State-Space Control Model of Tokamak Reactors

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Abstract: - This paper deals with the modeling of tokamak nuclear fusion reactors. In order to control the creation of unstable modes in fusion processes, it is necessary to derive numerical models that are suitable for control strategies. This model addresses flux and energy conservation issues and the mechanisms behind the creation of uncontrollable modes are discussed. The dynamics of the system are given by the energy functions which solve for the currents in the structure, the plasma current and plasma position. Thus, the equations for the state variables will be derived based on the Hamiltonian equation of motion. In order to solve numerically, this model will be linearised around an operation point by taking a Newton-Raphson step. Besides, the system output will be completed by considering one equation for the flux and another for the poloidal field. Finally, the resulting low order linear model is modified so as to obtain a standard state space model verified against measuring both time and frequency domain responses.

Key-Words: - Plasma Control, Tokamak Model, Grad-Shafranov, Newton-Raphson, Linear Control

1 Introduction

This paper deals with the modeling and control of tokamak nuclear fusion reactors (see [20-21] and [1]). Nuclear fusion is an attractive source of power since the fuels are abundant, there are no long term waste management issues and it is inherently safe [12]. The most common reaction occurs when the hydrogen isotopes deuterium and tritium fuse to release helium, a neutron and energy. The heat generated when the neutrons are slowed down by a blanket, produces electricity while the lithium of the blanket react with the neutrons to form tritium. Thus, lithium and deuterium are the primary fuels.

Current ramping is necessary to reach temperatures high enough to overcome the Coulomb barrier. At these temperatures the atomic nuclei are dissociated from their electrons resulting in a mixture called plasma. Currently, the most successful fusion reactors are toroidal devices called tokamaks that use magnetic fields to confine similarly shaped hot plasma [5]. The plasma is confined using electromagnetic forces generated by external magnetic fields: The toroidal field is produced by a set of poloidal coils and the smaller poloidal field from the induced plasma current.

During large plasma disturbances, such as sawteeth, ELMs, VDEs (see [6] and [17-18]) and minor disruptions, voltage saturation can occur and as a consequence the vertical position can be lost damaging the wall of the vessel (see [10] and [13]). Besides, superconducting plasma provokes the creation of uncontrollable modes because when the magnetic transverse field changes, the magnet generates two types of heat loss, the so called coupling loss and the so called hysteresis loss, grouped together as AC losses, which heat up the superconducting material. Once the superconductivity is lost, the electric currents in the coils produce an enormous heat lost due to the ohmic resistivity [5] and [2]. The objective is to design a controller that improves the stability and performance properties of a tokamak [7]. However, the performance of the control is limited by the open-loop plant model used in the design phase and it is therefore necessary to derive a suitable tokamak physical model (see [15] and [4]). In order to understand the tokamak physics and engineering linear and non-linear models will be studied (see [8-9]). Firstly, a non-linear model will be obtained simplifying the Grad-Shafranov equilibrium equation. Based on these equations, a lump parameter model will be derived and linearised about any prescribed equilibrium state.

This model of tokamak system relates to plasma position and shape control (see [14] and [25]), flux and energy conservation are treated explicitly, using a Lagrangian approach, whereas the adiabatic approximation is a natural result from massless plasma.

2 Tokamak Equilibrium

The Grad-Shafranov equation describes the shape and current profile of toroidally symmetric plasma in an externally applied magnetic field (see [26]). The equilibrium equations are found by considering the force balance between the thermodynamic pressure and magnetic force,

$$\mathbf{j} \times \mathbf{B} = \nabla p , \quad (1)$$

subject to toroidal symmetry, where $\mathbf{j}$ is the current density, $\mathbf{B}$ is the magnetic field and p is the pressure. Thus, the magnetic surfaces are surfaces of constant pressure and the lines of current lie on the magnetic surfaces.

A rigid plasma displacement model will be considered, i.e., the current density profile is assumed to be unchanged under spatial translation although each element might change proportionally to the changes on the total plasma current. The plasma is considered as a unit which is allowed to move either vertically or radially and the vessel eddy currents will use an eigenmode representation. A crude single filament model of the plasma vertical position response is adequate for the control problem. However, when a subsystem is reduced using the vessel eigenmode model reduction, there are no guaranteed stability properties.

The model is derived from the plasma and circuit equations, perturbing the vertical and radial force balance equations. Changes in current profile are treated as disturbances (explicit treatment in [22-23]). In this sense, the model is based on the one presented by A. S. Sharma in [17] and [24] and it has been developed with the aim of extending the control techniques used and the stability region of the resulting closed-loop system.

3 Tokamak Modeling

The main components of the model are the poloidal field coils, driven by external voltage sources, the passive structure, with electromagnetically induced eddy currents, and the plasma (see also [3], [16] and [27]). A cylindrical coordinate system is used (R, z, ϕ) with the following simplifying assumptions

- The system is symmetric around the z-axis.
- Any poloidal currents are ignored.
- The tokamak structure will be represented by a finite set of toroidal circuits fixed in space with finite resistance. The toroidal currents may vary in time.
- The plasma will be represented by a finite number of filaments free to move axisymmetrically with constant finite mass and resistance. The currents may vary in time.

3.1 The energy functions

The generalised coordinates are the plasma elements currents, structure currents and variation of the position of the plasma current elements

$$\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{I_e} \\ \mathbf{I_s} \\ \dot{\mathbf{r}} \end{bmatrix}, \text{ where } \mathbf{r} = \begin{bmatrix} \mathbf{R} \\ \mathbf{z} \end{bmatrix}.$$

We introduce the inductance matrix

$$\mathbf{T}(\mathbf{q}) = \begin{bmatrix} \mathbf{L_e} & \mathbf{M_{es}} & \mathbf{0} \\ \mathbf{M'_{es}} & \mathbf{L_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{m_e} \end{bmatrix} \quad (2)$$

with the self and mutual inductance matrices and a constant diagonal mass matrix.

The input vector contains the vector with the effective voltages applied to each plasma element and a vector of externally applied poloidal field coil voltages

$$\mathbf{U} = \begin{bmatrix} \mathbf{V_e} \\ \mathbf{V_s} \\ \mathbf{0} \end{bmatrix} \quad (3)$$

In turn, the resistance matrix is defined as

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{\Omega_e} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (4)$$

The kinetic energy is given by

$$T = \frac{1}{2} \dot{\mathbf{q}}' \mathbf{T} \dot{\mathbf{q}} + W \quad (5)$$

where the plasma internal energy is $W = \mathbf{q}' \mathbf{E} \dot{\mathbf{q}}^2 / 2$ and $\mathbf{E}$ is a constant matrix and the generalised potential is

$$V = -\mathbf{q}'\mathbf{U} + \frac{1}{2} \frac{d\left(\int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} \int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} d\dot{\mathbf{q}}' \mathbf{\Omega} d\dot{\mathbf{q}} \right)}{dt} \quad (6)$$

the integral denotes the total energy dissipated from time t_0 to time t.

The Lagrangian is the difference between the kinetic and potential energy $L = T - V$

$$L = \frac{1}{2} \dot{\mathbf{q}}' \mathbf{T} \dot{\mathbf{q}} + W + \mathbf{q}'\mathbf{U} - \frac{1}{2} \frac{d\left(\int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} \int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} d\dot{\mathbf{q}}' \mathbf{\Omega} d\dot{\mathbf{q}} \right)}{dt} \quad (7)$$

The Lagrangian is used to derive Hamilton´s equations of motion as $H(\mathbf{p}, \mathbf{q}, t) = \mathbf{p}'\dot{\mathbf{q}} - L$ where the generalised momentum is

$$\mathbf{p} = \partial L / \partial \dot{\mathbf{q}} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{q}'\mathbf{E}\dot{\mathbf{q}} \quad (8)$$

Thus, the Hamiltonian is the sum of the kinetic and potential energy of a closed system expressed in terms of momentum, position and time

$$H = \frac{1}{2} \dot{\mathbf{q}}' \mathbf{T} \dot{\mathbf{q}} + \frac{1}{2} \mathbf{q}' \mathbf{E} \dot{\mathbf{q}}^2 - \mathbf{q}'\mathbf{U} + \frac{1}{2} \frac{d\left(\int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} \int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} d\dot{\mathbf{q}}' \mathbf{\Omega} d\dot{\mathbf{q}} \right)}{dt} \quad (9)$$

The Hamiltonian the equations of motion may be derived from the relations $\dot{\mathbf{p}} = -\partial H/\partial \mathbf{q}$ and $\mathbf{q} = \partial H/\partial \mathbf{p}$. Besides, equation (8) implies that

$$\dot{\mathbf{q}} = (\mathbf{T} + \mathbf{q}'\mathbf{E})^{-1}\mathbf{p} \tag{10}$$

Now, eq. (10) serves to eliminate $\dot{\mathbf{q}}$ from the Hamiltonian

$$H = \frac{1}{2}\mathbf{p}'[\mathbf{T}+\mathbf{q}'\mathbf{E}]^{-1}\mathbf{p} - \mathbf{q}'\mathbf{U} + \frac{1}{2}\frac{d\left(\int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)}\int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} d\dot{\mathbf{q}}'\mathbf{\Omega} d\dot{\mathbf{q}}\right)}{dt} . \tag{11}$$

Thus, substituting equations (8) and (11) into the first equation of motion gives

$$\frac{d(\mathbf{T}\dot{\mathbf{q}} + \mathbf{q}'\mathbf{E}\dot{\mathbf{q}})}{dt} = \frac{\mathbf{E}\dot{\mathbf{q}}'^2}{2} + \frac{1}{2}\dot{\mathbf{q}}'\frac{\partial \mathbf{T}}{\partial \mathbf{q}}\dot{\mathbf{q}} + \mathbf{U} - \frac{1}{2}\frac{d\left(\int_{\mathbf{q}(t_0)}^{\mathbf{q}(t)} d\mathbf{q}'\mathbf{\Omega}\right)}{dt} \tag{12}$$

This equation can be extended into four vector equations, the Kirchoff voltage law for the plasma elements, for the structural and poloidal circuits and the force balance in the $\mathbf{R}$ and $\mathbf{z}$ directions respectively

$$\frac{d(\mathbf{L_e I_e} + \mathbf{M_{es} I_s} + \mathbf{REI_e})}{dt} + \mathbf{\Omega_e I_e} = \mathbf{V}_e \tag{13}$$

$$\frac{d(\mathbf{L}_s \mathbf{I}_s + \mathbf{M}_{se}\mathbf{I}_e)}{dt} + \mathbf{\Omega}_s \mathbf{I}_s = \mathbf{V}_s \tag{14}$$

$$\frac{d(\mathbf{m_e}\dot{\mathbf{R}})}{dt} = \frac{1}{2}\mathbf{I}_\mathbf{e}'\frac{\partial \mathbf{L_e}}{\partial \mathbf{R}}\mathbf{I_e} + \mathbf{I}_s'\frac{\partial \mathbf{M_{se}}}{\partial \mathbf{R}}\mathbf{I_e} + \frac{\mathbf{EI_e^2}}{2} \tag{15}$$

$$\frac{d(\mathbf{m_e}\dot{\mathbf{z}})}{dt} = \frac{1}{2}\mathbf{I}_\mathbf{e}'\frac{\partial \mathbf{L_e}}{\partial \mathbf{z}}\mathbf{I_e} + \mathbf{I}_s'\frac{\partial \mathbf{M_{se}}}{\partial \mathbf{z}}\mathbf{I_e} \tag{16}$$

3.2 Lumped System

A lumped model may be derived from equations (13-16) by defining averaged plasma quantities. The total plasma current will be denoted by I_p, the density distribution $j(R,z)$ is calculated from the Grad-Shafranov equation by an inverse equilibrium reconstruction code. The plasma mass is considered to be zero and its average radial position

$$R = \sum_k i_k R_k \Big/ \sum_k i_k , \tag{17}$$

its average vertical position is defined analogously.

The effective mutual inductance matrix is the addition of the mutual inductance matrices for all plasma elements and the vector structure currents

$$I_p \mathbf{M_{ps} I_s} = \sum_k i_k \mathbf{M_{ks} I_s} \tag{18}$$

The effective plasma self-induction is the addition of the mutual inductance between plasma elements

$$L_p I_p^2 = \sum_k \sum_h i_k M_{kh} i_h \tag{19}$$

To evaluate the internal energy of the plasma, the equations governing the plasma profile $\nabla p = \mathbf{j} \times \mathbf{B}$ and $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$ which substituting gives $\mu_0 \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}$ or equivalently

$$\mu_0 \nabla p = (\mathbf{B} \cdot \nabla)\mathbf{B} - 1/2\, \nabla \mathbf{B}^2 \tag{20}$$

so that

$$\nabla\left(p + \frac{\mathbf{B}^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} , \tag{21}$$

where the right hand side is zero for small aspect ratio, which implies that

$$p + \frac{\mathbf{B}^2}{2\mu_0} = \frac{\mathbf{B}_0^2}{2\mu_0} \tag{22}$$

being $\mathbf{B}_0$ the magnetic field outside the plasma. Besides, we may define the constant parameter $\beta = p 2\mu_0 / \mathbf{B}^2$ with an averaged poloidal beta $\beta_p = \frac{\int p dS}{S}\frac{2\mu_0}{\overline{\mathbf{B}}_\theta^2}$ where S is the plasma cross-section area and the averaged poloidal field is

$$\overline{\mathbf{B}}_\theta = \mu_0 I_p / l . \tag{23}$$

Thus, the internal energy of the plasma

$$W = pV = \int p dS 2\pi R = \frac{\overline{\mathbf{B}}_\theta^2}{2\mu_0}\beta_p S 2\pi R = \mu_0 \frac{I_p^2}{l^2}\beta_p S \pi R \tag{24}$$

4 A Linear Model

The four matrix equations (13-16) define the evolution of the variables $(R, z, I_p, \mathbf{I_s})$ for computation simplicity we solve for the increments and replace the variables (R,z) with (RI_p^0, zI_p^0), where I_p^0 is the constant equilibrium plasma current. Thus, we introduce the perturbations

$$\mathbf{x} = \begin{bmatrix} \mathbf{I_s} - \mathbf{I_s^0} \\ (z - z^0)I_p^0 \\ (R - R^0)I_p^0 \\ I_p - I_p^0 \end{bmatrix} = \begin{bmatrix} \delta\mathbf{I_s} \\ \delta z I_p^0 \\ \delta R I_p^0 \\ \delta I_p \end{bmatrix} \tag{25}$$

4.1 The structure circuit equation

The tokamak structure obeys the circuit equation

$$\frac{d(\mathbf{M_{sp}} I_p)}{dt} + \mathbf{L_s}\dot{\mathbf{I}}_\mathbf{s} + \mathbf{\Omega_s I_s} = \mathbf{V_s} . \tag{26}$$

We know that $\mathbf{L_s} = \mathbf{L_s^0}$ and $\mathbf{\Omega_s} = \mathbf{\Omega_s^0}$ are constant

$$\dot{\mathbf{M}}_\mathbf{sp} I_p + \mathbf{M_{sp}}\dot{I}_p + \mathbf{L_s}\dot{\mathbf{I}}_\mathbf{s} + \mathbf{\Omega_s I_s} = \mathbf{V_s} \tag{27}$$

using first order Taylor expansion

$$\begin{aligned} &\frac{d(\delta\mathbf{M_{sp}})}{dt}(I_p^0 + \delta I_p) + (\mathbf{M_{sp}^0} + \boldsymbol{\delta}\mathbf{M_{sp}})\frac{d(\delta I_p)}{dt} + \\ &\mathbf{L_s^0}\frac{d(\delta\mathbf{I_s})}{dt} + \mathbf{\Omega_s^0}(\mathbf{I_s^0} + \delta\mathbf{I_s}) = (\mathbf{V_s^0} + \delta\mathbf{V_s}) \end{aligned} \tag{28}$$

subtracting the equilibrium equation $\mathbf{L_s}\dot{\mathbf{I}}_\mathbf{s}^\mathbf{0} + \boldsymbol{\Omega}_\mathbf{s}^\mathbf{0}\mathbf{I}_\mathbf{s}^\mathbf{0} = \mathbf{V}_\mathbf{s}^\mathbf{0}$ and neglecting second order terms

$$\frac{d(\delta\mathbf{M_{sp}})}{dt}I_p^0 + \mathbf{M_{sp}^0}\frac{d(\delta I_p)}{dt} + \mathbf{L_s^0}\frac{d(\delta\mathbf{I_s})}{dt} + \boldsymbol{\Omega}_\mathbf{s}\delta\mathbf{I_s} = \delta\mathbf{V_s}. \quad (29)$$

Besides, expanding in terms of the state variables

$$\dot{\mathbf{M}}_\mathbf{sp} = \left.\frac{\partial\mathbf{M_{sp}}}{\partial zI_p^0}\right|_0 \frac{d(zI_p^0)}{dt} + \left.\frac{\partial\mathbf{M_{sp}}}{\partial RI_p^0}\right|_0 \frac{d(RI_p^0)}{dt} \quad (30)$$

so that the linearised equation in terms of the state vector and its time derivative is

$$\begin{bmatrix} \mathbf{L_s^0} & \left.\frac{\partial\mathbf{M_{sp}}}{\partial zI_p^0}\right|_0 I_p^0 & \left.\frac{\partial\mathbf{M_{sp}}}{\partial RI_p^0}\right|_0 I_p^0 & \mathbf{M_{sp}^0} \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} \boldsymbol{\Omega}_\mathbf{s} & 0 & 0 & 0 \end{bmatrix}\mathbf{x} = \delta\mathbf{V_s}.$$

Now, since the derivatives are computed at the equilibrium, the system may be simplified as

$$\begin{bmatrix} \mathbf{L_s^0} & \left.\frac{\partial\mathbf{M_{sp}}}{\partial z}\right|_0 & \left.\frac{\partial\mathbf{M_{sp}}}{\partial R}\right|_0 & \mathbf{M_{sp}^0} \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} \boldsymbol{\Omega}_\mathbf{s} & 0 & 0 & 0 \end{bmatrix}\mathbf{x} = \delta\mathbf{V_s} \quad (31)$$

4.2 The plasma circuit equation

The plasma circuit equation becomes

$$\frac{d\left(L_pI_p + \mathbf{M_{ps}}\mathbf{I_s} + \mu_0\pi S\beta_p RI_p/l^2\right)}{dt} + \Omega_pI_p = V_p \quad (32)$$

expanding the derivative of a product, using first order Taylor expansion and neglecting second order terms

$$\dot{L}_pI_p^0 + L_p^0\dot{I}_p + \dot{\mathbf{M}}_\mathbf{ps}\mathbf{I_s^0} + \mathbf{M_{ps}^0}\dot{\mathbf{I}}_\mathbf{s} + \Omega_p^0\delta I_p + \delta\Omega_pI_p^0 + \Omega_p^0I_p^0 + \frac{\mu_0 2\pi S\beta_p}{l^2}\left(\dot{R}I_p^0 + R^0\dot{I}_p\right) = \mathbf{V}_p^0 + \delta\mathbf{V}_p \quad (33)$$

using the chain's rule for the plasma inductances, considering the plasma resistance to be independent of $\mathbf{x}$, subtracting the equilibrium equation and since the radial magnetic field at equilibrium is zero

$$\left.\frac{\partial\mathbf{M_{ps}}}{\partial zI_p^0}\right|_0 \mathbf{I_s^0} = 0, \quad (34)$$

gives

$$\left.\frac{\partial L_p}{\partial zI_p^0}\right|_0 I_p^0\frac{d(zI_p^0)}{dt} + \left.\frac{\partial L_p}{\partial RI_p^0}\right|_0 I_p^0\frac{d(RI_p^0)}{dt} + L_p^0\dot{I}_p + \left.\frac{\partial\mathbf{M_{ps}}}{\partial RI_p^0}\right|_0 \mathbf{I_s^0}\frac{d(RI_p^0)}{dt} + \mathbf{M_{ps}^0}\dot{\mathbf{I}}_\mathbf{s} + \Omega_p^0\delta I_p + \frac{\mu_0 2\pi S\beta_p}{l^2}\left(\dot{R}I_p^0 + R^0\dot{I}_p\right) = \delta\mathbf{V}_p \quad (35)$$

Thus, if the plasma resistance is independent of $\mathbf{x}$,

$$\begin{bmatrix} \mathbf{M_{ps}^0} \\ \left.\frac{\partial L_p}{\partial z}\right|_0 \\ \left.\frac{\partial\mathbf{M_{ps}}}{\partial R}\right|_0 \frac{\mathbf{I_s^0}}{I_p^0} + \left.\frac{\partial L_p}{\partial R}\right|_0 + \frac{\mu_0 2\pi S\beta_p}{l^2} \\ L_p^0 + \frac{\mu_0 2\pi S\beta_p}{l^2}R^0 \end{bmatrix}'\dot{\mathbf{x}} + \begin{bmatrix} \mathbf{0} \\ 0 \\ 0 \\ \Omega_p^0 \end{bmatrix}'\mathbf{x} = \delta\mathbf{V_p} \quad (36)$$

4.3 The vertical force balance equation

The vertical force balance equation (16) becomes

$$\frac{I_p^2}{2}\frac{\partial L_p}{\partial z} + \frac{\partial\left(I_p\mathbf{M_{ps}}\mathbf{I_s}\right)}{\partial z} = 0. \quad (37)$$

Considering the variation of the equation with respect to time,

$$\frac{1}{2}\left.\frac{\partial^2 L_p}{\partial z^2}\right|_0 \dot{z}I_p^{0^2} + \frac{1}{2}\left.\frac{\partial^2 L_p}{\partial z\partial R}\right|_0 \dot{R}I_p^{0^2} + \left.\frac{\partial L_p}{\partial z}\right|_0 I_p^0\dot{I}_p + I_p^0\left.\frac{\partial\mathbf{M_{ps}}}{\partial z}\right|_0 \dot{\mathbf{I}}_\mathbf{s} + \left.\frac{\partial\mathbf{M_{ps}}}{\partial z}\right|_0 \mathbf{I_s^0}\dot{I}_p + \left.\frac{\partial^2\mathbf{M_{ps}}}{\partial z^2}\right|_0 \mathbf{I_s^0}\frac{d(zI_p^0)}{dt} + \left.\frac{\partial^2\mathbf{M_{ps}}}{\partial z\partial R}\right|_0 \mathbf{I_s^0}\frac{d(RI_p^0)}{dt} = 0 \quad (38)$$

Thus, using that the radial magnetic field at equilibrium is zero gives a generalized vertical force balance equation of the form

$$\begin{bmatrix} \left.\frac{\partial\mathbf{M_{ps}}}{\partial z}\right|_0 \\ \frac{1}{2}\left.\frac{\partial^2 L_p}{\partial z^2}\right|_0 + \left.\frac{\partial^2\mathbf{M_{ps}}}{\partial z^2}\right|_0 \frac{\mathbf{I_s^0}}{I_p^0} \\ \frac{1}{2}\left.\frac{\partial^2 L_p}{\partial z\partial R}\right|_0 + \left.\frac{\partial^2\mathbf{M_{ps}}}{\partial z\partial R}\right|_0 \frac{\mathbf{I_s^0}}{I_p^0} \\ \left.\frac{\partial L_p}{\partial z}\right|_0 \end{bmatrix}'\dot{\mathbf{x}} = 0 \quad (39)$$

4.4 The radial force balance equation

The radial force balance equation (15) becomes

$$\frac{1}{2}\frac{\partial L_pI_p^2}{\partial R} + \frac{\partial\left(I_p\mathbf{M_{ps}}\mathbf{I_s}\right)}{\partial R} + \mu_0\frac{\pi S}{l^2}\beta_pI_p^2 = 0 \quad (40)$$

Considering the variation of the equation with respect to time gives

$$
\begin{aligned}
&\left.\frac{\partial L_p}{\partial R}\right|_0 I_p^0 \dot{I}_p + \left.\frac{\partial^2 L_p}{\partial R^2}\right|_0 \dot{R}\frac{I_p^{0\,2}}{2} + \left.\frac{\partial^2 L_p}{\partial R \partial z}\right|_0 \dot{z}\frac{I_p^{0\,2}}{2} + \\
&I_p^0 \left.\frac{\partial \mathbf{M_{ps}}}{\partial R}\right|_0 \dot{\mathbf{I}}_\mathbf{s} + \left.\frac{\partial \mathbf{M_{ps}}}{\partial R}\right|_0 \mathbf{I_s^0} \dot{I}_p + \\
&\left.\frac{\partial^2 \mathbf{M_{ps}}}{\partial R^2}\right|_0 \mathbf{I_s^0} \frac{d\left(RI_p^0\right)}{dt} + \left.\frac{\partial^2 \mathbf{M_{ps}}}{\partial R \partial z}\right|_0 \mathbf{I_s^0} \frac{d\left(zI_p^0\right)}{dt} + \\
&\mu_0 \frac{\pi S}{l^2} \beta_p 2 I_p^0 \dot{I}_p = 0
\end{aligned}
\tag{41}
$$

Thus, the linearised radial force equation is

$$
\begin{bmatrix}
\left.\dfrac{\partial \mathbf{M_{ps}}}{\partial R}\right|_0 \\
\left.\dfrac{\partial^2 L_p}{\partial R \partial z}\right|_0 \dfrac{1}{2} + \left.\dfrac{\partial^2 \mathbf{M_{ps}}}{\partial R \partial z}\right|_0 \dfrac{\mathbf{I_s^0}}{I_p^0} \\
\left.\dfrac{\partial^2 L_p}{\partial R^2}\right|_0 \dfrac{1}{2} + \left.\dfrac{\partial^2 \mathbf{M_{ps}}}{\partial R^2}\right|_0 \dfrac{\mathbf{I_s^0}}{I_p^0} \\
\left.\dfrac{\partial L_p}{\partial R}\right|_0 + \left.\dfrac{\partial \mathbf{M_{ps}}}{\partial R}\right|_0 \dfrac{\mathbf{I_s^0}}{I_p^0} + \mu_0 \dfrac{\pi S}{l^2} \beta_p 2
\end{bmatrix}
\dot{\mathbf{x}} = 0
\tag{42}
$$

5 An Eigenmode Representation of the Passive Structure Currents

The non-linear circuit equations of the structure and plasma can be written as a matrix equation

$$
\begin{bmatrix} \mathbf{L_c} & \mathbf{M_{cv}} & \mathbf{M_{cp}} \\ \mathbf{M_{vc}} & \mathbf{L_v} & \mathbf{M_{vp}} \\ \mathbf{M_{pc}} & \mathbf{M_{pv}} & L_p \end{bmatrix}
\begin{pmatrix} \dot{\mathbf{I}}_\mathbf{c} \\ \dot{\mathbf{I}}_\mathbf{v} \\ \dot{I}_p \end{pmatrix}
+
\begin{bmatrix} \mathbf{\Omega_c} & \mathbf{0} & \dot{\mathbf{M}}_\mathbf{cp} \\ \mathbf{0} & \mathbf{\Omega_v} & \dot{\mathbf{M}}_\mathbf{vp} \\ \dot{\mathbf{M}}_\mathbf{pc} & \dot{\mathbf{M}}_\mathbf{pv} & \Omega_p + \dot{L}_p \end{bmatrix}
\begin{pmatrix} \mathbf{I_c} \\ \mathbf{I_v} \\ I_p \end{pmatrix}
=
\begin{pmatrix} \mathbf{V_c} \\ \mathbf{0} \\ 0 \end{pmatrix}
\tag{43}
$$

where the subscript v stands for vessel (passive structure) and c for coils. The eddy currents occur over different time scales associated with $\mathbf{\Omega_v^{-1} L_v}$.

The vessel states can be represented in terms of the eigenvectors of this matrix $\left(\mathbf{\Omega_v^{-1} L_v}\right)\mathbf{\Psi} = \mathbf{\Psi\lambda}$. Thus, pre-multiplying the second line by $\mathbf{\Psi^{-1}\Omega_v^{-1}}$ and replacing $\mathbf{I_v}$ with $\mathbf{\Psi I_\Lambda}$

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{L_c} & \mathbf{M_{cv}\Psi} & \mathbf{M_{cp}} \\ \mathbf{\Psi^{-1}\Omega_v^{-1}M_{vc}} & \mathbf{\lambda} & \mathbf{\Psi^{-1}\Omega_v^{-1}M_{vp}} \\ \mathbf{M_{pc}} & \mathbf{M_{pv}\Psi} & L_p \end{bmatrix}
\begin{pmatrix} \dot{\mathbf{I}}_\mathbf{c} \\ \dot{\mathbf{I}}_\mathbf{\Lambda} \\ \dot{I}_p \end{pmatrix} + \\
&\begin{bmatrix} \mathbf{\Omega_c} & \mathbf{0} & \dot{\mathbf{M}}_\mathbf{cp} \\ \mathbf{0} & \mathbf{1} & \mathbf{\Psi^{-1}\Omega_v^{-1}}\dot{\mathbf{M}}_\mathbf{vp} \\ \dot{\mathbf{M}}_\mathbf{pc} & \dot{\mathbf{M}}_\mathbf{pv}\mathbf{\Psi} & \Omega_p + \dot{L}_p \end{bmatrix}
\begin{pmatrix} \mathbf{I_c} \\ \mathbf{I_\Lambda} \\ I_p \end{pmatrix}
=
\begin{pmatrix} \mathbf{V_c} \\ \mathbf{0} \\ 0 \end{pmatrix}
\end{aligned}
\tag{44}
$$

Now the passive structure states in terms of the current eigenmodes rather than the currents so as to enable truncation of the appropriate eigenmodes.

6 System output

We can relate the system output to the state of the system by writing $\mathbf{y} = \mathbf{Cx}$. In the output vector appear some states of the tokamak, the poloidal magnetic field and the magnetic flux.

There are flux loops all around the tokamak. A change in flux within the loop is measured by the mutual inductances between the flux loop and plasma or structure $\Delta\Phi = \Delta\left(M_{pf} I_p + \mathbf{M_{sf} I_s}\right)$. This equation can be expanded about the plasma equilibrium state

$$
\Delta\Phi = \left[\left.\frac{\partial M_{pf}}{\partial z}\right|_0 \delta\left(zI_p^0\right) + \left.\frac{\partial M_{pf}}{\partial R}\right|_0 \delta\left(RI_p^0\right)\right] + M_{pf}^0 \delta I_p + \mathbf{M_{sf}^0} \delta\mathbf{I_s}
\tag{45}
$$

which in terms of the state vector gives

$$
\Delta\Phi = \begin{bmatrix} \mathbf{M_{sf}^0} & \left.\dfrac{\partial M_{pf}}{\partial z}\right|_0 & \left.\dfrac{\partial M_{pf}}{\partial R}\right|_0 & M_{pf}^0 \end{bmatrix} \mathbf{x}
\tag{46}
$$

The poloidal magnetic field probe is a loop of negligible area that measures the magnetic field normal to the loop, so that $\mathbf{B}\cdot\hat{\mathbf{n}} = \mathbf{B}_{pol}$ being $\mathbf{B} = B_R \hat{\mathbf{R}} + B_z \hat{\mathbf{z}}$. Expanding $\mathbf{B}$ in terms of the state variables about the tokamak equilibrium

$$
\mathbf{B}_{pol} = \begin{bmatrix} \left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})}{\partial \mathbf{I_s}}\right|_0 & \left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})}{\partial zI_p^0}\right|_0 & \left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})}{\partial RI_p^0}\right|_0 & \left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})}{\partial I_p}\right|_0 \end{bmatrix} \mathbf{x}
\tag{47}
$$

Thus, for a particular diagnostic, denoted with the subscript n, the relationship between the state of the system and the output $\mathbf{y} = \mathbf{Cx}$ is

$$
\begin{bmatrix} \delta(\mathbf{I}_s) \\ \delta\left(zI_p^0\right) \\ \delta\left(RI_p^0\right) \\ \delta(I_p) \\ \mathbf{B}_{pol_n} \\ \Delta\phi_n \end{bmatrix}
=
\begin{bmatrix}
[\mathbf{I} \;\; \mathbf{0}] & 0 & 0 & 0 \\
\mathbf{0} & 1 & 0 & 0 \\
\mathbf{0} & 0 & 1 & 0 \\
\mathbf{0} & 0 & 0 & 1 \\
\left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})_n}{\partial I_s}\right|_0 & \left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})_n}{\partial zI_p^0}\right|_0 & \left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})_n}{\partial RI_p^0}\right|_0 & \left.\dfrac{\partial(\mathbf{B}\cdot\hat{\mathbf{n}})_n}{\partial I_p}\right|_0 \\
\left(\mathbf{M_{sf}^0}\right)_n & \left.\dfrac{\partial(\mathbf{M_{sf}})_n}{\partial z}\right|_0 & \left.\dfrac{\partial(\mathbf{M_{sf}})_n}{\partial R}\right|_0 & \left(\mathbf{M_{pf}^0}\right)_n
\end{bmatrix}
\mathbf{x}
\tag{48}
$$

7 State space model of a tokamak

The four physic equations are now linearised about an equilibrium point $\mathbf{x^0} = \mathbf{0}$ to give the standard state-space control model

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\
\mathbf{y} &= \mathbf{Cx} + \mathbf{Du}
\end{aligned}
\tag{49}
$$

The structure circuit equation, vertical force balance, radial force balance and structure circuit equations in the matrix form given in Sections 4.1-4.4 lead to a system of the form $\mathbf{M\dot{x}} + \mathbf{Rx} = \mathbf{u}$, which may be rewritten as the first equation of the system (49).

8 Conclusion

A tokamak numerical model has been presented based on the Hamiltonian equation of motion. In order to solve this system numerically, the model has been linearised around an operation point taking a Newton-Raphson step. Besides, a state space control model has been derived by considering the flux and the poloidal field equations.

The resulting state space control model is currently being verified in order to extend the performance of the existing PID schemes guaranteeing closed loop stability and performance via an a priori bound.

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