Some Properties Referring to the Weight Selection in the
LQ Optimal Problem

CORNELIU BOTAN, FLORIN OSTAFI
Dept. of Automatic Control and Industrial Informatics
"Gh. Asachi" Technical University of Iasi
Bd. Mangeron. 53A, 700050 Iasi
ROMANIA

Abstract: The paper establishes some properties referring to the linear quadratic (LQ) optimal problem in continuous time-variant case. The established properties are useful especially for the weight selection in the adopted performance index. A connection between the problems with finite and infinite final time is indicated. Some illustrative examples are given.

Keywords: optimal control, linear quadratic, finite and infinite final time, continuous time, weight selection

1 Introduction
The present paper refers to the linear quadratic (LQ) optimal control problem for time variant systems, with finite final time and free end-point.

There are numerous books and papers devoted to this problem. We mention as references for LQ problems [1], [2], [3], [4] but many other books also contain the basic referring to this problem.

The paper establish some properties for the above mentioned problems, which refer especially to the influence of the weight matrices on the behaviour of the optimal system and on the value of the performance index. Different aspects regarding this problem are discussed, for instance in [1], [2], [5], [6]. In many cases, the problem is closely related to the solutions to Riccati differential equation (RDE), in connection with the solution to Riccati algebraic equation (RAE). There are different categories of methods for RDE solving. We mention here only that a comparison among different iterative procedures is given in [7], but the interest of the present paper refers to the analytical methods. Many methods from these categories are based on a partition of a \( 2n \times 2n \) matrix (\( n \) is the order of the system) [8]. It is also possible to find an analytical solution to RDE using \( n \times n \) transition matrices [9], [10], [11], [12], [13] and a method of this type will be applied in the sequel.

Starting from an analytical solution proposed by authors in a previous paper, a comparison between the optimal values of the performance indices in the LQ problems with finite and infinite final time is performed. Also, it is presented a property referring to the influence of the weight matrices on the performance index value and on the solution to RDE in the problem with finite final time. These properties are important because they offer designers the possibility of an adequate choice of the weight matrices in the performance index of the LQ optimal problem.

2 General aspects referring to the LQ problem

A linear time-variant system

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x^0
\]

\( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \)

and a quadratic performance index in the form

\[
J = \frac{1}{2} \int_{t_0}^{T} \left[ x^T(t)Q(t)x(t) + u^T(t)P(t)u(t) \right] dt
\]

or

\[
J_w = \frac{1}{2} \int_{t_0}^{T} \left[ x^T(t)Q(t)x(t) + u^T(t)P(t)u(t) \right] dt
\]

with \( S \geq 0, Q(t) \geq 0, P(t) > 0 \)

are considered. The matrices in the previous equations have appropriate dimensions; \( T \) denotes the transposition.

One formulates the following problems:

**P1** (LQ problem with finite final time): find \( u(t) \) which minimizes the index (2), subject the system (1).
**P2** (LQ problem with infinite final time): find \( u(t) \) which minimizes the index (3), subject the system (1), with \( A \) and \( B \) constant.

It is well-known that the solution to these problems is in the form
\[
u^*(t) = -P^{-1}(t)B^T(t)\tilde{R}(t)x(t)\tag{5}\]

For the **P1** problem, the matrix \( \tilde{R}(t) \) is symmetrical and it is solution to Riccati differential equation (RDE)
\[
\dot{\tilde{R}}(t) = \tilde{R}(t)B(t)P^{-1}(t)B^T(t)\tilde{R}(t) - \tilde{R}(t)A(t) - A^T(t)\tilde{R}(t) - Q(t),
\tag{6}
\]
with the final condition
\[
\tilde{R}(t_f) = S
\tag{7}
\]

For the **P2** problem, \( \tilde{R}(t) \) from (5) is replaced with the constant matrix \( R \), solution to Riccati algebraic equation (RAE)
\[
RNR - RA - A^T R - Q = 0
\tag{8}
\]

If the pair \((A,B)\) is stabilizable and the pair \((\sqrt{Q},A)\) is detectable, there is a unique positive defined solution to the equation (8) and the control (5) using this matrix is stabilizing. Also, \( R = \lim_{t_i \to \infty} \tilde{R}(t, t_f) \)

when \( \tilde{R}(t_f) = 0 \).

The implementation of the optimal controller for the **P1** problem implies some difficulties since the controller is time variant. Moreover, the RDE has to be solved in inverse time, starting from the final condition (7). Therefore, it is difficult to establish a recurrent procedure for on-line computing of the control vector (5).

There are different methods for obtaining the solution to RDE [8], [9], [13]. For the following needs, an analytical solution presents interest
\[
\tilde{R}(t) = \tilde{R}(t) + \Psi^T(t, t_f)(S - \bar{S})M^{-1}(t, t_f)
\tag{9}
\]
where \( \Psi \) is the transition matrix for
\[
F(t) = A(t) - B(t)P^{-1}(t)B^T(t)\tilde{R}(t),
\tag{10}
\]
which represents the matrix of the closed loop system, with the feedback matrix \( P^{-1}(t)B^T(t)\tilde{R}(t) \) and
\[
M(t, \theta) = \Psi(t, \theta) + \Omega(t, \theta)(S - \bar{S}),
\tag{11}
\]
where
\[
\Omega(t, \theta) = \int_0^{t_f} \Psi(t, \tau)B(\tau)P^{-1}(\tau)B^T(\tau)\Psi^T(\theta, \tau)d\tau
\tag{12}
\]

In (9), \( \tilde{R} \) is a particular solution to RDE which satisfies a certain final condition \( S \).

The solution (9) was proved by authors in [13] and it can be verified by straightforward computing.

Tacking into account (5) and (9), the optimal control can be expressed in the form
\[
u(t) = u_f(t) + u_c(t),
\tag{13}
\]
where \( u_f(t) \) is the feedback component and \( u_c(t) \) is a corrective one. The feedback component is
\[
u_f(t) = -P^{-1}(t)B^T(t)\tilde{R}(t)x(t)
\tag{14}
\]
and it is identical with the feedback control in the **P2** problem.

The corrective component is
\[
u_c(t) = -P^{-1}(t)B^T(t)\Psi(t_f, t)(S - \bar{S})M(t_f, t)x(t)
\tag{15}
\]

It is proved in [13] that
\[
x(t_f) = M(t_f, t_f)x(t_f),
\tag{16}
\]
or
\[
x(t_f) = M^{-1}(t_0, t_f)x(t_0)
\tag{17}
\]
so that, tacking also into account that \( \Psi^T(t_f, t) = \Psi^{-T}(t_f, t_0)\Psi^{-T}(t_0, t_f) \), one obtains
\[
u_c(t) = -P^{-1}(t)B^T(t)\Psi^{-T}(t, t_0)\Psi^{-T}(t_0, t_f)(S - \bar{S})M^{-1}(t_f, t_0)x(t_0)
\tag{18}
\]
\( \Psi^{-T} \) denotes \( (\Psi^{-1})^{-1} \).

The expression (18) is more convenient than (15) because it is avoided the on-line computing of the time-variant matrix \( M(t_f, t) \) and it remains to compute only one time variant element – the matrix \( \Psi^{-T}(t_0, t_f) \).

It has to remark that, although the above relations are rather complicated, the most part of the computing is performed off-line, in the design stage of the optimal controller. The real time computing implies only to establish a usual feedback component (14) and the corrective one (18). The last component can be easily computed since it contains only one time variant matrix, which can be recurrently established.

Moreover, a great number of the performed tests shows that a multi-time scale procedure can be applied for the computing of the optimal control vector \( u(t) \). Indeed, it has been ascertained that the sampling time for the corrective component can be chosen sensible greater than one for the feedback component with insignificant errors.

It is indicated also in [13] that the corrective component of the optimal control (18) can be written as
\[
u_c(t) = -P^{-1}(t)B^T(t)v(t),
\tag{19}
where the corrective vector \( v(t) \) satisfies the equation
\[
v(t) = -F^T(t)v(t),
\]
\( F \) is given by (10) having the solution
\[
v(t) = \Psi^T(t_f, t)v(t_f)
\]
with
\[
v(t_f) = (S - \tilde{S})x(t_f)
\]

3 Weight matrices selection

The indicated method offers the possibility for an easier implementation in comparison with classical procedures. Anyway, the application of the P1 problem is more complicated than for the P2 one. This is one of the principal reasons for the more frequent utilization of the P2 problem. In addition, one obtains by this way a controller that ensures a desired behaviour through an appropriate choice of the weight matrices \( Q \) and \( P \). On the other hand, there are cases when the interest is to reach more quickly the proximity of the state \( x = 0 \) and the performance index (2) (the P1 problem) is adopted in this situation. In the last case it is not lack of interest to improve the behaviour of the system in comparison with the one obtained in the P2 problem. This means, from the adopted criterion point of view, to obtain a smaller value for the optimal index in the P1 problem, that is, to obtain \( \Delta J < 0 \),

\[
\Delta J = J_T^* - J_w^*.
\]

An appropriate choice of the weight matrices in the criterion for the obtaining the condition (24) will be studied in the sequel. Also, another aspects referring to the effect of the weight matrices selection will be analyzed.

**Lemma 1:** If the initial state is the same, the difference \( \Delta J = J_T^* - J_w^* \)
\[
\Delta J = \frac{1}{2} x^T(t_0)(S - \tilde{S})x(t_0) + \frac{1}{2} v^T(t_f)L(t_0, t_f)v(t_f)
\]
where \( v(t_f) \) given by (22) and
\[
L(t_0, t_f) = \Psi(t_f, t_0)\Omega(t_0, t_f)
\]
\[
= \int_{t_0}^{t_f} \Psi(t_f, \tau)B(\tau)P^{-1}(\tau)B^T(\tau)\Psi(t_f, \tau)d\tau.
\]

**Proof:** Tacking into account that [1]
\[
J_T^* = \frac{1}{2} x^T(t_0)\tilde{R}(t_0)x(t_0) \text{ and } J_w^* = \frac{1}{2} x^T(t_0)\tilde{R}(t_0)x(t_0),
\]
and considering the same initial state \( x(t_0) \) in both cases, it results
\[
\Delta J = \frac{1}{2} x^T(t_0)\tilde{R}(t_0)x(t_0) - \frac{1}{2} x^T(t_0)\Omega(t_0, t_f)v(t_f)
\]
\[
\text{and } \Omega(t_0) = \Psi^T(t_0)\Omega(t_0, t_f)v(t_f) \text{ and } (30) \text{ becomes}
\]
\[
\Delta J = \frac{1}{2} x^T(t_0)(S - \tilde{S})x(t_0) + \frac{1}{2} v^T(t_f)\Omega(t_0, t_f)v(t_f)
\]
\[
\text{It results from the last expression and (26)}
\]
\[
L(t_0, t_f) = \Omega(t_0, t_f)\Psi(t_f, t_0) = \int_{t_0}^{t_f} \Psi(t_f, \tau)B(\tau)P^{-1}(\tau)B^T(\tau)\Psi(t_f, \tau)d\tau
\]
\[
= \Psi(t_f, t_0)\Psi(t_f, \tau)B(\tau)P^{-1}(\tau)B^T(\tau)\Psi(t_f, \tau)d\tau.
\]
which shows that \( L(t_0, t_f) \) is a symmetric matrix.

Introducing (32) in (31), it results (25).

**Lemma 2:** If \( (A, B) \) is completely controllable, then the matrix \( L(t_0, t_f) \) given by (26) is positive definite.

**Proof:** The sign of the symmetric matrix \( L(t_0, t_f) \) can be established based on the sign of the quadratic function
\[
z^T L(t_0, t_f)z, \ z \in \mathbb{R}^n, \text{ respectively of the function}
\]
\[
f = z^T \int_{t_0}^{t_f} \Psi(t_f, \tau)B(\tau)P^{-1}(\tau)B^T(\tau)\Psi(t_f, \tau)d\tau z
\]
If we denote \( h(\tau) = B^T(\tau)\Psi^T(\tau)z, \) thus
\[
h^T(\tau)P^{-1}(\tau)h(\tau) \geq 0,
\]
since \( P(\tau) > 0 \). The equality in (33) is possible only if
\[
h(\tau) = B^T(\tau)\Psi^T(\tau)z(\tau) = 0.
\]
If the pair \((A, B)\) is completely controllable, then the pair \((A, F)\), with \(F\) given by (10), is completely controllable [14]. In this case, (34) holds only for \(\varepsilon = 0\). Therefore, for \(\varepsilon \neq 0\), the inequality (33) becomes a strict one and thus \(f > 0\) and \(L(t_0, t_f) > 0\).

**Theorem 1:** If the pair \((A, B)\) is completely controllable, it is necessary to have \(S < R\) in order to obtain \(\Delta f > 0\).

*Proof:* The second term in (26) is positive definite according to Lemma 1. Thus, in order to obtain \(\Delta f > 0\), it is necessary (but not sufficient) to have \(S < R\).

**Remark 1:** If \(S > R\), both terms in (26) are positive definite, so that \(\Delta f > 0\). If \(S = R\), it results from (9) that the solutions to RDE and to RAE coincides, so that \(\Delta f = 0\).

**Remark 2:** The matrix \(S\) has in many cases a diagonal form. The previous results show that it is possible to obtain \(\Delta f < 0\) if its diagonal elements are chosen rather small. On the other hand, this choice can lead to rather big final norm \(\|x(t_f)\|\). The above relations allow the computing of \(x(t_f)\), \(\Delta f\) and to assign them proper value, depending on \(S\).

Let us now consider the two optimization problems with finite final time \(\textbf{P1.1}\) and \(\textbf{P1.2}\) on the interval \([t_0, t_f]\) for the system (1) and the criteria (2), denoted with \(J_{T_1}\) and \(J_{T_2}\) and described by the matrices \(S_1, Q_1, P_1\) and \(S_2, Q_2, P_2\), respectively. We denote with \(R_1(t)\) and \(R_2(t)\) the solutions to RDE for the corresponding problem.

The relations between the weight matrices, the optimal value of the performance indexes and the solutions to RDE’s are indicated by the following theorem (which extends a result from [5]).

**Theorem 2:** If \(S_1 \geq S_2\), \(Q_1 \geq Q_2\), \(P_1 \geq P_2\), then \(R_1(t) \geq R_2(t), t \in [t_0, t_f]\) and \(J_{T_1}^* \geq J_{T_2}^*\).

*Proof:* Let \(x^*(t), u^*(t)\) be the optimal trajectory and control for the \(\textbf{P1.1}\) problem. Starting from the non-optimal value of \(J_{T_2}\) for \(u^*(t)\) and \(x^*(t)\), one can successively write

\[
J_{T_2} = \frac{1}{2} \int_{t_0}^{t_f} x^T(t_s)S_2x^*(t_s) + \frac{1}{2} \int_{t_0}^{t_f} \left[ x^T(t_s)Q_2x^*(t_s) + u^T(t_s)P_2u^*(t_s) \right] dt = J_{T_1}^* + \frac{1}{2} \int_{t_0}^{t_f} x^T(t_s)(S_2 - S_1)x^*(t_s) + \frac{1}{2} \int_{t_0}^{t_f} x^T(t_s)Q_1x^*(t_s) + u^T(t_s)(P_2 - P_1)u^*(t_s) dt \leq J_{T_1}^*
\]

The last inequality is written in accordance of the hypothesis of the theorem.

It results from the above relations \(J_{T_2}^* \leq J_{T_2} \leq J_{T_1}^*\), or \(x^T(t_0)R_2(t_0)x(t_0) \leq x^T(t_0)R_1(t_0)x(t_0)\). The last inequalities are true for any \(x(t_0)\) and any \(t_0 \leq t_f\), so that \(J_{T_2} \leq J_{T_1}\), \(R_2(t) \leq R_1(t), \forall t \leq t_f\).

A similar theorem holds for the two \(\textbf{P2}\) type problems, with weight matrices \(Q_1, P_1\) and \(Q_2, P_2\), minimal performance indices \(J_{T_1}^*, J_{T_2}^*\) and solutions to Riccati equations \(R_1\) and \(R_2\), respectively:

**Theorem 3:** If \(Q_1 \geq Q_2\), \(P_1 \geq P_2\), then \(R_1 \geq R_2\) and \(J_{T_1}^* \geq J_{T_2}^*\).

*Proof:* is similar as for Theorem 3.

### 4 Illustrative examples

The above mentioned properties were verified on different examples for the system equation and performance index. Some results are presented in the sequel for a two order system, because this case is more suggestive. In all cases presented below, the equation (1) of the system has the matrices

\[
A = \begin{bmatrix} -0.04 & 20 \\ -4 & -19 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 7 \end{bmatrix}
\]

and the sampling period is \(T=0.002s\). The figures present the variations for state variables \(x_1(t), x_2(t)\) and control variable \(u(t)\). The corresponding optimal value of the performance index is indicated on each figure. The weight matrices of the performance indices are in the form: \(S = \text{diag}(s_1, s_2)\), \(Q = \text{diag}(q_1, q_2)\), \(P = p\). These values are mentioned in each case in the caption of the figure.

Fig. 1 presents the behaviour of the optimal system with the weight matrices indicated on the figure.

The property mentioned above regarding the sampling period of the corrective component of the optimal control is illustrated on the Fig.2. A sampling period 30 times greater than for the feedback component was used. The shape of the control variable has significant modifications, but its mean value is practically the same and this fact explains the very small changes in the form of state variables.
Fig. 1

\[ J_T^* = 57.4 \]

\[ s_1=10, s_2=1, q_1=1, q_2=3.1, p=1 \]

Fig. 2

\[ J_T^* = 60.57 \]

\[ s_1=10, s_2=1, q_1=1, q_2=3, p=1, T_c=30^*T \]

Fig. 3

\[ J_T^* = 41.98 \]

\[ q_1=1, q_2=3, p=1 \]

Fig. 4

\[ J_T^* = 39.4 \]

\[ s_1=0.01, s_2=0, q_1=1, q_2=3, p=1 \]

Fig. 5

\[ J_T^* = 49.95 \]

\[ s_1=1, s_2=0, q_1=1, q_2=3, p=1 \]

Fig. 6

\[ J_T^* = 37.2 \]

\[ s_1=10, s_2=1, q_1=0.5, q_2=1, p=1 \]
The Fig. 3 presents the behaviour of the optimal system with infinite final time (the P2 problem). The Fig. 4, and 5 are illustrative for the Theorem 1, being indicated the case S>R (Fig. 4) and S>R (Fig. 5) in comparison with Fig. 3. In all these cases,

\[ R = \begin{bmatrix} 0.1346 & 0.0822 \\ 0.0822 & 0.1402 \end{bmatrix} \]

The last figures illustrate the Theorem 2 in comparison with the behaviour presented in the first figure, for \( Q_2 > Q_1 \) (Fig. 6) and for \( P_2 > P_1 \) (Fig. 7).

4 Conclusions

The optimal control vector in the LQ problem can be expressed as a sum of a feedback component and of a corrective one, depending on the initial state.

The corrective component can be computed using a sampling period considerable greater than the feedback one, so that the computing effort for such a controller is not much bigger than that corresponding to a usual state feedback controller.

It is recommended to select the weight matrix \( S \) in the performance index \( J_f \) (for the problem with finite final time) so that \( S \leq R \) (the solution to RAE) in order to obtain a smaller value for the optimal index than in the problem with infinite final time (\( J_\infty \)).

The influences of the S, Q, P matrices on the minimal value for \( J_f \) or \( J_\infty \) or on the solution to RDE or RAE are also indicated.

References: