# A ROBUST OPTIMAL STATE FEEDBACK CONTROL DESIGN

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*Abstract:* - The control of uncertain systems has become an area of recent interest. The basic control techniques have been modified to work well in presence of perturbations. This paper presents a geometric relationship of performance specifications to pole assignment. The design freedom has been used in optimizing various cost functions for performance. Robustness issues have been incorporated in the design for specified ranges of parametric perturbations. An example has been included to illustrate the procedure and effectiveness of robust control design.

*Key-Words:* - Parameter uncertainties, Robust control, Optimization, State-feedback control, Cost functions, Performance specifications.

## 1 Introduction

Much of the modern control theory approach evolves around problems involving uncertainty. These pertain to a system to be controlled and a mathematical model which includes uncertain quantities in terms of physical parameter whose values are not exactly specified. Thus, in engineering systems it is of fundamental importance that control systems be designed so that the stability and performance is preserved in the face of various uncertainties. This gives rise to the concept of robustness which has witnessed a rapid advancement over last three decades. Various control system design techniques such as the H-∞ approach [1, 2], the generalized Kharitanov approach [3, 4] and pole assignment approach [5, 6] have been developed and applied to many practical systems. The aim of this paper is to present a robust pole assignment control based on the graphical relation to performance. A state-feedback controller is obtained to satisfy several performance cost functions such as minimum perturbation, settling time, rise time, damping ratio and composite functions. The performance patterns under perturbed conditions are presented and analysed.

Given a system described in state-space form as

$$\dot{x} = Ax + Bu \qquad \text{…(1)}$$

$$Y = Cx \qquad \text{...(2)}$$

where A, B, and C have dimensions nxn, nxm, lxn, respectively. The parametric uncertainties may be represented in the system model in terms of variations in elements of A and B. It is well established that, if the pair (A, B) is completely controllable, a state-feedback control law of the form

$$u = r - Kx \qquad \text{...(3)}$$

where r is an mx1 vector of reference.inputs and $K$ is a constant mxn state-feedback gain matrix, can be used to assign the open loop system poles $\{\lambda_i\}$, $i$ = [1, $n$] to any desired set of locations $\{\gamma_i\}$, $i$ = [1, $n$] subject to complex pairing. The poles (or eigen values) of the resulting closed loop system for the state-feedback control will be given by the roots of its characteristic polynomial

$$\left| sI - A + BK \right| = 0 \qquad \text{...(4)}$$

## 2. Pole assignment

The pole assignment problem through a fixed feedback controller $K$ for an uncertain system has a closed loop characteristic polynomial of the form

$$p(s,q,k) = \sum_{i=0}^{n} a_i(q,k)s' \qquad \text{...(5)}$$

where $q$ is the vector representing parameter uncertainties in the system (A, B) and $k$ is the parameterization vector incorporating those coefficients of $K$ which are independent of desired pole locations. The pole assignment state-feedback control $K$ is considered as [7]

$$K = fm' \quad ...(6)$$

where $f$ is mxl and $m$ is 1xn vectors. The choice for $f$ and $m$ have been taken as

$$f = [1, k] \quad ...(7)$$

when $k$ is a parameter of feedback controller gain matrix $K$ and

$$m = [\phi_c']X^{-1}\delta \quad ...(8)$$

where

$$\phi_c = [Bf, ABf, ..., A^{n-1}Bf]; \quad ...(9)$$

$$X = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ a_1 & 1 & ... & | & | \\ | & a_1 & ... & | & | \\ a_{n-2} & a_{n-1} & ... & | & | \\ a_{n-1} & a_{n-2} & ... & a_1 & 1 \end{bmatrix} \quad ...(10)$$

and

$$\delta = [d_1 - a_1, d_2 - a_2, ..., d_n - a_n]' \quad (11)$$

where $d_i$ are the coefficients of closed loop characteristic polynomial and $a_i$ are the coefficients of open loop characteristic polynomial as $i$ varies from 1 to n.

For fixed feedback controller design, the value of $k$ has to be fixed in accordance to the desired system performance. To achieve this, different cost functions, depending upon the specifications of interest, could be used. These cost functions restrict the allowable region in s-plane for pole location so as to always meet the desired specification even in presence of extreme uncertainties. Few of these cost functions [6, 8] considered in this paper are discussed next.

Each perturbed pole is desired to stay with in a disc centered about the nominal pole, the diameter of which may be different for different poles i.e. the dominant poles may be allowed smaller disc diameters compared to farther poles. Thus for each nominal pole there is an associated cost function based on minimum perturbation

$$J_{1i} = \frac{\left| \nu_{ni} - \nu_{pi} \right|}{r_i} \quad ...(12)$$

where $\nu_{ni}$ is a vector from the origin to the *ith* nominal pole, $\nu_{pi}$ is that to the *ith* perturbed pole and $r_i$ is the maximum allowable radius of the corresponding disc as shown in fig.(l). This cost function is zero when the perturbed pole overlaps the corresponding nominal pole and is one when it is on the border of the specified disc. Thus cost functions. greater than one are undesirable.

The settling time is approximately inversely proportional to the real part of the dominant poles, the cost function based on maximum and minimum settling times can be expressed as

$$J_{2i} = \frac{\mathrm{Re}(\nu_{ni}) - \mathrm{Re}(\nu_{pi})}{\mathrm{Re}(\nu_{ni}) - \sigma_{ri}} \quad ...(13)$$

$$J_{3i} = \frac{\mathrm{Re}(\nu_{pi}) - \mathrm{Re}(\nu_{ni})}{\sigma_{li} - \mathrm{Re}(\nu_{ni})} \quad ...(14)$$

respectively, where $\sigma_{ri}$ and $\sigma_{li}$ show the location of the right and left boundaries corresponding to the desired settling time.

Restrictions may, as well, be placed on the imaginary parts of the dominant poles in terms of upper and lower bounds as the rise time could roughly be considered as inversely proportional to it. The cost functions corresponding to the rise time, thus, could be expressed as

$$J_{4i} = \frac{\mathrm{Im}(\nu_{ni}) - \mathrm{Im}(\nu_{pi})}{\mathrm{Im}(\nu_{ni}) - \omega_{li}} \quad ...(15)$$

$$J_{5i} = \frac{\mathrm{Im}(\nu_{pi}) - \mathrm{Im}(\nu_{ni})}{\omega_{ui} - \mathrm{Im}(\nu_{ni})} \quad ...(16)$$

respectively, where $\omega_{li}$ and $\omega_{ui}$ are lower and upper allowed bounds.

Damping ratio is one of the basic design criterion to be restricted in most of the control problems. The cost function associated with minimum and maximum damping ratios could be expressed as

$$J_{6i} = \frac{\alpha_{ni} - \alpha_{pi}}{\alpha_{ni} - \alpha_{li}} \quad ...(17)$$

$$J_{7i} = \frac{\alpha_{pi} - \alpha_{ni}}{\alpha_{ui} - \alpha_{ni}} \quad ....(18)$$

respectively, where $\alpha_{ni}$ and $\alpha_{pi}$ are the angles between the negative real axis and $\nu_{ni}$ and $\nu_{pi}$ respectively and $\alpha_{li}$ and $\alpha_{ui}$ are the lower and upper bounds of damping ratios.

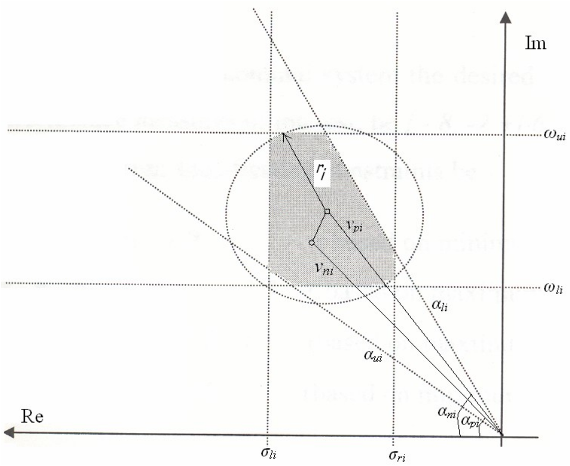


Fig.1 Allowable region for system poles

If some or all of these cost functions are to be considered simultaneously the concept of composite cost function could be applied as

$$J_i = \max_a (J_{ai}); \quad (a = [1.......7]) \quad ...(19)$$

depending upon the system specifications required to be met simultaneously.

**Controller Robustness :** The robustness has been included at the time of assigning the poles by optimizing a cost function. This cost function is a robustness measure and depends on the degree of variation of nominal system pole under specified perturbations.

## 3. Design Procedure

The design of the constant state-feedback robust controller for specified uncertainty bounds has been proposed through following steps:

1. Given a system (A, B) having some uncertain coefficients with specified bounds.
2. Given desired closed loop pole locations for prescribed performance and also other design constraints in terms of tolerance limits of performance specifications of interest.
3. Obtain the parameterized state-feedback controller gain *K* (which is a function of *k)* using equation (6).
4. The value of parameter *k* for optimum robustness is determined by optimizing the cost function corresponding to the performance specification of interest as per equations (12-19).
5. The value of constant feedback controller *K* is obtained for the optimum *k* determined in step (4) above.
6. The closed loop performance may be evaluated for validating the robustness of the proposed controller.

## 4. Example

Consider a open-loop system given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & q_1 & 0 \\ q_2 & 1 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 1 & q_3 \\ 0 & 1 \end{bmatrix} \quad \ldots(20)$$

where $q_i$ are interval uncertain parameters bounded as

$$\begin{aligned} &2.8 \le q_1 \le 3.2 \\ &4.5 \le q_2 \le 5.5 \qquad \ldots(21) \\ &2.95 \le q_3 \le 3.05 \end{aligned}$$

and have nominal values $q_1$ = 3, $q_2$ = 5 and $q_3$ = 3. The given open-loop system is unstable as its poles are {1, 2, 3}. The desired locations of closed loop poles, as per performance considerations, under nominal working conditions is {-8, -2+j6.5, -2-j6.5}. In addition, other design constraints be

$$\begin{aligned} &r_1 = 3.0, r_2 = r_3 = 1.0 \\ &\sigma_{r1} = -6, \sigma_{r2} = \sigma_{r3} = -1.5 \\ &\alpha_{12} = \alpha_{13} = \arctan[2.5] \\ &\alpha_{u2} = \alpha_{u3} = \arctan[5] \end{aligned} \quad ..(22)$$

The allowable region for poles of the example considered owing to the restrictions as mentioned in Eq.(22) is shown in Fig. 2. For the given uncertain system, under nominal working conditions, the parameterized state-feedback gain matrix is obtained as

$$K = \begin{bmatrix} \frac{3733k+921}{312k^2+200k+32} & \frac{-1087k+576}{312k^2+200k+32} & \frac{2959}{312k^2+200k+32} \\ \frac{3733k^2+921k}{312k^2+200k+32} & \frac{-1087k^2+576k}{312k^2+200k+32} & \frac{2959k}{312k^2+200k+32} \end{bmatrix} \quad .(23)$$

The variations in the respective cost functions, based on the design constraint of concern w.r.t. to *k* are shown in Fig. 3. Optimum value of *k,* for satisfying particular stated constraint, is that where the related cost function has the minimum value. The range of *k* for which the cost function is less then one is permissible range of $k$. The optimum value and the permissible range of $k$ for different stated

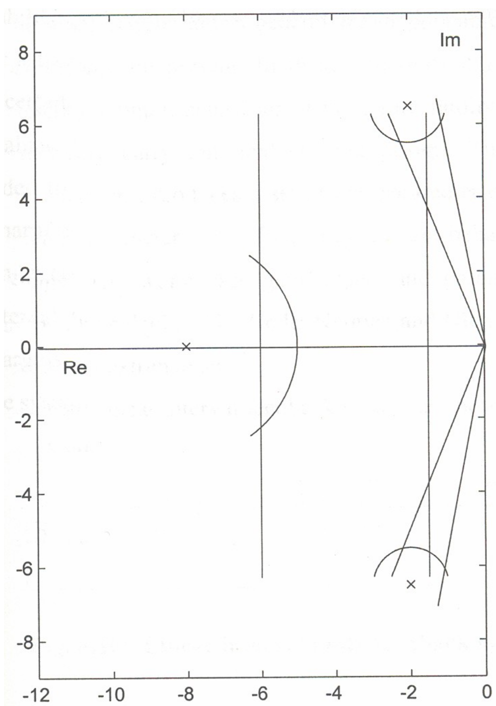


Fig.2 Allowable region for poles of design example

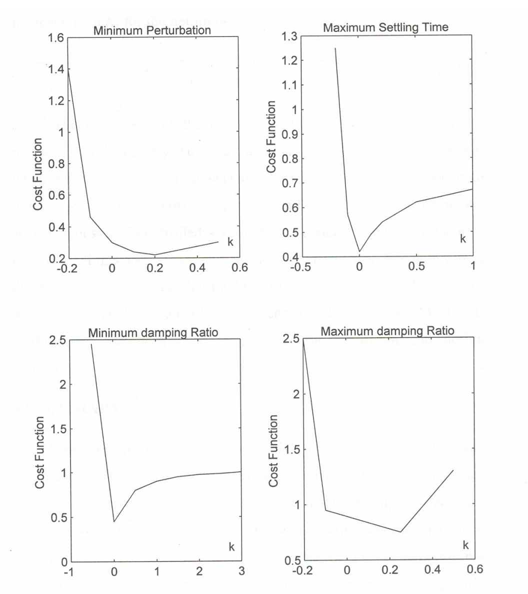


Fig. 3 Plots of costs functions

design constraints given in the problem are tabulated in Table 1.

**Table 1**

| Design Constraint | Optimum k | Permissible Range |
|---|---|---|
| Minimum perturbation | 0.2 | ≥ 0.2 |
| Settling Time | 0 | -0.2 to 3.0 |
| Damping Ratio (min.) | 0 | ≥ - 0.2 |
| Damping Ratio (max) | 0.25 | - 0.1 to 0.35 |

Selecting the optimum value of $k$ as 0.25, satisfying all the design constraints, the pole spread of closed loop system is shown in Fig. (4).

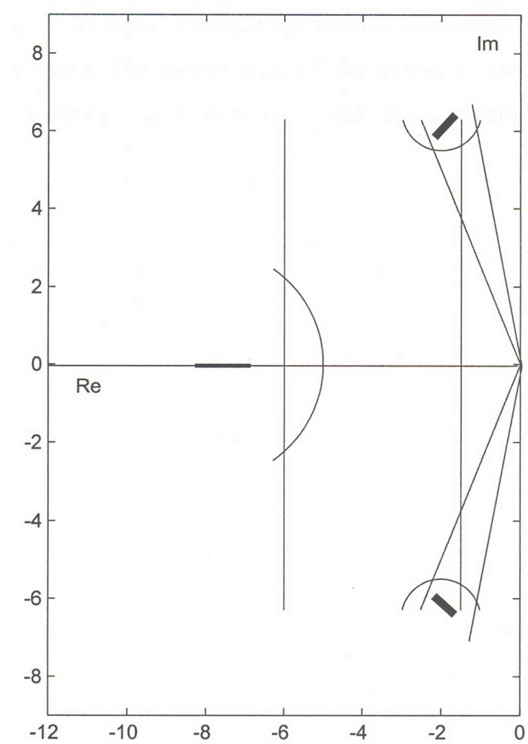


Fig.4 Pole spread diagram for design example

The constant state-feedback gain $K$ for optimum value of $k$ is determined as

$$K = \begin{bmatrix} 18.2685 & 2.9975 & 51.0172 \\ 4.5671 & 0.7494 & 12.7543 \end{bmatrix} \quad \text{.....(24)}$$

The robustness of the proposed controller for the given design problem is validaled by studying the behaviour of the state trajectories when the system (A, B) is subjected to various perturbations as specified in the problem. These plots are presented in Fig. 5. The MATLAB environment has been used to program the algorithm and design simulations.

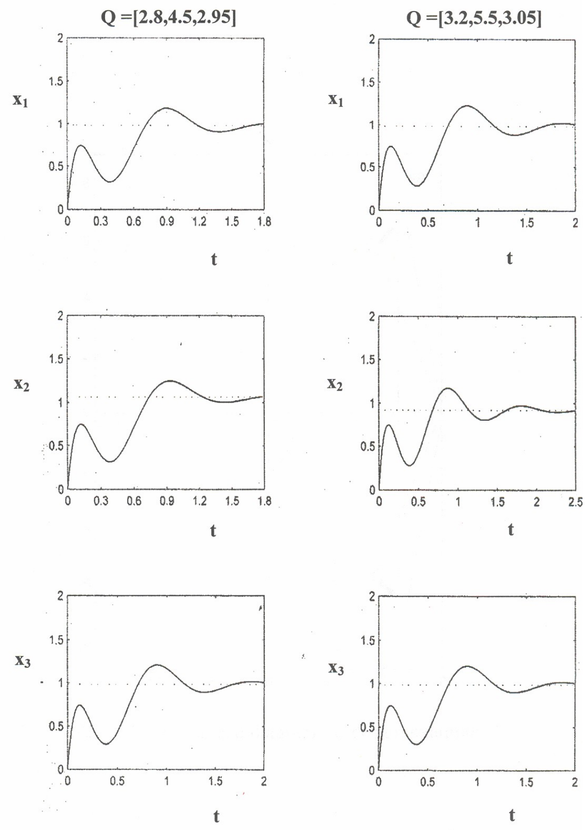


Fig. 5 State plot for design example

## 5. Conclusion

The state-feedback control is a potential design method using pole placement. The pole assignment is not sufficient to provide the desired performance in the presence uncertainties. This paper presents a control design method, which incorporates the uncertainties and the feedback gain is parameterized. The design freedom in pole assignment has been utilized in selecting the gain parameter by optimizing the performance measures. The optimum state-feedback control would provide the robustness for given range of uncertainties. This design procedure is illustrated with the help of an example. The results demonstrate the effectiveness and simplicity of the proposed method. This method can be extended to discrete systems with simple modifications.

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