The Deutsch-Josza algorithm for n-qudits

GABRIELA MOGOS
Al.I.Cuza University
Department of Computer Science
Carol I, 11, Iasi
ROMANIA

Abstract: Over the last decade has been a lot of interest in new types of computable machines, due partly over the fears on conventional, classical, computing. Research by leading physicists, mathematicians and computer scientists has shown quantum computers could be the future, as they may be able to solve some computationally difficult problems much faster than is currently possible on a classical computer. David Deutsch has already created a working model for a quantum computer, the quantum Turing machine or QTM Deutsch [1985]. This has been proved as a model for computation on a quantum and has allowed algorithms to be developed. Deutsch-Josza's algorithm, as all known quantum algorithms that provide exponential speedup over classical systems do, answers a question about a global property of a solution space. This paper describes the generalization of the Deutsch-Josza algorithm to n d-dimensional quantum systems or qudits.

Key–Words: quantum computing, quantum dits, quantum algorithm

1 Introduction

A qudit is a general state in a d-dimensional Hilbert space H_d i. e. $|\Psi\rangle = \sum_{m=0}^{d-1} c_m |m\rangle$, which reduces to $|\Psi\rangle = c_0 |0\rangle + c_1 |1\rangle$, for the qubit case space.

An n-qudit is a state in the tensor product Hilbert space. The computational basis of H is the orthonormal basis given by the d^n classical n-qudits:

$$|m_1\rangle \otimes |m_2\rangle \otimes ... \otimes |m_n\rangle = |m_1 m_2 ... m_n\rangle \quad (1)$$

where $0 \leq m_n \leq d-1$.

The general state in H is a superposition:

$$|\Psi\rangle = \sum \Psi_{m_1 m_2 .. m_n} |m_1 m_2 ... m_n\rangle$$

where $| |\Psi||^2 = \sum |\Psi_{m_1 m_2 .. m_n}|^2 = 1$.

We say Ψ is *decomposable* when it can be written as a tensor product of qudits:

$$|m_1 m_2 ... m_n\rangle = |m_1\rangle \otimes |m_2\rangle \otimes ... \otimes |m_n\rangle = \bigotimes_{i=0}^{n} |m_i\rangle = |m\rangle$$

where $|m_i\rangle$ is a general state in a d-dimesional Hilbert space for one qudit.

1.1 The Deutsch-Jozsa problem for n-qudits

The algorithm Deutsch-Josza are generalize Deutsch algorithm and the function is: $f : \{0, 1, ..., d-1\}^n \longrightarrow \{0, 1, ..., d-1\}$.

Taking matters the fact that exist n-qudits as input data, we put a global problems if the function $f(x)$ is constant or balanced. If the function will be balanced, then mean that the entire exit will be 0 for exactly half of the inputs.

Classic speacking that represent the evaluation of the function $f(x)$ for much more that half of inputs, because we must see with certitude when the function is balanced and when is constant.

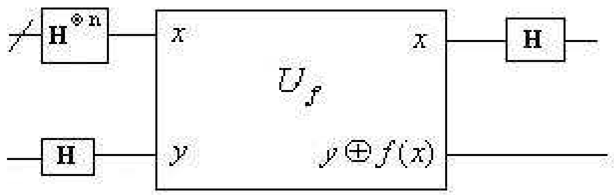

Figure 1: Quantum circuit to solve the Deutsch-Jozsa problem for n-qudits

Let us analyze this circuit now.

1. First we need to apply Hadamard gates H, to register with n qudits. We can now apply the Hadamard operator to each qudit of the product state:

$$H^{\otimes n} |m\rangle = H^{\otimes n}(|m_1\rangle \otimes |m_2\rangle \otimes ... \otimes |m_n\rangle) =$$
$$= H^{\otimes n} |m_1\rangle \otimes H^{\otimes n} |m_2\rangle \otimes ... \otimes H^{\otimes n} |m_n\rangle \quad (2)$$

In general, the Hadamard operator acting on a single qudit of dimension d is defined as:

$$H|x\rangle = \frac{1}{\sqrt{d}}\sum_{y=0}^{d-1}(-1)^{x\cdot y}|y\rangle \quad (3)$$

However, we generalize the latter to $H^{\otimes n}$ the notation pays off since the above form can immediately be generalized by summing over all possible combinations of qudit basis states, i.e., over all n-qudit states

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{d^n}}\sum_{i=0}^{n}\sum_{y_i=0}^{d-1}(-1)^{x_i\cdot y_i}|y_i\rangle$$

where

$$x_i\cdot y_i = x_1y_1 + x_2y_2 + ... + x_ny_n$$

For register with n qudits, the equation (3) become:

$$(\frac{1}{\sqrt{d^n}}\sum^{n}\sum_{y_1=0}^{d-1}(-1)^{x_1\cdot y_1}|y_1\rangle)\otimes$$

$$\otimes(\frac{1}{\sqrt{d^n}}\sum^{n}\sum_{y_2=0}^{d-1}(-1)^{x_2\cdot y_2}|y_2\rangle)\otimes$$

$$...\otimes(\frac{1}{\sqrt{d^n}}\sum^{n}\sum_{y_n=0}^{d-1}(-1)^{x_ny_n}|y_n\rangle) =$$

$$= \frac{1}{\sqrt{d^{n^2}}}\sum^{n}[(\sum_{y_1=0}^{d-1}(-1)^{x_1\cdot y_1}|y_1\rangle)\otimes$$

$$\otimes(\sum_{y_2=0}^{d-1}(-1)^{x_2\cdot y_2}|y_2\rangle)\otimes$$

$$...\otimes(\sum_{y_n=0}^{d-1}(-1)^{x_ny_n}|y_n\rangle)] =$$

$$= \frac{1}{\sqrt{d^{n^2}}}\sum_{i=0}^{n}(\bigotimes_{i=0}^{n-1}\sum_{y_i=0}^{d-1}(-1)^{x_i\cdot y_i}|y_i\rangle) =$$

$$= \bigotimes_{i=0}^{n-1}[\frac{1}{\sqrt{d^n}}\sum_{i=0}^{n}\sum_{y_i=0}^{d-1}(-1)^{x_i\cdot y_i}|y_i\rangle] = |\Psi_1\rangle$$

More generally, given $|\Psi\rangle = \bigotimes_{i=0}^{n-1}|\Psi_i\rangle$, where each $|\Psi_i\rangle \in H_d$, the state $H^{\otimes n}|\Psi\rangle$ can be computed in linear time by:

$$H^{\otimes n}|\Psi\rangle = \bigotimes_{i=0}^{n-1}H|\Psi_i\rangle$$

2. In next step of the Deutsch-Josza algorithm, we should evaluate the U_f operator effect. The operation of U_f gate is completely defined by its action on the computational basis for each qudit:

$$|x\rangle|y\rangle \xrightarrow{U_f} |x\rangle|y\oplus f(x)\rangle$$

where $|x\rangle$ and $|y\rangle \in \{|0\rangle, |1\rangle, ..., |d-1\rangle\}$ denote the state of control and auxiliary qudits.

Using U_f operator, we now transform the n-qudits of the upper lines and 1-qudit of the lower line, as:

$$[\bigotimes_{i=0}^{n-1}(\frac{1}{\sqrt{d^n}}\sum_{i=0}^{n}\sum_{x_i=0}^{d-1}(-1)^{z_i\cdot x_i}|x_i\rangle)](\frac{1}{\sqrt{d}}\sum_{y=0}^{d-1}(-1)^{t\cdot y}|y\rangle) \xrightarrow{U_f}$$

$$[\bigotimes_{i=0}^{n-1}(\frac{1}{\sqrt{d^n}}\sum_{i=0}^{n}\sum_{x_i=0}^{d-1}(-1)^{f(x)}(-1)^{z_i\cdot x_i}|x_i\rangle)]$$

$$(\frac{1}{\sqrt{d}}\sum_{y=0}^{d-1}(-1)^{t\cdot y}|y\rangle) = [\bigotimes_{i=0}^{n-1}(\frac{1}{\sqrt{d^n}}\sum_{i=0}^{n}\sum_{x_i=0}^{d-1}(-1)^{f(x)+x_i\cdot z_i}|x_i\rangle)]\cdot$$

$$\cdot[\frac{1}{\sqrt{d}}\sum_{y=0}^{d-1}(-1)^{y\cdot t}|y\rangle] = |\Psi_2\rangle$$

3. Finally we apply another $H^{\otimes n}$ transform to obtain:

$$|\Psi_3\rangle = \sum^{n}[\bigotimes_{i=0}^{n-1}(\frac{1}{\sqrt{d^n}}\sum_{i=0}^{n}\sum_{x_i=0}^{d-1}(-1)^{f(x_i)+x_i\cdot z_i}|x_i\rangle)]\cdot$$

$$\cdot(\frac{1}{\sqrt{d}}\sum_{y=0}^{d-1}(-1)^{y\cdot t}|y\rangle)$$

We measure the probability amplitude of $x_i = |m_i\rangle^{\otimes n}$.

For *constant* $f(x_i)$, the sum over x_i is independent of x_i and $x_i\cdot z_i$ must also be equal to zero and hence $(-1)^{x_i\cdot z_i+f(x_i)}$ is either -1 or $+1$ for all values of x_i, where -1 holds for $f(x_i) = 1$ and 1 holds for $f(x_i) = 0$.

In this case the amplitude for $x_i = |m_i\rangle^{\otimes n}$ is:

$$\bigotimes^{n}(\pm\sum_{x_i=0}^{d-1}\frac{1}{\sqrt{d^n}}) = \pm 1$$

since $|\Psi_3\rangle$ is normalized to 1 and the amplitude of $x_i = |m_i\rangle^{\otimes n}$ already gives probability 1, there can be no other component in $|\Psi_3\rangle$, all other amplitudes must be zero. Hence when we measure the first n qudits in the query register, we will obtain a zero $(|0_i\rangle^{\otimes n}$.

If $f(x)$ is *balanced* then $(-1)^{x_i \cdot z_i + f(x_i)}$ will be $+1$ for some values of x_i and -1 for other values of x_i. The amplitude of the all states $x_i = |m_i\rangle^{\otimes n}$ is then:

$$\overset{n}{\otimes}(+\sum_{x_{i_1}=0}^{d-1} \frac{1}{\sqrt{d^n}} - \sum_{x_{i_2}=0}^{d-1} \frac{1}{\sqrt{d^n}}) = 0$$

where x_{i_1} is the set of x_i's such that the function $f(x_i)$ has a plus sign and x_{i_2} is the set of x_i's where $f(x_i)$ has a minus sign.

We say that f has a balanced parity when an even value for exactly half of x_i and a odd value for the other half.

2 Conclusion

The above quantum circuit resolves deterministically the Deutsch-Jozsa problem performing a single application of U_f, while for the classical case it is necessary (in the worst case) $d^{n+1} + 1$ applications of f to assure that f is constant or balanced (f must be calculated with different entrances until finding two different values or until calculating the half plus one of the values). Because the complexity of an algorithm is measured by the complexity of the worst case, the deterministic classical solution to determine if f is constant or balanced has an exponential complexity, while the quantum algorithm to solve the same problem has a polynomial complexity.

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