Joint Structural Importance in Consecutive-*k* Systems

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Abstract: - The *joint structural importance* (JSI) is an important measure of how two components interact in contributing to the system reliability. The value of JSI is positive (negative) if and only if one component becomes more important (less important) when the other works. A *consecutive-k-out-of-n system* is a linear arrangement of n components such that the system is failed if and only if some consecutive k components are all failed. In this paper, we study joint structural importance $JSI(i, j)$ in the consecutive-*k*-out-of-*n* system.

We completely solve $JSI(i, j)$ for $k=1$ (the series system), $k=n$ (the parallel system), $k=n-1$, and $k=n-2$, respectively. For the other k, we prove that $JSI(1, j') = JSI(1, k) < 0 < JSI(1, n) = JSI(1, k+2) < JSI(1, j) < JSI(1, k+1)$, for $2 \le j' \le k-1$ and $k+3 \le j \le n-1$. For a fixed i, we prove that the graph of $JSI(i, j)$ has a W-shape property for $\max\{1, i-k-1\} \le j \le \min\{n, i+k+1\}$ with $JSI(i, i) = 0$. We also present exact formula for $JSI(i, j)$ and obtain many relations among them.

Key-Words: Joint structural importance, Consecutive-*k*-out-of-*n* system, Reliability, Birnbaum importance.

1 Introduction

The *Birnbaum reliability importance* $I(i)$ of a component i measures the improvement of the system reliability $R(P)$ over the improvement of the reliability p_i of that component [2]. The Birnbaum reliability importance $I(i)$ is defined as

$$I(i) = \partial R(P) / \partial p_i = R(1_i, P_i) - R(0_i, P_i),$$

where P_i denote all the component reliability except that of the component i, $R(1_i, P_i)$ and $R(0_i, P_i)$ are the system reliability with component i working and failed, respectively. Note that the system reliability $R(P)$ can be computed only when the reliabilities of all components are well defined. Without the information of component reliabilities, we need to know the relative importance of the locations so that more reliable components can be assigned to the more important locations to maximize the system reliability. The *structural Birnbaum importance* is a special kind of the Birnbaum reliability importance where all the component reliabilities are set equal to p so that the importance measure will depend only on the structure of a system. Many kinds of structural Birnbaum importance indices have been discussed [3-6, 16-18, 20, 21].

The *joint reliability importance* $JRI(i, j)$ of two components i and j measures how these two components in a system interact in contributing to the system reliability $R(P)$. The joint reliability importance $JRI(i, j)$ is defined as follows.

$$JRI\,(i, j) = \partial^2 R(P) / \partial p_i \partial p_j = R(1_i, 1_j, P_{i,j}) + R(0_i, 0_j, P_{i,j}) - R(1_i, 0_j, P_{i,j}) - R(0_i, 1_j, P_{i,j}),$$

where the $P_{i,j}$ will be omitted when no confusion is possible. Joint reliability importance was first proposed independently by Hagstrom [10] and by Hong and Lie [13]. Based on the definition of Birnbaum reliability importance, $JRI(i, j)$ can be interpreted as the change of the Birnbaum reliability importance of component i caused by component j's deteriorating from working to failed. The value of $JRI(i, j)$ is positive (negative) if and only if one component becomes more important (less important) when the other works. Similar to the problem in Birnbaum reliability importance, many researchers have studied *joint structural importance* (setting all $p_i = p$) in many systems: the fault tree [11], the two-terminal system [1, 13, 19], the *k*-out-of-*n* system [12, 15], etc. Jan first studied joint structural importance in the consecutive-2-out-of-*n* systems [15].

A *consecutive-k-out-of-n system* consists of an ordered sequence of n components where the system fails if and only if any k consecutive components are all failed. Relative to low reliability of a series system and high reliability but very expensive hardware of the parallel system, the consecutive-*k* system has attracted many researchers [5-9, 15-18, 21]. For the consecutive-2 system, Malon [17] and Du and Hwang [9] independently solved a problem called the invariant optimal assignment, which has the problem of finding the rank of structural Birnbaum importance as one of its special cases. Malon [18] also solved the assignment problem for $k = n-1, n-2$, and proved that an invariant optimal assignment does not exist for $3 \le k \le n-3$, which intensified the need to compare structural Birnbaum importance.

In this paper, we concentrate on the joint structural importance in the consecutive-k system. We first state several useful formulas for computing system reliability in Section 2. Then we study joint structural importance, denoted by $JSI(i, j)$, in the consecutive-k system in Section 3. We completely solve $JSI(i, j)$ in the consecutive-k-out- of-n system for $k=1, n-2, n-1, n$. For the other k, we first study the case of $i=1$ and make comparisons among $JSI(1, j)$. On the other hand, given a fixed i, we compare the values of $JSI(i, j)$ and present the W-shape property of the JSI function. We also find the exact formula for some $JSI(i, j)$ and discuss the relationship among them. Finally, we make a conclusion in Section 4.

2 System Reliability

For a consecutive-k-out-of-n system, let $R(n)$ denote the system reliability, and $R_n(1_i)$ and $R_n(0_i)$ denote the reliabilities of the system where the component i is working and failed, respectively. Note that $R(k)=1-q^k$ and $R(n)=1$ for $0 \le n \le k-1$, where $q=1-p$.

Lemma 2.1. (*See* [14]) *For* $n \ge k+1$, *the reliability* $R(n)$ *of a consecutive-k-out-of-n system satisfies the following recursive relations.*

(i) $R(n)=\sum_{m=1}^{k} pq^{m-1}R(n-m)$.

(ii) $R(n)=R(n-1)-pq^k R(n-k-1)$.

Note that for $n \ge k$, $R(n)$ is decreasing in n.

Corollary 2.2. $R(n)=[R(n+k)-R(n+k+1)]/pq^k$.

Proof. By Lemma 2.1 (ii). □

For convenience, by Corollary 2.2, we backward derive the reliabilities of a consecutive-k-out-of-n system for $-k-1 \le n \le -1$.

Definition 2.3. *The reliability of consecutive-k-out-of-n system for* $-k-1 \le n \le -1$ *is*

$$R(n)=\begin{cases} 1/p, & n=-1, \\ 0, & -k \le n \le -2, \\ 1/p^2q^{k-1}, & n=-k-1. \end{cases}$$

Lemma 2.4.

(i) $R(n)=pR_n(1_i)+qR_n(0_i)$.

(ii) $R_n(1_i)=R(i-1)R(n-i)$.

(iii) $R_n(0_i)=[R(n)-pR(i-1)R(n-i)]/q$.

Proof. Statements (i) and (ii) follows from the definition of the system reliability according to component i working or not. Statement (iii) follows immediately from (i) and (ii). □

Lemma 2.5. (*See* [20])

$$pR(i-1)R(n-i) < R(n) < R(i)R(n-i).$$

Lemma 2.6. $R(i)R(j) > R(i+j+1)$.

Proof. The difference between $R(i)R(j)$ and $R(i+j+1)$ is just the reliability of the cases that both the first i-component subsystem and the last j-component subsystem are working and the whole $(i+j+1)$-component system is failed due to the $(i+1)$-st component is failed. Hence

$$R(i)R(j)-R(i+j+1)$$
$$=\sum_{l=0}^{k-1} pq^l R(i-1-l)\sum_{m=k-l-1}^{k-1} pq^m R(j-1-m)>0. \quad □$$

3 Joint Structural Importance

In this section, we first consider the joint structural importance in the consecutive-k system for $k=1, n-2, n-1, n$. Note that a consecutive-k system is a series system for $k=1$, and a parallel system for $k=n$.

Theorem 3.1. *Consider a consecutive-k-out-of-n system.*

(i) *For* $k=1$ (*the series case*), $JSI(i, j)=p^{n-2}>0$ *for any* $i \ne j$.

(ii) *For* $k=n$ (*the parallel case*), $JSI(i, j)=-q^{n-2}<0$ *for any* $i \ne j$.

Proof. By definition, $JSI(i, j)=R(1_i, 1_j)-R(1_i, 0_j)-R(0_i, 1_j)+R(0_i, 0_j)$.

(i) For $k=1$, $JSI(i, j)=p^{n-2}-0-0+0=p^{n-2}>0$.

(ii) For $k=n$, $JSI(i, j)=1-1-1+(1-q^{n-2})=-q^{n-2}<0$. □

Theorem 3.2. *Suppose* $k=n-1$. $JSI(i, j)<0$ *for all* $i \ne j$, *except* $JSI(1, n)>0$.

Proof. By definition,

(i) $JSI(1, n)=1-(1-q^{n-2})-(1-q^{n-2})+(1-q^{n-2})=q^{n-2}>0$.

(ii) For $j \ne n$, $JSI(1, j)=1-(1-q^{n-2})-1+(1-pq^{n-3}-q^{n-2})=-pq^{n-3}<0$.

(iii) For $1<i<j<n$, $JSI(i, j)=1-1-1+(1-2pq^{n-3}-q^{n-2})=-2pq^{n-3}-q^{n-2}<0$.

Hence, in a consecutive-(n–1)-out-of-n system, $JSI(i, j)<0$ for all $i \ne j$, except $JSI(1, n)>0$. □

Theorem 3.3. *Suppose* $k=n-2$. $JSI(i, j)<0$ *for all* $i \ne j$, *except* $JSI(1, n)=0$, $JSI(1, n-1)>0$, *and* $JSI(2, n-1)$ *is positive, negative, and zero for* $p<1/2$, $p>1/2$, *and* $p=1/2$, *respectively.*

Proof. $JSI(i, j)$ are computed according to seven cases as follows. For the first three cases, we compute $JSI(i, j)$ by $JSI(i, j)=R(1_i, 1_j)-R(1_i, 0_j)-R(0_i, 1_j)+R(0_i, 0_j)$.

(i) $JSI(1, 2)=(1-q^{n-2})-(1-q^{n-3})-(1-q^{n-2})+(1-q^{n-4})=-pq^{n-4}<0$.

(ii) For $3 \le j \le n-2$, $JSI(1, j)=1-1-(1-q^{n-3}-pq^{n-3})+(1-q^{n-4}-pq^{n-3})=-pq^{n-4}<0$.

(iii) $JSI(1, n-1)=1-(1-q^{n-3}-pq^{n-3})-(1-q^{n-3})+(1-q^{n-3}-pq^{n-3})=q^{n-3}>0$.

In the following cases, we compute $JSI(i, j)$ by $JSI(i, j)=[R(1_i, 1_j)-R(0_i, 1_j)]-[R(1_i, 0_j)-R(0_i, 0_j)]$.

(iv) $JSI(1, n)=pq^{n-3}-pq^{n-3}=0$.

(v) For $3 \le j \le n-2$, $JSI(2, j)=0-(pq^{n-4}+pq^{n-3})<0$.

(vi) $JSI(2, n-1)=q^{n-3}-p(q^{n-3}+pq^{n-4})=q^{n-4}(1-2p)$. Thus $JSI(2, n-1)$ is positive, negative, and zero for $p<1/2$, $p>1/2$, and $p=1/2$, respectively.

(vii) For $2 < i < j < n-1$, $JSI(i, j) = 0 - (q^{n-4} + 2pq^{n-4}) < 0$. □

In the following, we state some properties of joint structural importance.

Lemma 3.4. *Consider a consecutive-k-out-of-n system.*

(i) $JSI(i, j) = JSI(n-i+1, n-j+1)$.

(ii) $JSI(i, j) = JSI(j, i)$.

(iii) $JSI(i, i) = 0$.

(iv) $JSI(i, j) = 0$ *for* $1 \le i, j \le n < k$.

Proof. Since a consecutive-*k*-out-of-*n* system is symmetric with respective to the middle location(s), we have Statement (i). Statements (ii) and (iii) follows immediately the definition of joint structural importance. If $n < k$, then $R(1_i, 1_j) = R(0_i, 1_j) = R(1_i, 0_j) = R(0_i, 0_j) = 1$ and thus $JSI(i, j) = 0$. □

By Lemma 3.4 (iv), throughout this thesis, we discuss JSI in the consecutive-*k*-out-of-*n* system for $n \ge k$. Given a fixed i, the following lemma simplifies the calculation of the difference between $JSI(i, l)$ and $JSI(i, j)$.

Lemma 3.5. *Given a fixed* i, *the difference between* $JSI(i, l)$ *and* $JSI(i, j)$ *is* $JSI(i,l) - JSI(i, j) = R(1_i, 0_j, 1_l) - R(1_i, 1_j, 0_l) + R(0_i, 1_j, 0_l) - R(0_i, 0_j, 1_l)$.

Proof. By the definition of JSI, we have

$$\begin{aligned} &JSI(i, l) - JSI(i, j) \\ &= [R(1_i, 1_l) - R(1_i, 1_j)] + [R(1_i, 0_j) - R(1_i, 0_l)] \\ &\quad + [R(0_i, 1_j) - R(0_i, 1_l)] + [R(0_i, 0_l) - R(0_i, 0_j)]. \end{aligned}$$

Extend each term to include all the reliabilities of three components. We have

$$\begin{aligned} &JSI(i, l) - JSI(i, j) \\ &= \{[pR(1_i, 1_j, 1_l) + qR(1_i, 0_j, 1_l)] - [pR(1_i, 1_j, 1_l) \\ &\quad + qR(1_i, 1_j, 0_l)]\} + \{[pR(1_i, 0_j, 1_l) + qR(1_i, 0_j, 0_l)] \\ &\quad - [pR(1_i, 1_j, 0_l) + qR(1_i, 0_j, 0_l)]\} + \{[pR(0_i, 1_j, 1_l) \\ &\quad + qR(0_i, 1_j, 0_l)] - [pR(0_i, 1_j, 1_l) + qR(0_i, 0_j, 1_l)]\} \\ &\quad + \{[pR(0_i, 1_j, 0_l) + qR(0_i, 0_j, 0_l)] - [pR(0_i, 0_j, 1_l) \\ &\quad + qR(0_i, 0_j, 0_l)]\} \\ &= R(1_i, 0_j, 1_l) - R(1_i, 1_j, 0_l) + R(0_i, 1_j, 0_l) - R(0_i, 0_j, 1_l). \end{aligned}$$

□

In a consecutive-2 system, Jan [15] proved that $JSI(1, 2) < 0$ and $JSI(1, j) > 0$ for $3 \le j \le n$. She also proved that $JSI(1, j) = JSI(1, n-j+4)$ in the consecutive-2-out-of-*n* system. In the following, we extend these results to the consecutive-*k* system. First, consider $i = 1$.

Theorem 3.6.

(i) *For* $2 \le j \le k$, $JSI(1, j) = -pq^{k-2}R(n-k-1) < 0$.

(ii) *For* $j = k+1$, $JSI(1, j) > 0$. *Furthermore,* $JSI(1, k+1) = q^{k-1}R(n-k-1)$ *for* $n > 2k$, *and* $JSI(1, k+1) = q^{k-1}$ *for* $k+1 \le n \le 2k$.

(iii) *For* $j \ge k+2$, $JSI(1, j) = pq^{k-2}[R(j-k-2) \times R(n-j) - R(n-k-1)]$. *Furthermore,* $JSI(1, j) > 0$ *for* $n > 2k$, *and* $JSI(1, j) = 0$ *for* $k+2 \le j \le n \le 2k$.

Proof. For $2 \le j \le k$, by definition,

$$\begin{aligned} JSI(1, j) &= [R(1_1, 1_j) - R(0_1, 1_j)] - [R(1_1, 0_j) - R(0_1, 0_j)] \\ &= 0 - pq^{k-2}R(n-k-1) < 0. \end{aligned}$$

For $j \ge k+1$, by definition and Lemma 2.4 (iii),

$$\begin{aligned} &JSI(1, j) \\ &= pq^{k-1}R(j-k-2)R(n-j) - pq^{k-1}R_{n-k-1}(0_{j-k-1}) \\ &= pq^{k-2}[R(j-k-2)R(n-j) - R(n-k-1)]. \end{aligned}$$

Consider $j = k+1$. $JSI(1, k+1) = q^{k-1}R(n-k-1) > 0$ for $n > 2k$, and $JSI(1, k+1) = q^{k-1} > 0$ for $k+1 \le n \le 2k$. On the other hand, if $k+2 \le j \le n \le 2k$, then $JSI(1, j) = 0$; and if $n > 2k$, then $JSI(1, j) > 0$ by Lemma 2.6. □

Theorem 3.7. $JSI(1, j) = JSI(1, n+k+2-j)$ *for* $k+2 \le j \le n$.

Proof. Immediately from Theorem 3.6 (iii). □

Theorem 3.7 shows that the JSI of the first and the j-th locations is equal to that of the first and the $(n+k+2-j)$-th locations. Thus we can discuss $JSI(1, j)$ only for $j \le (n+k+2)/2$. In the following, we find an upper bound and a lower bound of $JSI(1, j)$ for $j \ge k+2$.

Theorem 3.8. *For* $j \ge k+3$,

$JSI(1, n) = JSI(1, k+2) < JSI(1, j) < JSI(1, 1+k)$.

Proof. By Theorem 3.6 (ii) and (iii), for $j \ge k+2$,

$$\begin{aligned} &JSI(1, k+1) - JSI(1, j) \\ &= q^{k-2}[R(n-k-1) - pR(j-k-2)R(n-j)] \\ &= q^{k-1}R_{n-k-1}(0_{j-k-1}) > 0. \end{aligned}$$

Similarly, by Theorem 3.6 (iii), for $j \ge k+2$,

$$\begin{aligned} &JSI(1, j) - JSI(1, n) \\ &= pq^{k-2}[R(j-k-2)R(n-j) - R(n-k-2)]. \end{aligned}$$

By Lemma 2.5, $JSI(1, j) > JSI(1, n)$ for $j \ge k+2$, and $JSI(1, k+2) = JSI(1, n)$ by Theorem 3.7. Hence $JSI(1, n) = JSI(1, k+2) < JSI(1, j) < JSI(1, 1+k)$ for $j \ge k+3$. □

Corollary 3.9. *For* $2 \le j \le \min\{i+k+1, n\}$ *and* $i = 1$, *the rank of* $JSI(1, j)$ *are as follows.* $JSI(1, j') = JSI(1, k) < 0 < JSI(1, n) = JSI(1, k+2) < JSI(1, j) < JSI(1, k+1)$, *for* $2 \le j' \le k-1$ *and* $k+3 \le j \le n-1$.

Proof. By Theorems 3.6-3.8. □

Figure 1 shows a graph of the joint structural importance $JSI(i, j)$ for $i = 1$.

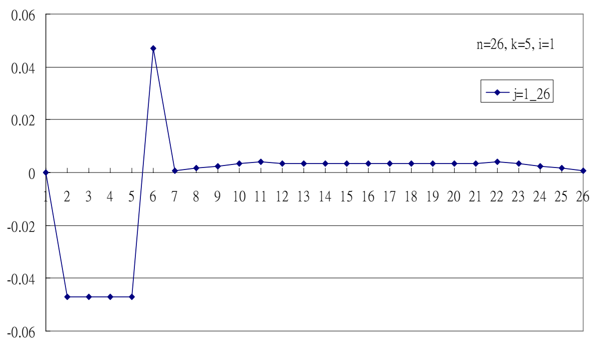

Fig. 1. $JSI(i, j)$ for $n = 26$, $k = 5$, $i = 1$, and $1 \le j \le 26$.

In the following, we study $JSI(i, j)$ for $i > 1$. First, consider $\max\{1, i-k+1\} \le j \le i-1$.

Theorem 3.10. *For* $2 \le j < i \le k+1$, $JSI(i, j-1) - JSI(i, j) = pq^{k-2}R(n-j-k)$. *Furthermore,* $JSI(i, j-1) > JSI(i, j)$ *for* $n \ge j+k-1$ *and* $JSI(i, j-1) = JSI(i, j)$ *for* $n \le j+k-2$.

Proof. For $2 \le j < i \le k+1$, by Lemma 3.5,

$$\begin{aligned} &JSI(i, j-1) - JSI(i, j) \\ &= [R(1_i, 1_{j-1}, 0_j) - R(1_i, 0_{j-1}, 1_j)] \\ &\quad + [R(0_i, 0_{j-1}, 1_j) - R(0_i, 1_{j-1}, 0_j)] \\ &= 0 + pq^{k-2}R(n-j-k). \end{aligned}$$

Hence we have Theorem 3.10 immediately. □

Theorem 3.11. *Suppose* $1 \le i, j \le k$ *and* $j \ne i$.

(i) *For* $n \le i+k-1$,

$$JSI(i, j) = \begin{cases} -jpq^{k-2}, & 1 \le j \le n-k, \\ -(n-k+1/p)pq^{k-2}, & n-k+1 \le j \le k. \end{cases}$$

(ii) *For* $i+k \le n \le 2k$, $JSI(i, j) = -\min\{i, j\}pq^{k-2}$.

(iii) *For* $n > 2k$, $JSI(i, j) = -pq^{k-2}\sum_{l=1}^{j} R(n-k-l)$.

Proof. Consider $1 \le i < j \le k \le n$. By definition,

$$JSI(i, j) = -pq^{k-2}\sum_{l=1}^{j} R(n-k-l).$$

If $i+k \le n \le 2k$, then $0 \le n-k-l \le k-1$ and thus $JSI(i, j) = -ipq^{k-2}$. If $k \le n \le i+k-1$, we have $n-k-1 \ge -1 \ge n-k-i$ and $n-i+1 \le k$. Since $R(-1) = 1/p$ and $R(n) = 1$ for $0 \le n \le k-1$, we have $JSI(i, j) = -(n-k+1/p)pq^{k-2}$ for $i+1 \le j \le k$. Moreover, by Lemma 3.4 (i), $JSI(i, j) = JSI(n-i+1, n-j+1) = JSI(n-i+1, j') = JSI(i, n-j'+1)$ for $n-i+2 \le j' \le k$, i.e., $i-1 \ge n-j'+1 \ge n-k+1$. Hence if $n \le i+k-1$, then

$$JSI(i, j) = -(n-k+\tfrac{1}{p})pq^{k-2} \text{ for } n-k+1 \le j \le k.$$

On the other hand, consider $1 \le j < i \le k$. By Theorems 3.6 (i) and 3.10,

$$\begin{aligned} JSI(i, j) &= JSI(i, 1) - \textstyle\sum_{m=2}^{j}[JSI(i, m-1) - JSI(i, m)] \\ &= -\textstyle\sum_{m=1}^{j} pq^{k-2}R(n-m-k). \end{aligned}$$

If $n \le i+k-1$ and $1 \le j \le n-k$, we have $JSI(i, j) = -jpq^{k-2}$; and if $i+k \le n \le 2k$, we have $JSI(i, j) = -jpq^{k-2}$. Hence Theorem 3.11 is proved. □

Remark 3.12. *In Theorem 3.11, we prove that the values of* $JSI(i, j)$ *is linear for fixed* $i \le k$, $n \le 2k$, *and* $1 \le j \le \min\{n-k, i\}$. *For* $n > 2k$, $JSI(i, j)$ *is no more a multiple of* $JSI(i, 1)$ *since* $R(n-k-j) \ne 1$.

Theorem 3.13. *For fixed* n *and* k, *if* $i \ne j$ *and* $n-k+1 \le i, j \le k$, *then*

$$JSI(i, j) = -(n-k+1/p)pq^{k-2}.$$

Proof. Immediately follow from Theorem 3.11 which depends only on n, k, and p. □

Theorem 3.14. *Suppose* $k+2 \le i < 2k$.

(i) $JSI(i, j-1) > JSI(i, j)$ *for* $k+1 \le j \le i-1$. *Furthermore,*

$$\begin{aligned} &JSI(i, j-1) - JSI(i, j) \\ &= pq^{k-2}R(j-2)R(n-j-k) \\ &\quad + pq^{k-2}R(j-k-2)[R(n-i) - R(n-j)]. \end{aligned}$$

(ii) $JSI(i, j-1) > JSI(i, j)$ *for* $i-k+1 \le j \le k$. *Furthermore,*

$$JSI(i, j-1) - JSI(i, j) = pq^{k-2}R(n-j-k) > 0.$$

Proof. (i) By Lemmas 2.4 (iii) and 3.5, for $k+1 \le j \le i-1$,

$$\begin{aligned} &JSI(i, j-1) - JSI(i, j) \\ &= pq^{k-1}R(j-k-2)R(n-i) \\ &\quad + \{[R(j-2) - pq^{k-1}R(j-k-2)]pq^{k-2}R(n-j-k) \\ &\quad - pq^{k-1}R(j-k-2)[R_{n-j}(0_{i-j}) - pq^{k-2}R(n-j-k)]\} \\ &= pq^{k-2}R(j-2)R(n-j-k) \\ &\quad + pq^{k-2}R(j-k-2)[R(n-i) - R(n-j)]. \end{aligned}$$

(ii) By Lemma 3.5, for $i-k+1 \le j \le k$, $JSI(i, j-1) - JSI(i, j) = 0 + pq^{k-2}R(n-j-k) > 0$. □

Theorem 3.15. *Suppose* $i \ge 2k$. $JSI(i, j-1) > JSI(i, j)$ *for* $i-k+1 \le j \le i-1$. *Furthermore,*

$$\begin{aligned} &JSI(i, j-1) - JSI(i, j) \\ &= pq^{k-2}R(j-2)R(n-j-k) \\ &\quad + pq^{k-2}R(j-k-2)[R(n-i) - R(n-j)]. \end{aligned}$$

Proof. Similar to the proof of Theorem 3.14 (i) □

Corollary 3.16. *Given a fixed* i, $JSI(i, j)$ *is decreasing for* $\max\{1, i-k\} \le j \le i-1$.

Proof. By Theorems 3.10-3.15. □

In the following, we consider $i+1 \le j \le \min\{i+k, n\}$.

Theorem 3.17. *Suppose* $i < k$. $JSI(i, j) < JSI(i, j+1)$ *for* $k \le j \le \min\{i+k-1, n-1\}$. *Furthermore,*

$$\begin{aligned} &JSI(i, j+1) - JSI(i, j) \\ &= pq^{k-2}R(j-k-1)R(n-j-1) \\ &\quad + pq^{k-2}R(n-j-k-1)[1 - R(j-1)]. \end{aligned}$$

In fact, $JSI(i, k+1) - JSI(i, k) = q^{k-2}R(n-k-1)$.

Proof. By Lemma 3.5, for $i < k$ and $k+1 \le j \le \min\{i+k-1, n-1\}$,

$$\begin{aligned} &JSI(i, j+1) - JSI(i, j) \\ &= pq^{k-1}R(n-j-k-1) + \{pq^{k-2}R(j-k-1) \\ &\quad \times [R(n-j-1) - pq^{k-1}R(n-j-k-1)] \\ &\quad - pq^{k-1}R(n-j-k-1)[R_{j-1}(0_i) - pq^{k-2}R(j-k-1)]\} \\ &= pq^{k-2}R(j-k-1)R(n-j-1) \\ &\quad + pq^{k-2}R(n-j-k-1)[1 - R(j-1)]. \end{aligned}$$

Note that $R(-1) = 1/p$. For $i < k = j$, $JSI(i, j+1) - JSI(i, j) = q^{k-2}R(n-j-1) > 0$. Hence, for $i < k$ and $k \le j \le \min\{i+k-1, n-1\}$, $JSI(i, j) < JSI(i, j+1)$. □

Corollary 3.18. *Suppose* $i < k$. *For* $k+1 \le j \le \min\{i+k, n\}$,

$$\begin{aligned} &JSI(i, j) \\ &= pq^{k-2}\textstyle\sum_{m=1}^{j-k} R(m-2)R(n-k-m) \\ &\quad + p^2q^{2k-2}\textstyle\sum_{m=2}^{j-k} R(n-2k-m)\sum_{l=2}^{m} R(m-l-1) \\ &\quad + \begin{cases} -(n-k+1/p)pq^{k-2}, & n \le i+k-1, \\ -ipq^{k-2}, & i+k \le n \le 2k, \\ -pq^{k-2}\sum_{l=1}^{k} R(n-k-l), & n > 2k. \end{cases} \end{aligned}$$

Proof. Note that $JSI(i, j) = JSI(i, k) + \sum_{m=k}^{j-1}[JSI(i, m+1) - JSI(i, m)]$. Corollary 3.18 follows immediately from Theorems 3.11 and 3.17. □

Theorem 3.19. *Suppose* $i \ge k$. $JSI(i, j) < JSI(i, j+1)$ *for* $i+1 \le j \le \min\{i+k-1, n-1\}$. *Furthermore,*

$$\begin{aligned} &JSI(i,\ j+1)-JSI(i,\ j)\\ &= pq^{k-2}R(j-k-1)R(n-j-1)\\ &+ pq^{k-2}R(n-j-k-1)[R(i-1)-R(j-1)]. \end{aligned}$$

Proof. By Lemma 3.5, for $i \ge k$ and $i+1 \le j \le \min\{i+k-1, n-1\}$,

$$\begin{aligned} &JSI(i,\ j+1)-JSI(i,\ j)\\ &= pq^{k-1}R(i-1)R(n-j-k-1)+\{pq^{k-2}R(j-k-1)\\ &\times[R(n-j-1)-pq^{k-1}R(n-j-k-1)]\\ &-pq^{k-1}R(n-j-k-1)[R_{j-1}(0_i)-pq^{k-2}R(j-k-1)]\}\\ &= pq^{k-2}R(j-k-1)R(n-j-1)\\ &+pq^{k-2}R(n-j-k-1)[R(i-1)-R(j-1)]. \end{aligned}$$

Hence $JSI(i,\ j) < JSI(i,\ j+1)$ for $i \ge k$ and $i+1 \le j \le \min\{i+k-1, n-1\}$. $\square$

Corollary 3.20. *Given a fixed i, $JSI(i,\ j)$ is nondecreasing for $i+1 \le j \le \min\{i+k, n\}$.*

Proof. By Theorems 3.11, 3.17, and 3.19. $\square$

In the following, we make two more comparisons.

Theorem 3.21. *$JSI(i, i+k) > JSI(i, i+k+1)$ for $n \ge i+2k$.*

Proof. By Lemmas 2.4 (iii) and 3.5,

$$\begin{aligned} &JSI(i, i+k)-JSI(i, i+k+1)\\ &=\{R(i-1)q^{k-1}[R(n-i-k-1)-pq^{k-1}R(n-i-2k-1)]\\ &-R(i-1)(1-q^{k-1})pq^{k-1}R(n-i-2k-1)\}\\ &+R_{i+k-1}(0_i)pq^{k-1}R(n-i-2k-1)\\ &=q^kR(i-1)R_{n-i-k-1}(0_k)\\ &+pq^{k-1}R_{i+k-1}(0_i)R(n-i-2k-1). \end{aligned}$$

$\square$

Theorem 3.22. *For $n \ge i+2k-1$ and $i \ge k$, $JSI(i, i+k) > 0$. Furthermore,*

$$\begin{aligned} JSI(i, i+k) = &\textstyle\sum_{j=0}^{k-2} pq^j[\sum_{l=0}^{k-2-j} pq^{k-1}R(i-k+l-1)\\ &\textstyle\times\sum_{m=0}^{j} pq^{k-m-1}R(n-i-2k+m)]\\ &+q^{k-1}R(i-1)R(n-i-k). \end{aligned}$$

Proof.

$$\begin{aligned} &JSI(i, i+k)\\ &=[\sum_{l=0}^{k-2} pq^l R_{i+k-2-l}(1_i)\sum_{m=0}^{l} pq^{k-1-m}R(n-i-2k+m)\\ &+q^{k-1}R(i-1)R(n-i-k)]\\ &-[\sum_{l=0}^{k-2} pq^l R_{i+k-2-l}(0_i)\sum_{m=0}^{l} pq^{k-1-m}R(n-i-2k+m)]\\ &=q^{k-1}R(i-1)R(n-i-k)\\ &+\textstyle\sum_{l=0}^{k-2} pq^l[R_{i+k-2-l}(1_i)-R_{i+k-2-l}(0_i)]\\ &\times\textstyle\sum_{m=0}^{l} pq^{k-1-m}R(n-i-2k+m)\\ &>0. \end{aligned}$$

$\square$

Theorem 3.23. *$JSI(i, i-k) > JSI(i, i-k-1)$ for $n \ge i+2k$.*

Proof. By Lemma 3.4 (i) and Theorem 3.21,

$$\begin{aligned} &JSI(i, i-k)-JSI(i, i-k-1)\\ &=JSI(n+1-i, n+1-i+k)\\ &-JSI(n+1-i, n+1-i+k+1)\\ &>0 \end{aligned}$$

$\square$

For a fixed i, in the following, we discuss the graph of JSI for $\max\{1, i-k-1\} \le j \le \min\{i+k+1, n\}$ according to $i<k$, $i=k$, and $i>k$.

Corollary 3.24. *For a fixed $i<k$ and $1 \le j \le i+k+1$, the rank of JSI are as follows.*

(i) $0 > JSI(i,1) > JSI(i,2) > \cdots > JSI(i,i-1) > JSI(i,i+1) = JSI(i,i+2) = \cdots = JSI(i,k)$,

(ii) $JSI(i,i+1) = \cdots = JSI(i,k) < JSI(i,k+1) < JSI(i,k+2) < \cdots < JSI(i,i+k)$, *and*

(iii) $JSI(i,i+k) > JSI(i,i+k+1)$.

Furthermore, the graph of $JSI(i,\ j)$ has a W-shape with a flat segment for $1 \le j \le i+k+1$.

Proof. By Theorems 3.10, 3.11, 3.17, and 3.21. $\square$

Corollary 3.25. *For a fixed $i=k$ and $1 \le j \le \min\{i+k+1, n\}$, the rank of JSI are as follows.*

(i) $0 > JSI(i,1) > JSI(i,2) > \cdots > JSI(i,i-1)$,

(ii) $JSI(i,i+1) < JSI(i,i+2) < \cdots < JSI(i,i+k)$, *and*

(iii) $JSI(i,i+k) > JSI(i,i+k+1)$.

Furthermore, the graph of $JSI(i,\ j)$ has a W-shape for $1 \le j \le i+k+1$.

Proof. By Theorems 3.10, 3.11, 3.19, and 3.21. $\square$

Corollary 3.26. *For $i>k$ and $i-k-1 \le j \le \min\{i+k+1, n\}$, the rank of JSI are as follows.*

(i) $JSI(i,i-k) > JSI(i,i-k+1) > \cdots > JSI(i,i-1)$,

(ii) $JSI(i,i+1) < JSI(i,i+2) < \cdots < JSI(i,i+k)$,

(iii) $JSI(i,i+k) > JSI(i,i+k+1)$, *and*

(iv) $JSI(i,i-k-1) < JSI(i,i-k)$.

Furthermore, the graph of $JSI(i,\ j)$ has a W-shape for $i-k-1 \le j \le i+k+1$.

Proof. By Theorems 3.14, 3.15, 3.19, 3.21, and 3.23. $\square$

Figure 2 shows a graph of the joint structural importance $JSI(i,\ j)$ for $i>k$ and $i-k-1 \le j \le i+k+1$.

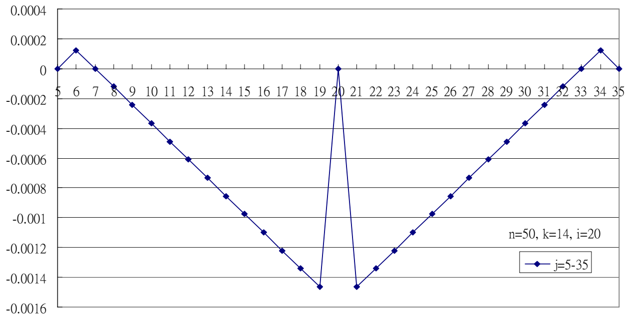

Fig. 2. $JSI(i,\ j)$ for $n=50$, $k=14$, $i=20$, and $5 \le j \le 35$.

In the following, we consider the JSI of the last k components for a fixed i.

Theorem 3.27. *$JSI(i,\ j) > JSI(i,\ j+1)$ for $j \ge i+k$ and $n-k+1 \le j \le n-1$.*

Proof. For $j-i \ge k+1$, by Lemmas 2.5, 3.4 (iii), and 3.5,

$$\begin{aligned} &JSI(i,\ j)-JSI(i,\ j+1)\\ &=R(i-1)pq^{k-1}R(j-k-i-1)-pq^{k-1}R_{j-k-1}(0_i)\\ &=R(i-1)pq^{k-1}R(j-k-i-1)-pq^{k-1}\\ &\times\tfrac{1}{q}[R(j-k-1)-pR(i-1)R(j-k-i-1)]\\ &=pq^{k-2}[R(i-1)R(j-k-i-1)-R(j-k-1)]>0. \end{aligned}$$

On the other hand, for $j-i=k$,

$$JSI(i,\ j) - JSI(i,\ j+1) = R(i-1)q^{k-1} > 0\,. \quad \Box$$

We compare Theorem 3.27 with Theorems 3.17 and 3.19 as follows.

Remark 3.28.

(i) *If* $j \ge i+k$ *and* $n-k+1 \le j \le n-1$, *then* $JSI(i,\ j) > JSI(i,\ j+1)$.

(ii) *If* $i < j < i+k$ *and* $n-k+1 \le j \le n-1$, *then* $JSI(i,\ j) < JSI(i,\ j+1)$.

Next, we compare $JSI(i,n)$ with $JSI(i,n-k)$.

Theorem 3.29. $JSI(i,n) < JSI(i,n-k)$ *for* $i \le n-3k$.

Proof. By Lemma 3.5,

$$\begin{aligned} & JSI(i,n-k) - JSI(i,n) \\ &= \sum_{l=0}^{k-2} R_{n-2k-1+l}(1_i)\, pq^{k-1-l} \sum_{m=0}^{k-2-l} pq^{k-2-m} \\ &\quad - \sum_{l=0}^{k-2} R_{n-2k-1+l}(0_i)\, pq^{k-1-l} \sum_{m=0}^{k-2-l} pq^{k-2-m} \\ &= \sum_{l=0}^{k-2} [R_{n-2k-1+l}(1_i) - R_{n-2k-1+l}(0_i)] pq^{k-1-l} \sum_{m=0}^{k-2-l} pq^{k-2-m} \\ &> 0. \end{aligned}$$

$\Box$

4 Conclusion

In this paper, we study joint structural importance in the consecutive-*k*-out-of-*n* system. We introduce the definitions of Birnbaum reliability importance, joint Birnbaum importance, and joint structural importance and state several useful formulas for computing the reliability of consecutive-*k* system. In Section 3, we first completely solve $JSI(i,\ j)$ for $k=1$ (the series system), $k=n$ (the parallel system), $k=n-1$, and $k=n-2$, respectively. For the other k, we study $JSI(i,\ j)$ for $i=1$ and show that the values of joint structural importance is symmetric to $\lfloor (n+k+2)/2 \rfloor$ and $\lceil (n+k+2)/2 \rceil$. We also prove that $JSI(1,\ j') = JSI(1,k) < 0 < JSI(1,n) = JSI(1,k+2) < JSI(1,\ j) < JSI(1,k+1)$, for $2 \le j' \le k-1$ and $k+3 \le j \le n-1$. On the other hand, given a fixed i, we prove that the graph of $JSI(i,\ j)$ has a W-shape property for $\max\{1, i-k-1\} \le j \le \min\{n, i+k+1\}$ with $JSI(i,i) = 0$. We show that the values of $JSI(i,\ j)$ is decreasing for the last k components. Note that the results of JSI is related to those of Birnbaum structural importance.

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References:

[1] M. J. Armstrong, “Joint reliability-importance of components,” *IEEE Transactions on Reliability*, Vol. 44, 1995, pp. 408-412.

[2] Z. W. Birnbaum, “On the importance of different components in a multicomponent,” in *Multivariate Analysis II*, P.R. Krishnaiah (Ed.), Academic: NY, 1969, pp.581-592.

[3] G. J. Chang, L. Cui, and F. K. Hwang, “New comparisons in Birnbaum importance for the consecutive-*k*-out-of-*n* system,” *Probability in the Engineering and Informational Sciences*, Vol. 13, 1999, pp. 187-192.

[4] G. J. Chang, L. Cui, and F. K. Hwang, “Corrigenda on new comparisons in Birnbaum importance for the consecutive-*k*-out-of-*n* system,” *Probability in the Engineering and Informational Sciences*, Vol. 14, 2000, pp. 405.

[5] H.-W. Chang, R. J. Chen and F. K. Hwang, “The structural Birnbaum importance of consecutive-*k* systems,” *Journal of Combinatorial Optimization*, Vol. 6, 2002, pp. 183-197.

[6] H. W. Chang and F. K. Hwang, “Rare-event component importance in the consecutive-*k* system,” *Naval Research Logistics*, Vol. 49, 2002, pp. 159-166.

[7] M. T. Chao, J. C. Fu, and M. V. Koutras, “Survey of reliability studies of consecutive-*k*-out-of-*n*: F & Related Systems,” *IEEE Transactions on Reliability*, Vol. 44, 1995, pp. 120-127.

[8] C. Derman, G. J. Lieberman, and S. M. Ross, “On the consecutive-*k*-out-of-*n*:F system,” *IEEE Transactions on Reliability*, Vol. 31, 1982, pp. 57-63.

[9] D. Z. Du and F. K. Hwang, “Optimal consecutive-2-out-of-*n* systems,” *Mathematics of Operations Research*, Vol. 11, 1986, pp. 187-191.

[10] J. N. Hagstrom, “Redundancy, substitutes and complements in system reliability,” Technical report, College of Business Administration, University of Illinois, 1990.

[11] J. S. Hong, H.Y. Koo, and C. H. Lie, “Computation of joint reliability importance of two gate events in a fault tree,” *Reliability Engineering and System Safety*, Vol. 68, 2000, pp. 1-5.

[12] J. S. Hong, H. Y. Koo, and C. H. Lie, “Joint reliability importance of *k*-out-of-*n* systems,” *European Journal of Operational Research*, Vol. 142, 2002, pp. 539-547.

[13] J. S. Hong and C. H. Lie, “Joint reliability-importance of two edges in an undirected network,” *IEEE Transactions on Reliability*, Vol. 42, 1993, pp. 17-23.

[14] F. K. Hwang, “Fast solution for consecutive-*k*-out-of-*n*:F system,” *IEEE Transactions on Reliability*, Vol. R-31, 1982, pp. 447-448.

[15] S. Jan, *The study on joint reliability importance in several systems*, Master thesis, Department of Applied Mathematics, Tatung University, Taipei, Taiwan, 2006.

[16] F. H. Lin, W. Kuo, and F. K. Hwang, “Structure importance of consecutive-*k*-out-of-*n* systems,” *Operations Research Letters*, Vol. 25, 1999, pp. 101-107.

[17] D. M. Malon, “Optimal consecutive-2-out-of-*n*:F component sequencing,” *IEEE Transactions on Reliability*, Vol. R33, 1984, pp. 414-418.

[18] D. M. Malon, “Optimal consecutive-*k*-out-of-*n*:F component sequencing,” *IEEE Transactions on Reliability*, Vol. R34, 1985, pp. 46-49.

[19] S. Wu, “Joint importance of multistate systems,” *Computers and Industrial Engineering*, Vol. 49, 2005, pp. 63-75.

[20] M. Zuo, “Reliability and component importance of a consecutive-*k*-out-of-*n* system,” *Microelectronics and Reliability*, Vol. 33, 1993, pp. 243-258.

[21] M. Zuo and W. Kuo, “Design and performance analysis of consecutive-*k*-out-of-*n* structure,” *Naval Research Logistics*, Vol. 37, 1990, pp. 203-230.