On the solitons and nonlinear wave equations

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Abstract: -. The paper is focused on the solitons and nonlinear equations for an uniaxial deformation problem. The aim is to determine a parametrical representation for a class of constitutive laws for nonhomogeneous media for which the motion equations attached to a material system, is associated to a pseudospherical surface (with negative Gaussian curvature K). A subclass of these constitutive laws can be associated to a Tzitzeica surface, for which the ratio K/d^4 (d is the distance from the origin to the tangent plane at an arbitrary point), is constant. A genetic algorithm is performed to study three inverse problems associated to some experimental results.

Key-Words: soliton theory, uniaxial deformation, nonhomogeneours material, constitutive laws, Tzitzeica surfaces.

1 Introduction

The discovery of the physical soliton is attributed to John Scott Russell. In 1834, Russell was observing a boat being drawn along a narrow channel by a pair of horses. He followed it on horseback and observed an amazing phenomenon: when the boat suddenly stopped, a bow wave detached from the boat and rolled forward with great velocity, having the shape of a large solitary elevation, with a rounded well-defined heap of water. The solitary wave continued its motion along the channel without change of form or velocity [1]. The wave of translation was regarded as a curiosity until the 1960s, when scientists began to use computers to study nonlinear wave propagation. The discovery of mathematical solutions started with the analysis of nonlinear partial differential equations, such as the work of Boussinesq and Rayleigh, independently, in the 1870s. Boussinesq and Rayleigh explained theoretically the Russell observation and later reproduction in a laboratory experiment. Korteweg and de Vries derived in 1895 the equation for water waves in shallow channels, and confirmed the existence of solitons [2].

The study of affine differential geometry was initiated by Gheorghe Tzitzeica (1873–1939) in 1907 by studying a particular class of hyperbolic surfaces. Tzitzeica proved that the surfaces for which the ratio K/d^4 (K is the Gaussian curvature and d, the distance from the origin to the tangent plane at an arbitrary point) is constant, are invariants under the group of centroaffine transformations. The Tzitzeica property proves to be invariant under affine transformations, and his surfaces are called Tzitzeica surfaces by Gheorghiu, or affine spheres by Blaschke [3], [4].

A privileged surface related to the certain nonlinear equations that admit solitonic solutions, is the Tzitzeica surface (1910). Developments in the geometry of such surface gave a gradual clarification of predictable properties in natural phenomena [5]-[7].

2 Pseudospherical reduction of the problem

Consider the 1D problem of uniaxial deformation of a nonhomogeneous rod. We present in this section the pseudospherical reduction of the problem in the spirit of [8], [9]. The governing equations in a Lagrangian system of coordinates (X,t) are written as

$$\varepsilon_t = v_X, \quad \rho_0 v_t = \sigma_X. \tag{1}$$

The constitutive law is given by

$$\sigma=\sigma(\varepsilon,X), \tag{2}$$

where σ and ρ are the uniaxial stress and respectively, the density of the material, $\varepsilon=\frac{\rho_0}{\rho}-1$ is the stretch, ρ_0 is the density of the material in the underformed state, and $v(X,t)$ is the material velocity. In terms of the Eulerian coordinates $x=x(X,t)$, we have $\mathrm{d}x=(\varepsilon+1)\mathrm{d}X=v\mathrm{d}t$, so that

$$\rho_0\mathrm{d}X=\rho\mathrm{d}x-\rho v\mathrm{d}t. \tag{3}$$

In (3, X corresponds to the particle function ψ of the Martin formulation. The independent variables are chosen to be σ and ψ, and we suppose $\rho_0=1$. In this case we obtain the Monge–Ampère equation

$$\xi_{\sigma\sigma}\xi_{\psi\psi}-\xi_{\sigma\psi}^2=\varepsilon_\sigma, \tag{4}$$

where $t=\xi_\sigma$, $v=\xi_\psi$, $\mathrm{d}x=\xi_\psi\xi_{\sigma\sigma}+(\xi_\psi\xi_{\sigma\psi}+\varepsilon)\mathrm{d}\psi$. If a solution $\xi(\sigma,\psi)$ of this equation is specified, then the particle trajectories are calculated from

$$x=\int[\xi_\psi\xi_{\sigma\sigma}+(\xi_\psi\xi_{\sigma\psi}+\varepsilon)\mathrm{d}\psi],\ t=\xi_p, \tag{5}$$

in terms of σ, for $\psi=\text{const.}$ By solving (5) the solution $\sigma(\psi,t)$ is obtained, and the original solution of (1), (2) is parametrically determined in terms of the Lagrangian variables $x=x(\psi,t)$, $v=v(\psi,t)$, $\sigma=\sigma(\psi,t)$. To made the geometric connection to this problem, let us consider a surface Σ in R^3 written the Monge parametrisation

$$r=xe_1+ye_2+z(x,y)e_3, \tag{6}$$

where $r=r(x,y,z)$ the position vector of a point P on the surface. The first and second fundamental forms are defined as

$$I=E\mathrm{d}x^2+2F\mathrm{d}x\mathrm{d}y+G\mathrm{d}y^2=(1+z_x^2)\mathrm{d}x^2$$
$$+2z_xz_y\mathrm{d}x\mathrm{d}y+(1+z_y^2)\mathrm{d}y^2,$$
$$II=e\mathrm{d}x^2+2f\mathrm{d}x\mathrm{d}y+g\mathrm{d}y^2$$
$$=\frac{1}{\sqrt{1+z_x^2+z_y^2}}(z_{xx}\mathrm{d}x^2+2z_{xy}\mathrm{d}x\mathrm{d}y+z_{yy}\mathrm{d}y^2). \tag{7}$$

The Gaussian curvature of Σ is

$$\mathrm{K}=\frac{eg-f^2}{EG-F^2}=-\frac{z_{xx}z_{yy}-z_{xy}^2}{(1+z_x^2+z_y^2)^2}.$$

If Σ is a hyperbolic surface, then total curvature is negative and the asymptotic lines on Σ may be taken as parametric curves.

By introducing the same independent variables as before, σ and ψ, $\sigma=z_x$, $\psi=z_y$, and the dependent variable ξ, $\xi_\sigma=x$, $\xi_\psi=y$, we have

$$\xi_{\sigma\sigma}=\frac{z_{yy}}{z_{xx}z_{yy}-z_{xy}^2},\ \xi_{\psi\psi}=\frac{z_{xx}}{z_{xx}z_{yy}-z_{xy}^2},$$
$$\xi_{\sigma\psi}=\frac{z_{xy}}{z_{xx}z_{yy}-z_{xy}^2}.$$

The Gaussian curvature (7) yields

$$K=\frac{1}{(1+\sigma^2+\psi^2)^2(\xi_{\sigma\sigma}\xi_{\psi\psi}-\xi_{\sigma\psi}^2)}.$$

The Gaussian curvature may be set into correspondence with the Martin's Monge–Ampère equation (4) by $\varepsilon_\sigma=\frac{1}{\mathrm{K}(1+\sigma^2+\psi^2)^2}$, and

$$\mathrm{K}=\frac{A^2}{(1+\sigma^2+X^2)^2}, \tag{8}$$

where $A^2=\frac{\partial\sigma}{\partial\varepsilon|_X}$, with A the Lagrangian wave velocity. The surface Σ is restricted to be pseudospherical, that is $K=-\frac{1}{a^2}$, $a=\text{const.}$

In this case the relation (8) gives

$$\frac{\partial^2\sigma}{\partial\varepsilon^2|_X}=\frac{2}{a^2}(1+\sigma^2+X^2)\sigma\frac{\partial\sigma}{\partial\varepsilon}|_X>0,\ \ \sigma>0. \tag{9}$$

By integrating (9) we have

$$\varepsilon=\frac{a^2}{2(1+X^2)^{3/2}}F+\alpha(X),$$
$$F=\left[\arctan\left(\frac{\sigma}{1+X^2}\right)+\frac{\sigma\sqrt{1+X^2}}{1+\sigma^2+X^2}\right], \tag{10}$$

with $\alpha(X)$ arbitrary. For $\sigma|_{\varepsilon=0}=0$, it results $\alpha(X)$=0. The relation (10) represents a class of constitutive laws for which (1) are associated to a pseudospherical surface Σ.

Starting from (10) we can obtain several constitutive laws for specified practical problems.

In particular, let us introduce into (9) the stress representation

$$\sigma=\sqrt{1+X^2}\tan A,\ \ A=\left[\frac{\sqrt{1+X^2}}{a}(c-c_0)\right]. \tag{11}$$

In this case we obtain

$$\varepsilon = \frac{a^2}{2(1+X^2)}\left[\frac{c-c_0}{a} + \frac{1}{\sqrt{1+X^2}}\right]\sin(2A). \quad (12)$$

Thus, relations (11) and (12) represent a parametric representation for the constitutive laws $\sigma = \sigma(\varepsilon, X)$, for which the equations (1) are associated to a pseudospherical surface Σ. These equations lead to $\sigma_{XX} = \varepsilon_{tt}$. We have

$$\sigma_{XX} = \left[\frac{a^2}{(1+\sigma^2+X^2)^2}\sigma_t\right]_t. \quad (13)$$

The equation (13) has a solitonic behavior and admits soliton solutions. These solutions known as solitons have the form of localized functions that conserve their properties even after interaction among them, and then act somewhat like particles.

3 Tzitzeica surfaces

Let $D \subset R^2$ be an open set and consider a surface Σ in R^3 defined by the position vector $r(u,v)$

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$
$$(u,v) \in D.$$

The vector $r(u,v)$, which satisfies the condition

$$(r, r_u, r_v) \neq 0, \quad (14)$$

is the solution of the second-order partial differential equations system that defines a surface

$$r_{uu} = ar_u + br_v + cr,\ r_{uv} = a'r_u + b'r_v + c'r,$$
$$r_{vv} = a''r_u + b''r_v + c''r,$$

which is completely integrable, that is

$$(r_{uu})_v = (r_{uv})_u,\ (r_{uv})_v = (r_{vv})_u, \quad (15)$$

where $a, a', a''...$ are the centroaffine invariant functions of u and v. For $c = c'' = 0$, the surface Σ is related to the asymptotic lines. If Σ is a surface related to the asymptotic lines. The ration $I = \frac{K}{d^4}$ is a constant if and only if $a' = b' = 0$. Therefore, the Tzitzeica surfaces are defined by the system of equations

$$r_{uu} = ar_u + br_v,\ r_{uv} = hr,\ r_{vv} = a''r_u + b''r_v, \quad (16)$$

where $c' = h$. The integrability conditions (15) become

$$ah = h_u,\ a_v = ba'' + h,\ b_v + bb'' = 0,$$
$$b''h = h_v,\ a'' + aa'' = 0,\ b''_u + a''b = h. \quad (17)$$

If h satisfies the Liouville–Tzitzeica equation $(\ln h)_{uv} = h$, the Tzitzeica surfaces which are not ruled surfaces are defined by

$$r_{uu} = \frac{h_u}{h}r_u + \frac{\varphi(u)}{h}r_v,\ r_{uv} = hr,\ r_{vv} = \frac{h_v}{h}r_v. \quad (18)$$

If h satisfies the Tzitzeica equation $(\ln h)_{uv} = h - \frac{1}{h^2}$, the Tzitzeica surfaces which are not ruled surfaces are defined by

$$r_{uu} = \frac{h_u}{h}r_u + \frac{1}{h}r_v,\ r_{uv} = hr,\ r_{vv} = \frac{1}{h}r_u + \frac{h_v}{h}r_v. \quad (19)$$

The system (3) can be written in the form

$$\theta_{uu} = a\theta_u + b\theta_v,\ \theta_{uv} = h\theta,\ \theta_{vv} = a''\theta_u + b''\theta_v, \quad (20)$$

with the condition that the three independent solutions $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ of (20) and (17) define a Tzitzeica surface. An equivalent form of (16) is given

$$x_{uu} = ax_u + bx_v,\ x_{uv} = hx,\ x_{vv} = a''x_u + b''x_v,$$
$$y_{uu} = ay_u + by_v,\ y_{uv} = hy,\ y_{vv} = a''y_u + b''y_v,$$
$$z_{uu} = az_u + bz_v,\ z_{uv} = hz,\ z_{vv} = a''z_u + b''z_v,$$

with the conditions (14) and (17). This form is useful for studying the symmetries of the system (16).

If Σ is a ruled Tzitzeica surface given by (18), the completely integrable conditions (17) turn in

$$a = \frac{h_u}{h},\ b = \frac{\varphi(u)}{h},\ a'' = 0,\ b'' = \frac{h_v}{h}, \quad (21)$$

with h a solution of the Liouville–Tzitzeica equation $(\ln h)_{uv} = h$.

$$\zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0,\ \zeta^3 = \frac{k}{\varphi},$$
$$hh_{uv} - h_u h_v h^3. \quad (22)$$

Writing $\zeta = \frac{1}{U'}$, $\eta = -\frac{1}{V'}$, where $U = U(u)$ and $V = V(v)$, then the first equation of (22) is

$$h = U'V'\mu(U+V). \quad (23)$$

Substituting (23) in the last equation (22) we have the equation

$$\mu\mu'' - \mu'^2 = \mu^3. \quad (24)$$

The general solution of (24) is

$$\mu(U+V)=\begin{cases}\dfrac{2}{(U+V+C)^2}, & k=0,\\ \dfrac{l^2}{2\cos^2[0.5l(U+V)+C]}, & k=-l^2,\\ \dfrac{l^2}{2\sinh^2[0.5l(U+V)+C]}, & k=l^2,\ l>0.\end{cases} \quad (25)$$

From (23) and the change of the functions $\tilde{U}=F(U)$, $\tilde{V}=G(V)$, we have

$$\tilde{U}=U+C,\ \tilde{V}=V,\ \text{for } k=0,$$

$$\tilde{U}=\tanh\frac{l}{2}(U+C),\ \tilde{V}=\tanh\frac{l}{2}V,\ \text{for } k=l^2,$$

$$\tilde{U}=\text{cotan}\frac{l}{2}(U+C),\ \tilde{V}=\tan\frac{l}{2}V,\ \text{for } k=-l^2. \quad (26)$$

Therefore, the general solution of the Liouville–Tzitzeica equation is

$$h(u,v)=\frac{2\tilde{U}'\tilde{V}'}{(\tilde{U}+\tilde{V})^2}. \quad (27)$$

This solution is expressed in terms of solitons for $k=-l^2$, and

$$\tilde{U}'=\frac{l}{2}\text{sech}^2\frac{l}{2}(U+C),\ \tilde{V}'=\frac{l}{2}\text{sech}^2\frac{l}{2}V.$$

4 The constitutive laws. Results

Materials we are going to model are Berea sandstone, Kayenta sandstone [10] and discontinuous random Polyethylene fiber reinforced cement [11]. For these materials and also, for metals, sintered ceramic and cracked solids, the scientists have discovered essential effects given by nonlinearities such as the slow dynamics, a creep-like behavior induced by mechanical excitation at small amplitudes (strains $10^{-5}-10^{-8}$). Slow dynamics is manifests by a significant and persistent alteration in the material dissipation and modulus after mechanical disturbance, a memory of the disturbed strain state. The modulus and wave dissipation progressively recover to their original values as log (time) after 10^3-10^4 seconds. Slow dynamics is destined to become a sensitive probe of the micromechanics of the system [12]-[14].These materials (fig.1) are aggregate of grains which act as rigid vibrating units, while the contacts between them – the bond system – constitute a set of interfaces that control the behaviour of the material. The interfaces are mesoscopic, with a typical size of $1\,\mu\text{m}$ [15].

This class of materials includes pearlitic steel, fiber-reinforced metal matrix composites, cement, concrete, ceramics, rocks, sand, soil etc. .

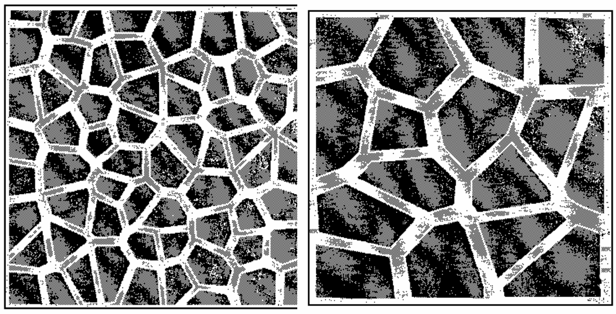

Fig.1. Schematic picture of a granular material.

In this paper, we consider three materials subjected to a standard loading path in the conventional triaxial configuration:1. Berea sandstone [10], Kayenta sandstone [10], and discontinuous random Polyethylene fiber reinforced cement [11]. We assume that the unknown three parameters $p=\{a,c,c_0\}$ are discretised into discrete values with the step width $\Delta a,\Delta c,\Delta c_0$. The set of parameters for an arbitrary problem are $p=\{a_{,i},c_{,j},c_{0,k}\}$ is expressed as the combination number $M_{ij}=(i-1)JK+(j-1)K+k$, where I,J,K are total number of discretised values for each parameter p [16]. This number is counted from the first set of parameter $p=\{a_{,1},c_{,1},c_{0,1}\}$ .We consider a square sum of differences between the measured stress-strain results for some selected cases of uniaxial tension problem for non-homogeneous materials $W=\sum_{m=1}^{M}(\sigma_i^m-\bar{\sigma}_i^m)^2$, where $\bar{\sigma}_i^m$ denotes the measured stress at point m on the strain-stress diagram $\sigma-\varepsilon$, and σ_i^m denotes the computed strain-stress $\sigma-\varepsilon$ at the same poin t m. M is the number of points from the measured uniaxial diagram $\sigma-\varepsilon$. We define fitness $\tilde{F}$ as a reciprocal number of the function $\tilde{F}=\frac{W_0}{W}$ where $W_0=\sum_{m=1}^{M}(\bar{\sigma}_i^m)^2$. The convergence criterion is given by the non-dimensional expression $Z=\frac{1}{2}\log_{10}\frac{W}{W_0}$.

For all considered examples, the number of populations is 25, ratio of reproduction 1, number of multi-point crossovers 1, probability of mutation 0.25 and the maximum number of generations 300.

The constitutive law for Berea sandstone is illustrated by our theory in fig.2. For $a = 0.2$, the stress marks a peak representing the failure. An increase $a = 0.5$ shifts the macroscopic failure mode from brittle to ductile. At $a = 1.7$ the strain hardening persists up. This behavior is qualitatively the same to the experimental results reported in [10]. The failure means to be a function of a, for specified $c = 2.3$ and $c_0 = 1.4$. The inelastic behavior is macroscopically ductile, and the conventional approach is to pick its failure strength to be the stress level at an arbitrary axial strain. The strength so determined is compiled as a function a, for specified c and c_0. In [10] the effective pressure plays the same role as the parameter.

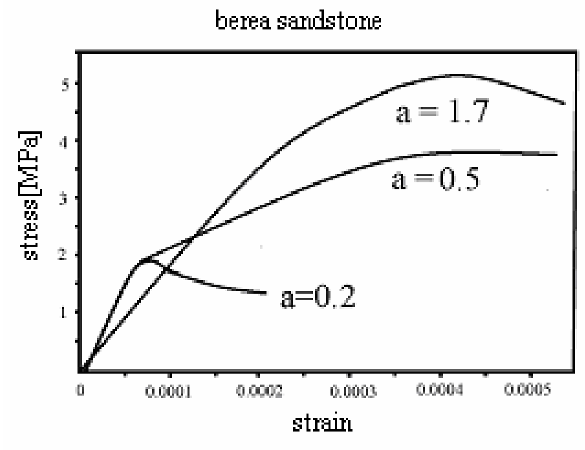

Fig. 2. The stress-strain law for Berea sandstone

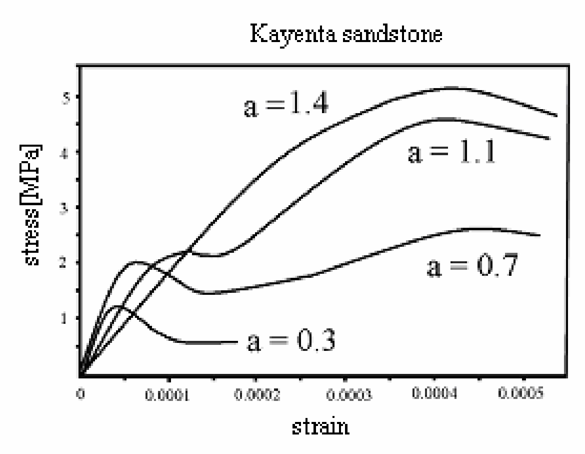

Fig. 3.The stress-strain law for Kayenta sandstone.

The constitutive law for Kayenta sandstone is illustrated in fig. 3. For $a = 0.3$, the stress marks a failure peak. For $a = 0.7$ and $a = 1.1$, the macroscopic failure mode is changed from brittle to ductile. At $a = 1.4$ the strain hardening is observed. This behavior is qualitatively the same to the experimental results reported in [10]. Also, the failure means to be a function of the parameter a, for specified $c = 1.9$ and $c_0 = 0.9$.

The constitutive law for discontinuous random polyethylene fiber reinforced cement is illustrated in fig. 4, for two values for the fiber volume fraction (0.1 % and 1%).

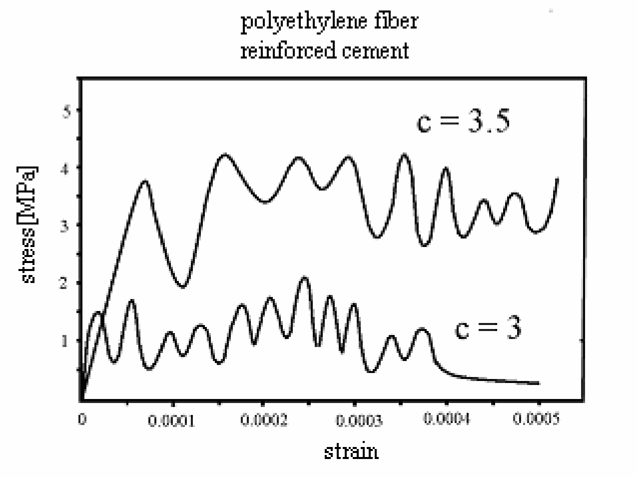

Fig. 4. The stress-strain law for polyethylene fiber reinforced cement.

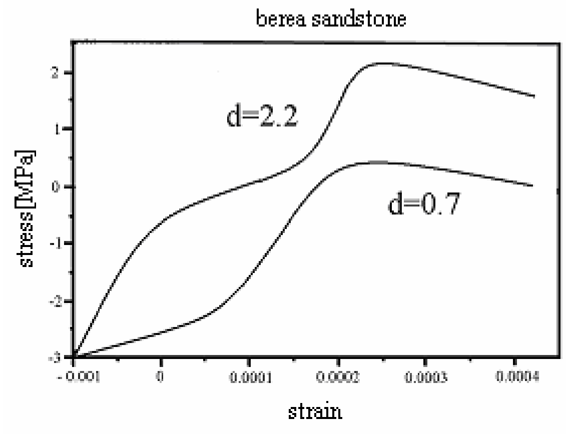

Fig. 5. The stress-strain law for Berea sandstone.

As shown in the figure, the composite with 0.1 % fiber volume fraction ($c = 3$) have a catastrophic failure. For 1% fiber volume fraction ($c = 3.5$) the figure shows a significant ductility and accentuated fracture toughness. The others parameters are $a = 0.7$ and $c_0 = 0.2$. The results reveal the importance of the parameter c in describing the stress-strain curve for fiber reinforced composites. We can say that this parameter is proportional to the fiber volume fraction for specified a and c_0. An interesting case of a constitutive law, associated to a Tzitzeica surface (K / d^4 is constant), is considered for the same data, as in the previous case, for Berea sandstone. We obtain the diagrams illustrated in fig.5, that show a significant ductility and fracture toughness.

5 Conclusions

The goal of the paper is to determine a parametrical representation for a class of constitutive laws for which the motion equations attached to a material system is associated to a pseudospherical surface. The uniaxial deformation problem for non-homogeneous materials is discussed via the pseudospherical reduction technique.

A genetic algorithm is performed to study four inverse problems associated to experimental results. For Berea sandstone and Kayenta sandstone, the strength of material can be determined as a function a, for specified c and c_0. The relation of a to the Gaussian curvature is $K = -\frac{1}{a^2}$. So, we can conclude that if the motion equations can be associated to a pseudospherical surface Σ, of Gaussian curvature K, the strength of material can be described as a function of K.

For discontinuous random polyethylene fiber reinforced cement, the results yields to the conclusion that the parameter c is important in describing the stress-strain curve for fiber reinforced composites. We can conclude that this parameter may be proportional to the fiber volume fraction for specified a and c_0.

A subclass of the constitutive laws is associated to a Tzitzeica surface, for which the ratio K/d^4 (d is the distance from the origin to the tangent plane at an arbitrary point), is constant. For Berea sandstone, the parameter a is related to d, and it is important in describing the ductility and fracture toughness properties.

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