Numerical simulation of the free convection flow in porous medium

AUREL CHIRIȚĂ
Department of Mathematics
High-School Ion Minulescu, Slatina
ROMANIA
HORIA ENE
Department of Fluids Mechanics
Institute of Research of Romanian Academy
Bucureşti
ROMANIA
BOGDAN NICOLESCU
University of Pitesti, Piteşti
ROMANIA
ION CARSTEA
Department of computers engineering and communication
University of Craiova, Craiova
ROMANIA
http://www.IonCarstea.webdrisign.net

Abstract: - The scope of this paper is to present, both from a theoretical and a practical viewpoint, the problem of the thermal boundary-layer approximation for free convection flow in a porous medium bounded by a vertical flat plate (wall). The influence of the wall temperature and heat flux on the problem solution is discussed.

We present numerical results for the problem of steady free convection using approximation models of Navier-Stokes equations. As target example we consider an unbounded porous medium in a gravitational field, saturated with a fluid at temperature T_∞ containing a longitudinal line heat source. We limit our study to a heated flat plate. The mathematical models were integrated numerically using the fourth_order Runge-Kutta method. Also we present the numerical results of the velocity distribution in brief.

Keywords: - Fluid mechanics; Similarity solutions; Numerical simulation.

1 Introduction

Thermal flow or natural convection occurs in a lot of practical problems in many branches of engineering and geophysical applications (geothermal energy, building thermal insulation, enhanced oil recovery, solid-matrix heat exchangers etc). Such problems arise when a heated vertical plate is embedded in an unbounded porous medium. At high Rayleigh number the most part of the convection is in a thin layer around the heated source.

There is a large literature in the area of approximation of boundary layer in porous medium so that we shall not present it in details. Our goal is to present the results of the numerical simulation of the system in some practical assumptions. In this work we present lumped-parameter models involving ordinary differential equations. In this way we reduce a hyperbolic equations system to a system of ordinary differential equations where there is a lot of numerical models.

In our target example, the mathematical model of the system is described by Navier-Stokes equations that are partial derivatives equations. The solution of this model can be obtained by numerical methods. The difficulties in the solution of Navier-Stokes equations were strong motivations for development of new simplified models. In some assumptions, the mathematical model is reduced to a lumped-parameter non linear equation so that a practical solution can be obtained by numerical procedures. But in the last case there is a large literature so that we have the advantage to use it.

In this paper we present some computational aspects for the classical two-dimensional laminar incompressible boundary layer flow past a flat plate. The thermal boundary-layer approximation is based on the assumption that convection takes place in a thin layer around the heating surface. The parameter that gives the possibility of this approach is the Rayleigh number. In our case the motion takes place at high Rayleigh numbers.

It is proved by physical considerations that the vertical velocity and temperature distributions are of the same shape. At the wall the vertical velocity varies in the same way as prescribed wall temperature. This fact is a consequence of the conditions imposed at infinity for velocity and temperature.

We simulated the phenomena arising in the thermal boundary layer by taking some classical models and examples.

2 Mathematical modelling

Our target example is a heated semi-infinite vertical flat plate embedded in an unbounded porous medium in a gravitational field, saturated with a fluid at temperature T_∞at rest. The heat source is the plate temperature and has a line form.

Let us consider a rectangular Cartesian co-ordinate system with the origin fixed at the leading edge of the vertical surface with the x-axis directed upwards along the wall, and y-axis directed to normal to the surface. Mathematical model is defined by a system of partial derivative equations [1]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x} + \rho g\right) \tag{2}$$

$$v = -\frac{k}{\mu}\frac{\partial p}{\partial y} \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

$$\rho = \rho_\infty\left[1 - \beta\left(T - T_\infty\right)\right] \tag{5}$$

The significances of the variables from Eq. (1) to (5) are:

- u, v – Darcy speed components in the directions Ox and Oy
- ρ, μ, β – density, viscosity and the coefficient of the thermal expansion of the fluid
- k – permeability of th6 porous medium
- λ – coefficient of thermal diffusivitty
- p - pressure
- T(x,y)- temperature in the point (x,y)
- g – acceleration due to gravity

The subscript ∞ means the value of the temperature at infinity.

It is proved that at large Rayleigh numbers, the most important part of convection takes place in a thin layer around the heated plate. In this case the mathematical model is a third order non-linear ordinary differential equation depending on a parameter related to the temperature on the wall.

We distinguish some practical cases:

- The wall temperature is uniform
- The wall temperature is nonuniform with a prescribed law
- The heat flux is uniform
- The heat flux is nonuniform

For example, ones of the boundary conditions for the model (1)-(4) are:

$$v(x,0) = 0 \tag{6}$$

$$T(x,0) = T_\infty + A.x^\alpha, \quad A > 0 \tag{7}$$

In other words the wall temperature is a power function of distance from the origin. From physical considerations, the boundary conditions at a great distance from the wall are:

$$u(x,\infty) = 0 \tag{8}$$

$$T(x,\infty) = T_\infty \tag{9}$$

In the professional literature the mathematical model described by (1)-(4) may be simplified by using by using boundary layer approximations similar to the method of Prandtl for the classical theory of a boundary layer in a free viscous fluid.

In the fluid mechanics theory, the Blasius equation appears in some boundary layer problems and we look for solutions having a similarity form. More, the Blasius equation is a particular case of Falkner-Skan model. For example, if the temperature of the plate is constant, the mathematical model for the motion of an incompressible viscous fluid near a semi-infinite flat plate is Blasius equation [2]:

$$f''' + \frac{1}{2}ff'' = 0 \quad y \in [0,\infty) \tag{10}$$

with the boundary conditions:

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{11}$$

Eq. (10) was obtained by dropping some terms in the Navier-Stokes equations.

In the case the prescribed temperature of the plate is by the form (7), a method similar to those proposed by Prandtl leads to the equation [2]:

$$f''' + \frac{\alpha+1}{2}ff'' - \alpha(f')^2 = 0 \tag{12}$$

It is obviously that Blasius equation is a particular case of Eq. (12) for α=0.

In these mathematical models we identify some physical constraints that limit the space of the solutions. One of these physical constraints is:

$$0 \le f'(t) \le 1 \qquad \forall t \in [0,\infty) \qquad (13)$$

Consequently, we have two solution types:

- A mathematical solution for the problem with the boundary conditions described by (10)
- A physical solution for the problem with boundary conditions (11) and (13)

It is obviously that the mathematical solution does not depend on physical solution, but from the engineer's viewpoint we must find the physical solution. The non-existence and uniqueness of the solution remains an open problem.

In this work we investigate the boundary value problem from numerical viewpoint. We have a bilocal problem so that an iterative procedure must be used for the numerical solution. We transform the bilocal problem in a Caughy problem and a fourth_order Runge-Kutta method was used.

In a previous work we presented the numerical results for two practical case: the uniform temperature of the wall and the case with nonuniform temperature of the wall.

Interesting results are obtained if the vertical flat plate embedded in the porous medium is assumed to have a non-uniform temperature or if a non-uniform heat flux is assumed. In these interesting cases the mathematical models in dimensionless variables and functions that take account of the non-uniformity of the temperature or heat flux on the plate, are ordinary differential equations. In professional literature the transformations of the initial variables and variables of similarity are presented in details so that in this work we use the final mathematical models.

Case a. The non-uniform temperature of the wall

This interesting case corresponds to a non-uniform wall temperature defined by formula (7), assumption that leads to the mathematical model defined by Eq. (12) with the boundary conditions:

$$f(0) = 0$$
$$f'(0) = 1 \qquad (14)$$
$$f'(\infty) = 0$$

We introduce the initial values Caughy problem by conditions:

$$f(0) = 0$$
$$f'(0) = 1 \qquad (15)$$
$$f''(0) = b$$

The value of the parameter ***b*** must be selected so that the final value of f' must be zero.

Case b. The non-uniform heat-flux of the wall

In this particular case the mathematical model is described by the ordinary differential equation:

$$f''' + \left(\frac{2}{3} + \frac{\alpha}{2}\right) ff'' - \left(\frac{1}{3} + \frac{2\alpha}{3}\right)(f')^2 = 0 \quad (16)$$

with specified boundary conditions:

$$f(0) = 0, \quad f'(0) = -1, \quad f'(\infty) = 0 \quad (17)$$

3 Numerical results

The numerical simulation of the heat transfer guided the work from a surface embedded in a porous medium through which a liquid is flowing. The surface temperature and the heat flux are non-uniform.

Case a. The non-uniform temperature of the wall

The differential equation (12) can be rewritten as a system of differential equations of the first order. For this we define the set {z1, z2, z3}={f, f', f''}. The equations system is the following:

$$z1' = z2$$
$$z2' = z3 \qquad (18)$$
$$z3' = -\frac{1+\alpha}{2} z1 \cdot z3 + \alpha \cdot (z2)^2$$

with the boundary conditions:

z1(0)=0

z2(0)=1

z3(0)=b

The real parameter b must be determined so that the conditions (14) are fulfilled.

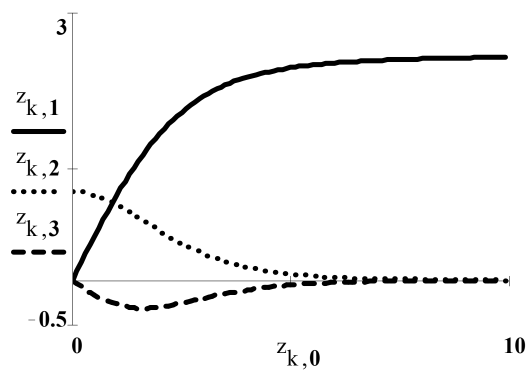

Fig.1- *Case of non-uniform temperature*

With the notations $z1(t_k)=z_{k,1}$, $z2(t_k)=z_{k,2}$ and $z3(t_k)=z_{k,3}$, the curves for the unknowns are plotted in Fig. 1. We used the program Mathcad [4]. We considered the particular case α= - 0.33 and the results are for the initial condition f''(0)=-0.01.

In Fig. 2 the curve for f' is plotted. It can be seen that the conditions defined by relationships (14) are fulfilled.

This case was treated in many works for different values of the parameter α. The problem is not the value of this parameter and the approach to select the value of the parameter μ. The problem is open.

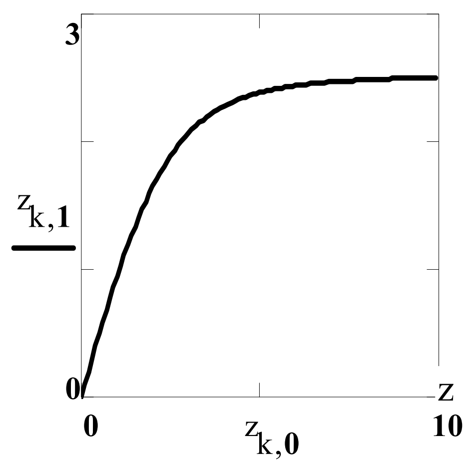

Fig.2- *Variation of the function f*

Case b. The non-uniform heat-flux of the wall
The differential equation (16) with the boundary conditions (17) was solved by fourth_order Runge-Kutta method. In Fig. 3 the curves for {f, f', f''} are plotted in the case α=-0.5.

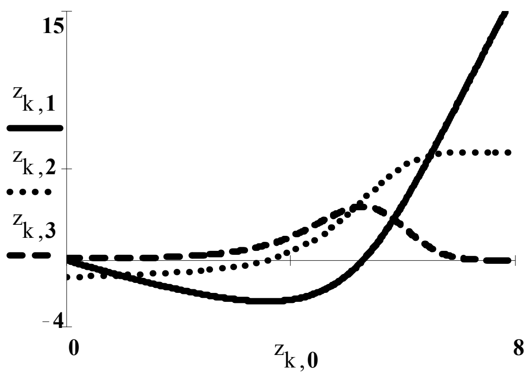

Fig.3 - *Case of non-uniform heat flux*

4 Velocity distributions

Another aspect in the boundary layer flow is the velocity distributions outside the boundary layer. We limit our discussion to some numerical results for two-dimensional incompressible, laminar, steady-state boundary layer. The mathematical model is described by the Prandtl equations [4] :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=u_e\frac{\partial u_e}{\partial x}+\upsilon\frac{\partial^2 u}{\partial y^2} \qquad (19)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (20)$$

where (x, y) denote the usual orthogonal Cartesian co-ordinates with axis Oy normal to the wall and axis Ox is the wall surface. The variables u and v are the corresponding velocity components, and the constant ν is the kinematic viscosity. The function u_e is a given exterior streaming velocity flow. In our example u_e is assumed to be a function of the single variable x and represents the limit of u (x, y) as $y\to\infty$.

In the history of this problem, the works of Blasius (1908) and Falkner and Skan (1931) were pioneering works. They considered the external velocity as being:

$$u_{e,}=u_\infty x^m, \quad u_\infty>0 \qquad (21)$$

The problem has solutions having a similarity form if the velocity distribution outside the boundary layer is proportional to x^m. Using the well-known stream function we can obtain similarity solutions solving an ordinary differential equation. A special case is for m= -1when the mathematical model is defined by the equation:

$$f'''+\gamma f''+(f')^2-1=0 \qquad (22)$$

with the boundary conditions:

$$f'(0)=\zeta, \quad f'(\infty)=1 \qquad (23)$$

where ζ is a subunitary real number, and γ is a positive real number that plays the role of suction/injection parameter [3].

4.1. Numerical results

For numerical solution of the Eq. (22) we replace f'=θ and Eq. (22) becomes a second order differential equation by the form [3]:

$$\theta''+\gamma\theta'+\theta^2-1=0 \qquad (24)$$

with boundary conditions:

$$\theta(0)=\zeta, \quad \theta(\infty)=1 \qquad (25)$$

This bilocal problem can be transformed in a Caughy problem if the boundary condition at infinity is replaced by the condition θ'(0)=d, where d is a real number that can be obtained by an iterative procedure so that the condition (25) is fulfilled. In the numerical simulation we can determine the maximal interval of existence, that is a finite interval $(0,T_d)$.

We shall not present the details of the analysis. In a numerical simulation we can do tests to find the velocity profiles and do analysis in the phase plane (θ,θ'). In this way we can do a stability analysis. Theoretically it was proved that the system is asymptotically stabile if $\gamma > 0$ and unstable for negative γ.

We used the fourth-order Runge-Kutta method. The mathematical model defined by Eq. (24) was written as a system of the first-order differential equations. The results were obtained with the software Mathcad where $z_{k,0}$ represents the argument

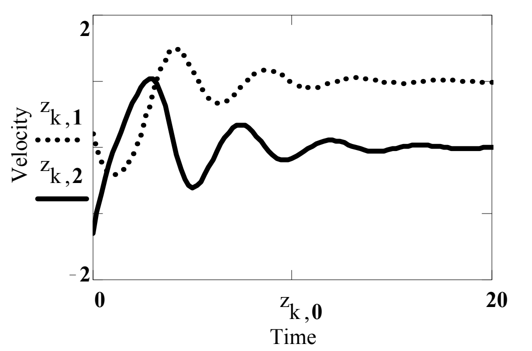

Fig.4- *Velocity profiles*

and $z_{k,1}$ and $z_{k,2}$ represent the state variables (θ,θ').

In Fig. 4 the curves are shown for some particular values of the initial conditions. The parameter values are the following: $\zeta=0.2$, $d=-1.3$ and $\gamma=0.5$.

In Fig. 5 the diagram in the phase plane is plotted. The point (1,0) is an equilibrium point of focus type.

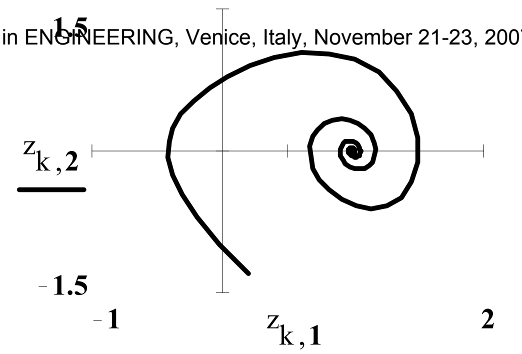

Fig.5- *A focus point*

5 Conclusions

In this work we presented some aspects in non-linear analysis of natural convection in a horizontal porous layer of infinite extent. The mathematical models describing the temperature distribution were solved by an iterative method as fourth_order Runge-Kutta method.

We presented simplified models for Navier-Stokes equations considering the thermal boundary-layer approximation for large Rayleigh numbers. The methods used to study the thermal boundary layer are similar to those from the theory of a viscous boundary layer. This similarity is formal because the physical significances of the model variables are different. A real interpretation of the physical significances must be carefully done. For example, in porous media the velocity near the wall is tangential and has a maximum value and in classical boundary-layer theory the velocity is zero at the wall.

We presented some examples for steady-state free convection in a porous medium adjacent to a vertical, semi-infinite flat wall. The problem of transient free convection with a step increase in wall temperature or surface heat flux is of great importance for the engineer. The complexity of the phenomena by singularities that appear in solution involves more computing resources and will be treated in some future works.

We limited presentation to numerical results obtained by a one-step method as fourth_order Runge-Kutta method. The bilocal problem was transformed in a Caughy problem. The numerical model was solved by an iterative procedure. The convergence of the method depends on the first initial approximation of Caughy problem.

A brief description of the numerical simulation of the solutions of Falkner-Skan boundary layers were presented for a particular velocity profiles.

References:

[1]. Belchami, Z., Brighi,B., Taous,K. On a family of differential equations for boundary layer approximations in porous medium. In: *Prepublications Mathematiques. Departement de Mathematiques, no. 9/99.* Universite de Metz.

[2]. Belchami, Z., Brighi, B., Taous, K. On the concave solutions of the Blasius equation. In: *Acta Math. Univ. Comenianae, vol. LXIX, 2(2000).* Pp.199-214

[3]. Guedda, M., Hammouch, Z., On similarity and pseudo-similarity solutions of Falkner-Skan boundary layers. In: *Fluid Dynamics Research 38 (2006).* The Japan Society of Fluid Mechanics and Elsevier B.V.

[4]. *** *Mathcad Professional 7. User's Manual.* MathSoft Inc. USA