Joint Structural Importance in Consecutive-\(k\) Systems

Hsun-Wen Chang and Jun-Da Chen
Department of Applied Mathematics
Tatung University
40 ChungShan North Road, 3rd Section
Taipei, Taiwan 104, R.O.C.

Abstract: - The joint structural importance (JSI) is an important measure of how two components interact in contributing to the system reliability. The value of JSI is positive (negative) if and only if one component becomes more important (less important) when the other works. A consecutive-\(k\)-out-of-\(n\) system is a linear arrangement of \(n\) components such that the system is failed if and only if some consecutive \(k\) components are all failed. In this paper, we study joint structural importance \(\text{JSI}(i, j)\) in the consecutive-\(k\)-out-of-\(n\) system.

We completely solve \(\text{JSI}(i, j)\) for \(k = 1\) (the series system), \(k = n\) (the parallel system), \(k = n - 1\), and \(k = n - 2\), respectively. For the other \(k\), we prove that \(\text{JSI}(i, j) = \text{JSI}(i, k) < 0 < \text{JSI}(1, n) = \text{JSI}(1, k + 2) < \text{JSI}(1, j) < \text{JSI}(1, k + 1)\), for \(2 \leq j' \leq k - 1\) and \(k + 3 \leq j \leq n - 1\). For a fixed \(i\), we prove that the graph of \(\text{JSI}(i, j)\) has a W-shape property for \(\max\{1, i - k - 1\} \leq j \leq \min\{n, i + k + 1\}\) with \(\text{JSI}(i, i) = 0\). We also present exact formula for \(\text{JSI}(i, j)\) and obtain many relations among them.

Key Words: Joint structural importance, Consecutive-\(k\)-out-of-\(n\) system, Reliability, Birnbaum importance.

1 Introduction

The Birnbaum reliability importance \(I(i)\) of a component \(i\) measures the improvement of the system reliability \(R(P)\) over the improvement of the reliability \(p_i\) of that component [2]. The Birnbaum reliability importance \(I(i)\) is defined as

\[
I(i) = \partial R(P)/\partial p_i = R(1, P) - R(0, P),
\]

where \(P\) denote all the component reliability except that of the component \(i\), \(R(1, P)\) and \(R(0, P)\) are the system reliability with component \(i\) working and failed, respectively. Note that the system reliability \(R(P)\) can be computed only when the reliabilities of all components are well defined. Without the information of component reliabilities, we need to know the relative importance of the locations so that more reliable components can be assigned to the more important locations to maximize the system reliability. The structural Birnbaum importance is a special kind of the Birnbaum reliability importance where all the component reliabilities are set equal to \(p\) so that the importance measure will depend only on the structure of a system. Many kinds of structural Birnbaum importance indices have been discussed [3-6, 16-18, 20, 21].

The joint reliability importance \(\text{JRI}(i, j)\) of two components \(i\) and \(j\) measures how these two components in a system interact in contributing to the system reliability \(R(P)\). The joint reliability importance \(\text{JRI}(i, j)\) is defined as follows.

\[
\text{JRI}(i, j) = \partial^2 R(P)/\partial p_i \partial p_j = R(1, 1, P_{ij}) + R(0, 0, P_{ij}) - R(1, 0, P_{ij}) - R(0, 1, P_{ij}),
\]

where the \(P_{ij}\) will be omitted when no confusion is possible. Joint reliability importance was first proposed independently by Hagstrom [10] and by Hong and Lie [13]. Based on the definition of Birnbaum reliability importance, \(\text{JRI}(i, j)\) can be interpreted as the change of the Birnbaum reliability importance of component \(i\) caused by component \(j\)'s deteriorating from working to failed. The value of \(\text{JRI}(i, j)\) is positive (negative) if and only if one component becomes more important (less important) when the other works. Similar to the problem in Birnbaum reliability importance, many researchers have studied joint structural importance (setting all \(p_i = p\) in many systems: the fault tree [11], the two-terminal system [1, 13, 19], the \(k\)-out-of-\(n\) system [12, 15], etc. Jan first studied joint structural importance in the consecutive-2-out-of-\(n\) systems [15].

A consecutive-\(k\)-out-of-\(n\) system consists of an ordered sequence of \(n\) components where the system fails if and only if any \(k\) consecutive components are all failed. Relative to low reliability of a series system and high reliability but very expensive hardware of the parallel system, the consecutive-\(k\) system has attracted many researchers [5-9, 15-18, 21]. For the consecutive-2 system, Malon [17] and Du and Hwang [9] independently solved a problem called the invariant optimal assignment, which has the problem of finding the rank of structural Birnbaum importance as one of its special cases. Malon [18] also solved the assignment problem for \(k = n - 1, n - 2\), and proved that an invariant optimal assignment does not exist for \(3 \leq k \leq n - 3\), which intensifies the need to compare structural Birnbaum importance.
In this paper, we concentrate on the joint structural importance in the consecutive-\(k\) system. We first state several useful formulas for computing system reliability in Section 2. Then we study joint structural importance, denoted by \(JSI(i,j)\), in the consecutive-\(k\) system in Section 3. We completely solve \(JSI(i,j)\) in the consecutive-\(k\)-out-of-\(n\) system for \(k = 1, n-2, n-1, n\). For the other \(k\), we first study the case of \(i = 1\) and make comparisons among \(JSI(i,j)\). On the other hand, given a fixed \(i\), we compare the values of \(JSI(i,j)\) and present the W-shape property of the \(JSI\) function. We also find the exact formula for some \(JSI(i,j)\) and discuss the relationship among them. Finally, we make a conclusion in Section 4.

2 System Reliability

For a consecutive-\(k\)-out-of-\(n\) system, let \(R(n)\) denote the system reliability, and \(R_i(1)\) and \(R_n(0)\) denote the reliabilities of the system where the component \(i\) is working and failed, respectively. Note that \(R(k) = 1 - q^k\) and \(R(n) = 1\) for \(0 \leq n \leq k - 1\), where \(q = 1 - p\).

**Lemma 2.1.** (See [14]) For \(n \geq k + 1\), the reliability \(R(n)\) of a consecutive-\(k\)-out-of-\(n\) system satisfies the following recursive relations.

(i) \(R(n) = \sum_{j=1}^{k} p q^{m-1} R(n-m)\).

(ii) \(R(n) = R(n-1) - p q^n R(n-k-1)\).

Note that for \(n \geq k\), \(R(n)\) is decreasing in \(n\).

**Corollary 2.2.** \(R(n) = [R(n+k) - R(n+k+1)]/p q^k\).

Proof. By Lemma 2.1 (ii).

For convenience, by Corollary 2.2, we backward derive the reliabilities of a consecutive-\(k\)-out-of-\(n\) system for \(-k-1 \leq n \leq -1\).

**Definition 2.3.** The reliability of consecutive-\(k\)-out-of-\(n\) system for \(-k-1 \leq n \leq -1\) is

\[
R(n) = \begin{cases} 
1/p, & n = -1, \\
0, & -k \leq n < -2, \\
p q^{n-1}, & n = -k-1. 
\end{cases}
\]

**Lemma 2.4.**

(i) \(R(n) = p R_i(1) + q R_n(0)\).

(ii) \(R_i(1) = R(i) - R(n-i)\).

(iii) \(R_n(0) = [R(n) - p R(i-1) R(n-i)]/q\).

Proof. Statements (i) and (ii) follows from the definition of the system reliability according to component \(i\) working or not. Statement (iii) follows immediately from (i) and (ii).

**Lemma 2.5.** (See [20])

\[p R(i-1) R(n-i) < R(n) < R(i) R(n-i).\]

**Lemma 2.6.** \(R(i) R(j) > R(i+j+1)\).

Proof. The difference between \(R(i) R(j)\) and \(R(i+j+1)\) is just the reliability of the cases that both the first \(i\)-component subsystem and the last \(j\)-component subsystem are working and the whole \((i+j+1)\)-component system is failed due to the \((i+1)\)-st component is failed. Hence

\[
R(i) R(j) - R(i + j + 1) = \sum_{m=0}^{\infty} p q^m R(i-1-m) \sum_{l=0}^{m-1} p q^l q^n R(j-1-l) > 0. \]

3 Joint Structural Importance

In this section, we first consider the joint structural importance in the consecutive-\(k\) system for \(k = 1, n-2, n-1, n\). Note that a consecutive-\(k\) system is a series system for \(k = 1\), and a parallel system for \(k = n\).

**Theorem 3.1.** Consider a consecutive-\(k\)-out-of-\(n\) system.

(i) For \(k = 1\) (the series case), \(JSI(i,j) = p^{n-2} > 0\) for any \(i \neq j\).

(ii) For \(k = n\) (the parallel case), \(JSI(i,j) = -q^{n-2} < 0\) for any \(i \neq j\).

Proof. By definition, \(JSI(i,j) = R((1,1,1)) - R((0,0,0)) - R((0,1,1)) + R((0,0,0))\).

(i) For \(k = 1\), \(JSI(i,j) = p^{n-2} - 0 + 0 = p^{n-2} > 0\).

(ii) For \(k = n\), \(JSI(i,j) = 1 - 1 + (1 - q^{n-2}) = -q^{n-2} < 0\).

**Theorem 3.2.** Suppose \(k = n-1\). \(JSI(i,j) < 0\) for all \(i \neq j\), except \(JSI(1,n) > 0\).

Proof. By definition,

(i) \(JSI(1,n) = 1 - (1 - q^{n-2}) - (1 - q^{n-2}) - (1 - q^{n-2}) = q^{n-2} > 0\).

(ii) For \(j \neq n\), \(JSI(1,j) = 1 - (1 - q^{n-2}) - 1 + (1 - q^{n-2}) - (1 - q^{n-2}) = -q^{n-2} > 0\).

(iii) For \(1 < j < n\), \(JSI(1,j) = 1 - 1 + (1 - 2 q^{n-2}) - (1 - q^{n-2}) = -2 q^{n-2} < 0\).

Hence, in a consecutive-(\(n-1\))-out-of-\(n\) system, \(JSI(i,j) < 0\) for all \(i \neq j\), except \(JSI(1,n) > 0\).

**Theorem 3.3.** Suppose \(k = n-2\). \(JSI(i,j) < 0\) for all \(i \neq j\), except \(JSI(1,n) = 0\), \(JSI(n-1,n) > 0\), and \(JSI(2,n-1)\) is positive, negative, and zero for \(p < 1/2\), \(p > 1/2\), and \(p = 1/2\), respectively.

Proof. \(JSI(i,j)\) are computed according to seven cases as follows. For the first three cases, we compute \(JSI(i,j)\) by

\(JSI(i,j) = R((1,1,1)) - R((0,1,1)) + R((0,0,0))\).

(i) \(JSI(1,2) = (1 - q^{n-2}) - (1 - q^{n-2}) - (1 - q^{n-2}) + (1 - q^{n-2}) = -p q^{n-4} < 0\).

(ii) For \(3 \leq j \leq n-2\), \(JSI(1,j) = 1 - (1 - q^{n-2}) - (1 - q^{n-2}) - (1 - q^{n-2}) = p q^{n-4} > 0\).

(iii) \(JSI(1,n) = 1 - (1 - q^{n-2}) - (1 - q^{n-2}) = q^{n-3} > 0\).

In the following cases, we compute \(JSI(i,j)\) by \(JSI(i,j) = [R((1,1,1)) - R((1,0,1)) - R((1,0,0)) - R((0,0,0))\).

(iv) \(JSI(1,n) = p q^{n-4} - p q^{n-3} = 0\).

(v) For \(3 \leq j \leq n-2\), \(JSI(2,j) = 0 - (p q^{n-4} + p q^{n-3}) < 0\).

(vi) \(JSI(2,n-1) = q^{n-3} - p (q^{n-3} + p q^{n-4}) = q^{n-3} - 2 p\). Thus \(JSI(2,n-1)\) is positive, negative, and zero for \(p < 1/2\), \(p > 1/2\), and \(p = 1/2\), respectively.
(vii) For $2 \leq i < j < n - 1$, $JSI(i, j) = 0 - (q^{-n} + 2pq^{-n-1}) < 0$.

In the following, we state some properties of joint structural importance.

**Lemma 3.4.** Consider a consecutive-$k$-out-of-$n$ system.

(i) $JSI(i, j) = JSI(n - i + 1, n - j + 1)$

(ii) $JSI(i, j) = JSI(j, i)$

(iii) $JSI(i, i) = 0$

(iv) $JSI(i, j) = 0$ for $1 \leq i, j \leq n - k$

**Proof.** Since a consecutive-$k$-out-of-$n$ system is symmetric with respect to the middle location(s), we have Statement (i). Statements (ii) and (iii) follows immediately the definition of joint structural importance. If $n < k$, then $R(1, 1) = R(0, 1) = R(1, 0, 0) = R(0, 0, 0) = 1$ and thus $JSI(i, j) = 0$.

By Lemma 3.4 (iv), throughout this thesis, we discuss $JSI$ in the consecutive-$k$-out-of-$n$ system for $n \geq k$.

**Lemma 3.5.** Given a fixed $i$, the difference between $JSI(i, l)$ and $JSI(i, j)$ is $JSI(i, l) - JSI(i, j) = R(1, 0, 1) - R(1, 0, 0) + R(0, 0, 1) - R(0, 0, 0)$.

**Proof.** By definition of $JSI$, we have

$JSI(i, l) - JSI(i, j) = [R(1, 1) - R(0, 1)] + [R(0, 1) - R(1, 0)]$.

Since $n < k$, the difference between $JSI(i, l)$ and $JSI(i, j)$ simplifies the calculation of the difference between $JSI(i, l)$ and $JSI(i, j)$.

**Theorem 3.7.** $JSI(i, j) = JSI(n + k - 2 - j, n - k + 1)$ for $1 \leq j \leq n$.  

**Proof.** Immediately from Theorem 3.6 (iii).

Theorem 3.7 shows that the $JSI$ of the first and the $j$-th locations is equal to that of the first and the $(n + k - 2 - j)$-th locations. Thus we can discuss $JSI(i, j)$ only for $j \leq (n + k - 2)/2$. In the following, we find an upper bound and a lower bound of $JSI(i, j)$ for $j \geq k + 2$.

**Theorem 3.8.** For $j \geq k + 2$, $JSI(i, j) = JSI(n + k - 2, i)$.

**Proof.** By Theorems 3.6 (ii) and (iii), for $j \geq k + 2$, $JSI(k, i) - JSI(i, j) = q^k - [R(n - k - 1) - R(n - j)] > q^k > R(0, 0, 0)$.

Similarly, by Theorems 3.6 (iii), for $j \geq k + 2$, $JSI(i, j) = JSI(n + k - 2, i)$.

**Corollary 3.9.** For $2 \leq i \leq n + k + 1$ and $i + 1$, the rank of $JSI(i, j)$ are as follows. $JSI(i, j') = JSI(n + k + 1, i)$.

**Proof.** By Theorems 3.6-3.8.

Figure 1 shows a graph of the joint structural importance $JSI(i, j)$ for $i = 1$.  

![Fig. 1.](image-url)
Theorem 3.10. For $2 \leq j < i \leq k + 1$, $JSI(i, j-1) - JSI(i, j) = pq^{k-2}R(n - j - k)$. Furthermore, $JSI(i, j-1) > JSI(i, j)$ for $n \geq j + k - 1$ and $JSI(i, j-1) = JSI(i, j)$ for $n = j + k - 2$.

Proof. For $2 \leq j < i \leq k + 1$, by Lemma 3.5,

$JSI(i, j-1) - JSI(i, j) = \sum_{j \leq j \leq k} [R(1, 1, j, 0) - R(1, 0, j, 1)] + [R(0, 0, j+1, 1) - R(0, 1, j+1, 0)]$.

Hence we have Theorem 3.10 immediately.

Theorem 3.11. Suppose $1 \leq i, j \leq k$ and $j \neq i$.

(i) For $n \leq i + k - 1$,

$JSI(i, j) = \begin{cases} \frac{1}{2} - j \leq n - \frac{k}{2}, \\ -(n - k + 1/p)q^{k-2}, & n - k + 1 \leq j \leq k. \end{cases}$

(ii) For $i + k \leq n \leq 2k$, $JSI(i, j) = -\min\{i, j\}q^{k-2}$. (iii) For $n > 2k$, $JSI(i, j) = -\frac{p}{k}q^{k-2}\sum_{j \leq j \leq k} R(n-k-1)$.

Proof. Consider $1 \leq j \leq k$. By definition, $JSI(i, j) = -\frac{p}{k}q^{k-2}\sum_{j \leq j \leq k} R(n-k-1)$.

If $i + k \leq n \leq 2k$, then $0 \leq n - k - 1 \leq k$ and thus $JSI(i, j) = -\frac{p}{k}q^{k-2}$. If $k \leq n \leq i + k - 1$, we have $n - k - 1 \leq n - k - i$ and $n - i + 1 \leq k$. Since $R(-1) = 1/p$ and $R(n) = 1$ for $0 \leq n \leq k - 1$, we have $JSI(i, j) = -(n - k + 1/p)q^{k-2}$ for $i + k \leq j \leq k$. Moreover, by Lemma 3.4 (i), $JSI(i, j) = JSI(n - i + 1, n - j + 1) = JSI(n - i + 1, j') = JSI(n - i - 1, j) + JSI(n - i - 1, j)$ for $n - i + 1 \leq j \leq k$.

Remark 3.12. In Theorem 3.11, $JSI(i, j)$ is linear for fixed $i \leq k$, $n \leq k$, and $1 \leq j \leq \min\{n - k, i\}$. For $n > 2k$, $JSI(i, j)$ is no more a multiple of $JSI(i, j)$ since $R(n-k-j) = 1$.

Theorem 3.13. For fixed $n$ and $k$, if $i \neq j$ and $n - k + 1 \leq i, j \leq k$, then $JSI(i, j) = -(n - k + 1/p)q^{k-2}$.

Proof. Immediately follow from Theorem 3.11 which depends only on $n$, $k$, and $p$.

Theorem 3.14. Suppose $2 \leq i \leq 2k$.

(i) $JSI(i, j-1) > JSI(i, j)$ for $k + 1 \leq j \leq i - 1$.

Furthermore, $JSI(i, j-1) - JSI(i, j) = pq^{k-2}R(n - j - k)$.

(ii) $JSI(i, j-1) > JSI(i, j)$ for $i - k + 1 \leq j \leq k$.

Furthermore, $JSI(i, j-1) - JSI(i, j) = pq^{k-2}R(n - j - k) > 0$.

Proof. (i) By Lemmas 2.4 (iii) and 3.5, for $k + 1 \leq j \leq i - 1$,

$JSI(i, j-1) - JSI(i, j) = pq^{k-2}R(n - j - k)$.

Furthermore, $JSI(i, j-1) > JSI(i, j)$ for $n \geq j + k - 1$ and $JSI(i, j-1) = JSI(i, j)$ for $n = j + k - 2$.

(ii) By Lemma 3.5, for $i - k + 1 \leq j \leq k$, $JSI(i, j) = 0 + pq^{k-2}R(n - j - k) > 0$.

Theorem 3.15. Suppose $i \geq 2k$.

$JSI(i, j-1) > JSI(i, j)$ for $i - k + 1 \leq j \leq i - 1$.

Furthermore, $JSI(i, j-1) - JSI(i, j) = pq^{k-2}R(n - j - k)$.

Proof. Similar to the proof of Theorem 3.14 (i).

Corollary 3.16. Given a fixed $i$ , $JSI(i, j)$ is decreasing for max $\{i, i - k\} \leq j \leq i - 1$.

Proof. By Theorems 3.10-3.15.

Theorem 3.17. Suppose $i \leq k$. $JSI(i, j) < JSI(i, j+1)$ for $j \leq \min\{i + k - 1, n - 1\}$. Furthermore, $JSI(i, j+1) - JSI(i, j) = pq^{k-2}R(n - j - 1) + pq^{k-2}R(n - j - 1)[1 - R(j - 1)]$.

In fact, $JSI(i, k+1) = JSI(i, k) = q^{k-2}R(n - k - 1)$.

Proof. By Lemma 3.5, for $i + k \leq n \leq \min\{i + k - 1, n - 1\}$, $JSI(i, j + 1) - JSI(i, j) = pq^{k-2}R(n - j - 1) + pq^{k-2}R(n - j - 1)[1 - R(j - 1)]$.

Note that $R(-1) = 1/p$. For $i \leq k$, $JSI(i, j + 1) - JSI(i, j) = q^{k-2}R(n - j - 1) > 0$. Hence, for $i + k \leq n \leq \min\{i + k - 1, n - 1\}$, $JSI(i, j) < JSI(i, j + 1)$.

Corollary 3.18. Suppose $i \leq k$. For $k + 1 \leq j \leq \min\{i + k, n\}$, $JSI(i, j) = pq^{k-2}R(n - k - m) + pq^{k-2}R(n - k - 1) + \sum_{j = 1}^{i} \sum_{j = 1}^{i} R(m - l - 1)$.

Proof. Note that $JSI(i, j) = JSI(i, k) + \sum_{j = 1}^{i} [JSI(i, j + 1) - JSI(i, j)]$.

Corollary 3.19. Suppose $i \leq k$. $JSI(i, j) < JSI(i, j + 1)$ for $i + 1 \leq j \leq \min\{i + k - 1, n - 1\}$.

Furthermore,
where JSI(i, j + 1) = JSI(i, j).

\[ p_{ij} \leq R(j - k - 1)R(n - j - 1) + p_{ij}^{-\alpha}R(n - j - k - 1)R(j - 1) - R(j - 1). \]

\[ p_{ij}^{-\alpha}R(n - j - k - 1)R(j - 1) - p_{ij}^{-\alpha}R(n - j - k - 1)R(j - 1). \]

Hence, JSI(i, j) < JSI(i, j + 1) for i ≥ k and i + 1 ≤ j ≤ n.

**Theorem 3.20.** Given a fixed i, JSI(i, j) is nondecreasing for i + 1 ≤ j ≤ min{\( i + k - 1, n \)}.

**Proof.** By Theorems 3.11, 3.17, and 3.19.

In the following, we make two more comparisons.

**Theorem 3.21.** JSI(i, i + k) > JSI(i, i + k + 1) for \( n \geq i + 2k \).

**Proof.** By Lemmas 2.4 (iii) and 3.5:

\[ JSI(i, i + k) = \sum_{i=0}^{n} p_{ij} \left[ \sum_{j=0}^{i} p_{ij}^{k-1}R(n - i - 2k + m) \right] + q_{ij}^{k-1}R(i - 1)R(n - i - k). \]

For a fixed i, in the following, we discuss the graph of JSI for \( max\{1, i - k - 1\} \leq j \leq \min\{i + k + 1, n\} \) according to \( i < k, i = k, \) and \( i > k \).

**Corollary 3.24.** For a fixed i < k and 1 ≤ j ≤ i + k + 1, the rank of JSI are as follows.

(i) \( 0 > JSI(i, 1) > JSI(i, 2) > \cdots > JSI(i, i - 1) \) \( JSI(i, 1) = JSI(i, 2) = \cdots = JSI(i, k) \),

(ii) \( JSI(i, i + 1) = \cdots = JSI(i, k) < JSI(i, k + 1) < JSI(i, k + 2) < \cdots < JSI(i, i + k) \), and

(iii) JSI(i, i + k) > JSI(i, i + k + 1).

Furthermore, the graph of JSI(i, j) has a W-shape with a flat segment for \( 1 \leq j \leq i + k + 1 \).

**Proof.** By Theorems 3.10, 3.11, 3.17, and 3.21.

**Corollary 3.25.** For a fixed i = k and 1 ≤ j ≤ \( i + k + 1, n \), the rank of JSI are as follows.

(i) \( 0 > JSI(i, 1) > JSI(i, 2) > \cdots > JSI(i, i - 1) \) \( JSI(i, 1) = JSI(i, 2) = \cdots = JSI(i, k) \),

(ii) \( JSI(i, i + 1) < JSI(i, i + 2) < \cdots < JSI(i, i + k) \), and

(iii) JSI(i, i + k) > JSI(i, i + k + 1).

Furthermore, the graph of JSI(i, j) has a W-shape for \( 1 \leq j \leq i + k + 1 \).

**Proof.** By Theorems 3.10, 3.11, 3.19, and 3.21.

**Corollary 3.26.** For i < k and \( i + k - 1 \leq j \leq i + k + 1 \), the rank of JSI are as follows.

(i) \( JSI(i, i - k) > JSI(i, i + k + 1) > \cdots > JSI(i, i - 1) \),

(ii) \( JSI(i, i + 1) < JSI(i, i + 2) < \cdots < JSI(i, i + k) \), and

(iii) JSI(i, i + k) > JSI(i, i + k + 1).

Furthermore, the graph of JSI(i, j) has a W-shape for \( i - k - 1 \leq j \leq i + k + 1 \).

**Proof.** By Theorems 3.14, 3.15, 3.19, 3.21, and 3.23.

Figure 2 shows a graph of the joint structural importance JSI(i, j) for i > k and \( i - k - 1 \leq j \leq i + k + 1 \).

![Figure 2](image-url)
\[ JSI(i, j) - JSI(i, j+1) = R(i-1)q^{k-1} > 0. \]

We compare Theorem 3.27 with Theorems 3.17 and 3.19 as follows.

**Remark 3.28.**
(i) If \( j \geq i + k \) and \( n - k + 1 \leq j \leq n - 1 \), then \( JSI(i, j) > JSI(i, j+1) \).
(ii) If \( i < j < i + k \) and \( n - k + 1 \leq j \leq n - 1 \), then \( JSI(i, j) < JSI(i, j+1) \).

Next, we compare \( JSI(i, n) \) with \( JSI(i, n-k) \).

**Theorem 3.29.** \( JSI(i, n) < JSI(i, n-k) \) for \( i \leq n \leq 3k \).

**Proof.** By Lemma 3.5,
\[
JSI(i, n-k) - JSI(i, n) = \sum_{i=0}^{n-k} R_n \sum_{i=0}^{n-k} q^{n-i} \sum_{i=0}^{n-k} q^{n-k-i} > 0.
\]

\[ JSI(i, n) < JSI(i, n-k) \] for \( i \leq n \leq 3k \).

\[ JSI(i, n-k) = \sum_{i=0}^{n-k} R_n \sum_{i=0}^{n-k} q^{n-i} \sum_{i=0}^{n-k} q^{n-k-i} > 0. \]

\[ JSI(i, n) < JSI(i, n-k) \] for \( i \leq n \leq 3k \).

4 Conclusion

In this paper, we study joint structural importance in the consecutive-\( k \)-out-of-\( n \) system. We introduce the definitions of Birnbaum reliability importance, joint Birnbaum importance, and joint structural importance and state several useful formulas for computing the reliability of consecutive-\( k \) system. In Section 3, we first completely solve \( JSI(i, j) \) for \( k = 1 \) (the series system), \( k = n \) (the parallel system), \( k = n - 1 \), and \( k = n - 2 \), respectively. For the other \( k \), we study \( JSI(i, j) \) for \( i = 1 \) and show that the values of joint structural importance are symmetric to \( \{(n + k + 2)/2\} \) and \( \{(n + k + 3)/2\} \). We also prove that \( JSI(1, j') = JSI(1, k < 0 < JSI(1, n) = JSI(i, k + 2) < JSI(i, j) < JSI(i, k + 1) \), for \( 2 \leq j' \leq k - 1 \) and \( k + 3 \leq j \leq n - 1 \). On the other hand, given a fixed \( i \), we prove that the graph of \( JSI(i, j) \) has a W-shape property for \( \max\{1, i - k - 1\} \leq j \leq \min\{n, i + k + 1\} \) with \( JSI(i, i) = 0 \). We show that the values of \( JSI(i, j) \) is decreasing for the last \( k \) components. Note that the results of \( JSI \) is related to those of Birnbaum structural importance.

Acknowledgement

This work was supported in part by the National Science Council of the Republic of China under Contract NSC 94-2213-E-036-013-.

References:


