1 + \frac{1}{\sqrt{\beta} [(\alpha - 1)b - 1]} \right)$

If $\beta \in \left[ \frac{b+1}{[(\alpha - 1)b - 1]} , \infty \right)$ then

$s_1 \in \left[ \ln \left( 1 + \frac{1}{\sqrt{\beta} [(\alpha - 1)b - 1]} \right) \right] - \ln \left( 1 - \frac{1}{\sqrt{\beta} [(\alpha - 1)b - 1]} \right)$ and

$s_2 \in \left[ \ln \left( 1 - \frac{1}{\sqrt{\beta} [(\alpha - 1)b - 1]} \right) \right] - \ln \left( 1 + \frac{1}{\sqrt{\beta} [(\alpha - 1)b - 1]} \right)$

From Propositions 4 and 7 we get the following result:

**Proposition 8.** If $c > 1$, $b \in \left[ 0, \frac{1}{2c+1} \right]$, $\alpha \in \left( 1, +\infty \right]$ and

$\beta \in \left[ b^2(\alpha - 1) + 1 \right] \frac{b+1}{[(\alpha - 1)^2 b^2 - 1] [(\alpha - 1)b - 1]}$, then the fixed point

$(0, \ldots, 0) \in \mathbb{R}^k$ of the system is locally attractive.

### 4 Numerical simulations

We now recall some notions which will be used in this section. We say that a set $A$ is an attracting set with the fundamental neighbourhood $U$, if it verifies the following properties (see [5]):

1) **attractivity:** for every open set $V \supset A$, $F'U \subset V$ for all sufficiently large $t$.

2) **invariance:** $F'(A) = A$, for all $t$.

3) **$A$ is minimal:** there is no proper subset of $A$ that satisfies conditions 1 and 2.

The basin of attraction is the set of initial points $x$ so that $F'(x)$ is close to $A$ when $t \to \infty$.

It is possible to classify the different attractors: attracting fixed point, attracting n-cycle, quasiperiodic attractor and strange attractor. An attractor, as an experimental object, gives a global description of the asymptotic behavior of a dynamical system.

When a deterministic mechanism presents complex behavior with intermittence, we can conclude that the series evinces chaos under certain conditions.

The sensitive dependence on initial conditions is one of the most essential aspects in identifying the chaos. We recall that the sensitive dependence on initial conditions means that two trajectories starting very close together will rapidly diverge from each other. The strange attractor is associated with a chaotic state of time evolution and is characterized by the sensitive dependence on initial conditions.

A measure of the average rate of exponential divergence exhibited by a chaotic system is given by the Lyapunov exponents of the system; the positivity of one from these exponents can suggest the presence of chaos.

The Lyapunov exponents $\lambda_1, \lambda_2 \ldots \lambda_k$ are given by

\[
(11) \quad \left\{ \begin{array}{l}
\epsilon^{s_1}, \epsilon^{s_2}, \ldots, \epsilon^{s_k} = \lim_{n \to \infty} \left\{ \text{eigenvalues of} \left( \prod_{i=0}^{n} J(F(s_i, s_{i+1}, \ldots, s_{i+k})) \right)^{\frac{1}{n}} \right\}
\end{array} \right.
\]

where $J(F(s_i, s_{i+1}, \ldots, s_{i+k}))$ represents the Jacobian matrix of the function $F$. For a period-$p$ point the Lyapunov exponents $\lambda_1, \lambda_2 \ldots \lambda_k$ are given by

\[
(12) \quad \left\{ \begin{array}{l}
\epsilon^{s_1}, \epsilon^{s_2}, \ldots, \epsilon^{s_k} = \left( \text{eigenvalues of} \left( \prod_{i=0}^{n} J(F(s_i, s_{i+1}, \ldots, s_{i+k})) \right)^{\frac{1}{n}} \right)
\end{array} \right.
\]

We recall now that for an attracting period-$p$ cycle the Lyapunov exponents are negative; in case of a bifurcation point, at least one Lyapunov exponent is zero; for a limit cycle one Lyapunov exponent is zero and the others are negative and for a chaotic behavior the highest Lyapunov exponent is positive while the sum of all Lyapunov exponents is negative.

In order to compute the Lyapunov exponents, when system (10) displays a chaotic behavior, we use the method proposed in [1], based on the Householder QR factorization and the implementation method proposed in [8].

We have many numerical simulations and we have found many situations in which the system displays this types of attractors. In order to illustrate these, now, we give some examples. The implementation of the algorithms is made using VBA (Visual Basic for Applications) program in Excel, and the images from the figures are made using Mathematica.

**Example of chaotic attractors:**

In the case $k=2$, $c=0.2$, $b=0.95$, $\alpha = 2$, $\beta = 600$ and $(s_1, s_2) = (0.02, -0.02)$, the trajectory tends towards the chaotic attractor presented in Figure 1. The Lyapunov exponents are: $(\lambda_1, \lambda_2) = (0.4029, -1.2158)$.

**Figure 1:** Chaotic behavior for $k=2$, the space $(s_1, s_{i+1})$
In the case $k=3$, $c=2$, $b=0.95$, $\alpha = 2$, $\beta = 600$ and $(s_1, s_2, s_3) = (0.02, -0.02, -0.0794)$, the trajectory tends towards the chaotic attractor presented in Figure 2. The Lyapunov exponents are: $(\lambda_1, \lambda_2, \lambda_3) = (0.2328, -0.2468, -0.4297)$.

**Figure 2:** Chaotic behavior for $k=3$, the space $(s_1, s_{11})$

Example of limit cycle (quasiperiodic attractor)

In the case $k=2$, $c=2$, $b=0.95$, $\alpha = 0.3$, $\beta = 600$ and $(s_1, s_2) = (0.02, -0.02)$, the trajectory tends towards the limit cycle presented in Figure 3. The Lyapunov exponents are: $(\lambda_1, \lambda_2) = (0, -0.3449)$.

**Figure 3:** Limit cycle for $k=2$, the space $(s_1, s_{11})$

Attracting period- $p$ cycles

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$s_0$</th>
<th>$s_1$</th>
<th>Period</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40.5528</td>
<td>0.22816</td>
<td>-0.10586</td>
<td>1024</td>
<td>-0.0012</td>
<td>-0.6448</td>
</tr>
<tr>
<td>40.5513</td>
<td>-0.49251</td>
<td>0.23373</td>
<td>512</td>
<td>-0.0063</td>
<td>-0.6387</td>
</tr>
<tr>
<td>40.512</td>
<td>-0.49253</td>
<td>0.2338</td>
<td>128</td>
<td>-0.0177</td>
<td>-0.6257</td>
</tr>
<tr>
<td>40.515</td>
<td>-0.48868</td>
<td>0.22887</td>
<td>128</td>
<td>-0.0248</td>
<td>-0.5963</td>
</tr>
<tr>
<td>40.384</td>
<td>-0.49377</td>
<td>0.24238</td>
<td>64</td>
<td>-0.0568</td>
<td>-0.5882</td>
</tr>
<tr>
<td>40.3</td>
<td>-0.49496</td>
<td>0.23532</td>
<td>32</td>
<td>-0.0242</td>
<td>-0.6208</td>
</tr>
<tr>
<td>39.981</td>
<td>-0.50179</td>
<td>0.23606</td>
<td>16</td>
<td>-0.0062</td>
<td>-0.6398</td>
</tr>
<tr>
<td>36.52696</td>
<td>0.24804</td>
<td>8</td>
<td>-0.0920</td>
<td>-0.5538</td>
<td></td>
</tr>
<tr>
<td>32.5617</td>
<td>0.26258</td>
<td>4</td>
<td>-0.2447</td>
<td>-0.4125</td>
<td></td>
</tr>
<tr>
<td>28.3853</td>
<td>-0.58727</td>
<td>0.26821</td>
<td>4</td>
<td>-0.0002</td>
<td>-0.6963</td>
</tr>
</tbody>
</table>

**Table 2:** A sequence of period-doubling bifurcations (the first period is 4)

In Table 2, for the case $k=2$, $c=2$, $b=0.95$, $\alpha = 2$ and $(s_1, s_2) = (0.02, -0.02)$, we present some situations in which the trajectory tends towards the period-$p$ attracting cycles. Here, we have found a sequence of period-doubling bifurcations.

5 Conclusion

This study leads to the conclusion that there are similarities between the dynamics of the studied systems. These results are interesting from a mathematical viewpoint. But also, these results lead to economic interpretations.

References:


