Three Dimensional Finite Volume Solutions of Seepage and Uplift in Homogonous and Isotropic Foundations of Gravity Dams

SAEED-REZA SABBAGH-YAZDI*, BABAK BAYAT **
*Associate Professor, ** M.Sc. Graduate
Civil Engineering Department,
KN Toosi University of Technology,
No.1346 Valiasr Street, 19697- Tehran-IRAN

and
NIKOS E. MASTORAKIS
Military Insitutes of University Education (ASEI)
Hellenic Naval Academy
Terma Chatzikyriakou 18539, Piraues, GREECE

Abstract: In this paper, a three-dimensional version of NASIR[1] finite volume seepage solver developed for tetrahedral mesh is introduced. The numerical analyzer is utilized for solving the seepage in homogeneous and isotropic porous media and uplift under gravity dams with upstream cut off wall. The results of numerical solver in terms of uplift pressure underneath of the dam base with upstream cut off wall are compared with analytical solutions obtained by application of from conformal mapping technique for a constant unit ratio of foundation depth over dam base ($T/b = 1$). The accuracy of the results computed uplift pressure present acceptable agreements with the analytical solutions for various ratios of cut off wall over the dam base length (s/b).

Key-Words: - Numerical Analysis, Dam Base Seepage and Uplift, NASIR Software, Finite Volume Method

1 Introduction

The problem of seepage flow underneath of gravity dams can be formulated in terms of a non-linear partial differential equation. The equation describes a constant density fluid flow in a heterogeneous and isotropic porous media [1].

Although empirical formulations are suggested for simple cases, due to the inherently complex boundary conditions and intricate physical geometries in any practical problem, an analytical solution is not possible for complicated dam foundations [2].

This paper presents a finite volume mesh method for modeling water flow in a saturated heterogeneous porous media with complex boundary systems. The solution domain is discretized with tetrahedral cells and the control volumes are constructed around the tetrahedral vertices. Using this strategy the partial differential of fluid volume conservation equations are discretized into a system of differential/algebraic equations. These equations are then resolved in time. These methods are suitable for intricate physical geometries and flow through three dimensional saturated porous media with constant volume. Simulation results for two cases of homogeneous and heterogeneous and isotropic porous media underneath of a gravity dam with upstream cut off are presented and compared with analytical solutions obtained by application of from conformal mapping technique for a constant unit ratio of foundation depth over dam base ($T/b = 1$). The accuracy of the results computed uplift pressure are assessed by comparison of computed results for various ratios of cut off wall over the dam base length (s/b) with the analytical solutions obtained using conformal mapping technique by Pavlovsky,1956 [3].

2 Problem Formulation

The problem of seawater seepage is governed by a partial differential equation for the ground water flow that describes the head distribution in the heterogeneous zone of interest underneath of a gravity dam. The flow equation for a confined saturated porous media can be written as [1]:

$$\frac{\partial}{\partial x_i}\left(k_i \frac{\partial h}{\partial x_i}\right) = S_s \frac{\partial h}{\partial t} \quad (i = 1,2,3) \tag{1}$$

[1] Numerical Analyzer for Scientific and Industrial Requirements

Where h is the reference hydraulic head referred to as the freshwater head; k_i is a component of the hydraulic conductivity tensor; S_s is the specific storage; t is time. The parameter p is the fluid pressure; g is the gravitational acceleration and z is elevation.

If head gradient flux in i direction (secondary variable) is defined as,

$$F_i^d = k_i \partial h / \partial x_i \qquad (i=1,2,3) \tag{2}$$

And hence, the equation takes the form:

$$S_s \frac{\partial h}{\partial t} - \left(\frac{\partial}{\partial x_i} F_i^d \right) = 0 \qquad (i=1,2,3) \tag{3}$$

The boundary conditions for this equation may be stated as follows [1]:

-Dirichlet boundary condition:

$$h(x_b, z_b; t) = h_d(x, z; t) \quad \text{In } B_d \tag{4}$$

-Neumann boundary condition:

$$V.n_i = V_n(x_b, z_b; t) \quad \text{In } B_n \tag{5}$$

where n_i is the outward unit vector normal to the boundary; (x_b, z_b) is a spatial coordinate on the boundary; h_d and V_n are the Dirichlet functional value and Neumann flux, respectively.

It should be noted that, for the homogeneous and isotropic porous media the following relations are valid.

$$k_x = k_y = k_z = k$$

$$\frac{\partial k_x}{\partial x} = 0, \frac{\partial k_y}{\partial y} = 0, \frac{\partial k_z}{\partial z} = 0 \tag{6}$$

3 Numerical Formulation

During the last twenty years there has been a strong focus upon the utilization of the Finite Volume methods for solving fluid flow and heat transfer problems or, as it is more generally known, problems in Computational Fluid Dynamics (CFD). This success is mostly due to the conservative nature of the scheme and the fact that the terms appearing in the resulting algebraic equations have a specific physical interpretation. In fact, the straightforward formulation and low computational cost compared with other methods have made Finite Volume Method the preferred choice for most CFD practitioners [4].

Over the last ten years, several control volume based-unstructured mesh (FVUM) methods have in many way overcome the structured nature of the original control volume method. In general, the FVUM methods can be categorized into two approaches, namely, vertex-centered or cell-centered. The classification of the approach is based on the relationship between the control volume and the finite element like unstructured mesh. The approach described here is the vertex-centered, which uses linear shape function of tetrahedral elements as the interpolation function within the Control Volumes formed by gathering all the elements sharing a nodal point. This approach is very similar to the Galerkin Finite Element Method with linear elements [5,6].

In a finite element mesh, the sub-regions are called elements, with the vertices of the elements being the nodal locations. For the vertex centered approach only the basic three dimensional elements, which tetrahedron with four nodes are considered [7].

Therefore, each node in the solution domain is associated with one control volume. Consequently, each control volume consists of some tetrahedral elements, as illustrated in Figure 1.

The CV can be assembled in a straightforward and efficient manner at the element level. The flow across each control surface must be determined by an integral.

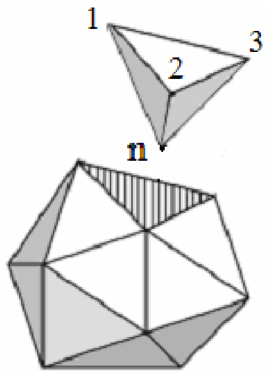

Figure 1 - Sub-domain Ω associated with node *n* of the computational field

The FVUM discretisation process is initiated by utilizing the integrated form of the equation (1). By application of the Variational Method, after multiplying the residual of the above equation by the test function ϕ and integrating over a sub-domain Ω, we have,

$$S_s \int_{\Omega} \frac{\partial h}{\partial t} \phi \, d\Omega + \int_{\Omega} \frac{\partial F_i^d}{\partial x_i} \phi \, d\Omega = 0 \quad (i=1,2,3) \tag{7}$$

The terms containing spatial derivatives can be integrated by part over the sub-domain Ω and then equation (5) may be written as,

$$S_s \int_\Omega \frac{\partial h}{\partial t} \phi \, d\Omega + \int_\Omega F_i^d \phi \, d\Omega - \int_\Omega F_i^d \frac{\partial \phi}{\partial x_i} d\Omega = 0 \qquad (i=1,2,3) \qquad (8)$$

Using gauss divergence theorem the equation takes the form:

$$S_s \int_\Omega \frac{\partial h}{\partial t} \phi \, d\Omega + k_i \oint_\Gamma \phi \, h(n.d\Gamma) - \int_\Omega F_i^d \frac{\partial \phi}{\partial x_i} d\Omega = 0 \qquad (i=1,2,3) \qquad (9)$$

Where Γ is the boundary of domain Ω.

Following the concept of weighted residual methods, by considering the test function equal to the weighting function, the dependent variable inside the domain Ω can be approximated by application of a linear combination, such as $h = \sum_{k=1}^{N_{nodes}} h_k \varphi_k$ [8].

According to the Galerkin method, the weighting function ϕ can be chosen equal to the interpolation function φ. In finite element methods this function is systematically computed for desired element type and called the shape function. For a tetrahedral type element (with four nodes), the linear shape functions, φ_k, takes the value of unity at desired node *n*, and zero at other neighboring nodes *k* of each triangular element ($k \neq n$) [8].

Extending the concept to a sub-domain to the control volume formed by the elements meeting node *n* (Figure 1), the interpolation function φ_n takes the value of unity at the center node n of control volume Ω and zero at other neighboring nodes m (at the boundary of the control volume Γ). Noteworthy that, this is an essential property of weight function, φ, which should satisfy homogeneous boundary condition on T at boundary of sub-domain [3]. That is why the integration of the linear combination $h = \sum_{k=1}^{N_{nodes}} h_k \varphi_k$ (as approximation) over elements of sub-domain Ω takes the value of h_n (the value of the dependent variable in central node n). By this property of the shape function φ ($\varphi_n = 0$ on boundary Γ of the sub-domain Ω), the boundary integral term in equation (7) takes zero value for a control volume which the values of T assumed known at boundary nodes.

After omitting zero term, the equation (7) takes the form,

$$S_s \frac{\partial}{\partial t} \int_\Omega h\varphi \, d\Omega - \int_\Omega F_i^d \frac{\partial \varphi}{\partial x_i} d\Omega = 0 \quad (i=1,2,3) \quad (10)$$

In order to drive the algebraic formulation, every single term of the above equation first is manipulated for each element then the integration over the control volume is performed. The resulting formulation is valid for the central node of the control volume.

For the terms with no derivatives of the shape function φ, an exact integration formula is used as, $\int_\Lambda \varphi_1^a \varphi_2^b \varphi_3^c \varphi_4^d = 6\Lambda(a!b!c!d!)/(a+b+c+d+3) = \Lambda/4$ (for *a*=1 and *b*=*c*=*d*=*0*), where Λ is the volume of the tetrahedral element [6]. This volume can be computed by the integration formula as,

$$\Lambda = \int_\Lambda x_i (d\Lambda)_i \approx \sum_k^4 [\bar{x}_i \, \delta\ell_i]_k \qquad (11)$$

where $\bar{x}_i$ and $\delta\ell_i$ are the average i direction coordinates and projected area (normal to i direction) for every side face opposite to node k of the element.

Therefore, the transient term $\partial/\partial t \int_\Omega \phi \, h \, d\Omega$ for each tetrahedral element Λ (inside the sub-domain) can be written as,

$$\partial/\partial t \int_\Lambda \phi \; h \; d\Lambda = (\Lambda/4) \, dh/dt$$

Consequently, the transient term of equation (10) for the sub-domain Ω (with central node *n*) takes the form,

$$S_s \frac{\partial}{\partial t} \int_\Omega \varphi \, h \, d\Omega = S_s \frac{\Omega_n}{4} \frac{dh_n}{dt} \qquad (12.a)$$

Now we try to discrete the terms containing spatial derivative, $\int_\Omega F_i^d (\partial\phi/\partial x_i) \, d\Omega$ in equation (10). Since the only unknown dependent variable is $h = \sum_k^4 h_k \varphi_k$ and the shape functions, φ_k, are chosen piece wise linear in every tetrahedral element, the heat gradient flux (F_i^d is formed by first derivative) is constant over each element and can be taken out of the integration. On the other hand, the integration of the shape function spatial derivation over tetrahedral element can be converted to boundary integral using Gauss divergence theorem [9], and hence, $\int_\Lambda \partial\varphi/\partial x_i \, d\Lambda = -\oint_\Delta \varphi \, (d\Delta)_i \cdot$

Here Δ is component of the side face element normal to the i direction. The discrete form of the line integral can be written as, $\oint_\Delta \varphi.(d\Delta)_i \approx 1/\Lambda \sum_k^4 [\bar{\varphi} \, \delta\ell_i]_k$, where $[\bar{\varphi} \, \delta\ell_i]_k$ is formed by considering the side of the element opposite to the node *k*, and then, multiplication of its component perpendicular to the i direction by $\bar{\varphi}$ the average shape function value of its three end nodes. Hence, the term

$\int_\Omega F_i^d (\partial\phi / \partial x_i)\, d\Omega \approx -\sum_1^N [F_i^d / \Lambda \sum_k^4 (\bar{\varphi}\,\delta\ell_i)_k]$ for a control volume Ω (containing N elements sharing its central node). Since the shape function φ takes the value of unity only at central node of control volume and is zero at the nodes located at the boundary of control volume, $\bar{\varphi} = 1/3$ for the faces connected to the central node of control volume and $\bar{\varphi} = 0$ for the boundary faces of the control volume. On the other hand the sum of the projected area (normal to i direction) of three side faces of every tetrahedral element equates to the projected area of the fourth side face, hence the term containing spatial derivatives in i direction of the equation (10), can be written as,

$$\int_\Omega F_i^d \frac{\partial\varphi}{\partial x_i}\, d\Omega = -\frac{1}{3}\sum_{m=1}^{M}\left[F_i^d\,\delta\ell_i\right]_m \qquad (12.b)$$

Where $[\delta\ell_i]_m$ is the component of the boundary face m (opposite to the central node of the control volume Ω) perpendicular to i direction. Note that, F_i^d is computed at the center of tetrahedral element of the control volume, which is associated with side m. The head gradient flux in i direction, $F_i^d = k_i \partial h / \partial x_i$, at each tetrahedral element can be calculated using Gauss divergence theorem, $\int_\Omega F_i^d\, d\Omega = k_i \int_\Lambda \partial h / \partial x_i\, d\Lambda = -k_i \oint_\Delta T\,(d\Delta)_i$, where $(d\Delta)_i$ is the projection of side faces of the element perpendicular to i direction. By expressing the boundary integral in discrete form as, $\oint_\Delta h\,(d\Delta)_i \approx \sum_k^3 (\bar{h}\,\delta\ell_i)_k$, for each element inside the control volume Ω. Therefore, we have,

$$[F_i^d]_m = -\frac{k_i}{\Lambda_m}\sum_{k=1}^{3}\left(\bar{h}\,\delta\ell_i\right)_k \qquad (13)$$

Where, $\delta\ell_i$ is the component of kth face of a tetrahedral element (perpendicular to the i direction) and $\bar{h}$ is the average head of that face and Λ is the volume of the element.

Note worthy that for control volumes at the boundary of the computational domain, central node n of the control volume Ω locates at its own boundary. For the boundary sides connected to the to the node n there are no neighboring element to cancel the contribution. Hence, their contributions remain and they act as the boundary sides of the sub-domain. Therefore, there is no change to the described procedure for computation of the spatial derivative terms $\int_\Omega F_i^d (\partial\varphi / \partial x_i)\, d\Omega$.

Finally, using expressions (12.a) and (12.b) the equation (10) can be written for a control volume Ω (with center node n) as:

$$S_s \frac{dh_n}{dt}\,\Omega_n = -\frac{4}{3}\sum_{m=1}^{M}\left[F_i^d\,\delta\ell_i\right]_m \qquad (i=1,2,3) \qquad (14)$$

The volume of control volume, Ω can be computed by summation of the volume of the elements associated with node n.

The resulted numerical model, which is similar to Non-Overlapping Scheme of the Cell-Vertex Finite Volume Method on unstructured meshes, can explicitly be solved for every node n (the center of the sub-domain Ω which is formed by gathering elements sharing node n). The explicit solution of head at every node of the domain of interest can be modeled as,

$$h_n^{t+\Delta t} = h_n^t - \frac{\delta t}{S_s}\left[\frac{4}{3\Omega_n}\left(\sum_{k=1}^{N} F_i^d \Delta l_i\right)_n\right] \qquad (i=1,2) \qquad (15)$$

Now we need to define a limit for the explicit time step, δt. Considering thermal diffusivity as $\alpha=\kappa/\rho C$ with the unit (m^2/s), the criterion for measuring the ability of a material for head change. Hence the rate of head change can be expressed as, $\Omega_n / \delta t \approx k$. Therefore, the appropriate size for local time stepping can be considered as,

$$\delta t = \beta\, \Omega_n / k \qquad (\beta \le 1) \qquad (15)$$

β is considered as a proportionality constant coefficient, which its magnitude is less than unity. For the steady state problems this limit can be viewed as the limit of local computational step toward steady state.

However, there are different sizes of control volumes in unstructured meshes. This fact implies that the minimum magnitude of the above relation be considered. Hence, to maintain the stability of the explicit time stepping the global minimum time step of the computational field should be considered, so,

$$\delta t = \beta\left(\frac{\Omega_n}{k}\right)_{min} \qquad (\beta \le 1) \qquad (16)$$

Noteworthy that for the solution of steady state problems on suitable fine unstructured meshes, the use of local computational step instead of global minimum time step may considerably reduce the computational efforts.

In order to stablizing the numerical solution, time step is restricted by:

$$\Delta t = \left(\frac{S_s \Omega_n}{max(k_i)} \right)_{min} \tag{17}$$

Where Ω_n is area of each control volume and k_i (i=1,2,3) is hydraulic conductivity in i direction.

4 Verification Test Cases

To verify the above described numerical model, a test case considered, for which analytical solution is available. The analytical solutions of the seepage and uplift pressure through the homogeneous and isotropic dam foundation results are obtained for a number of ratios of cut off wall over the dam base length (s/b) using conformal mapping technique. The parameters were chosen so that the analyzed cases correspond to those analytically solved by Pavlovsky, 1956 [3].

The geometry of the dam foundation with a upstream cut off wall at the dam base test case is schematically described in figure 2. The boundary conditions employed in present numerical simulation are also shown in Figure 2.

The foundation region considered to be as homogeneous and isotropic sand with $k_x = k_y = k_z = 5 \times 10^{-5}\ {m}/{Sec}$ and $S_s = 8 \times 10^{-5}\ {1}/{m}$ is represented in a discrete form by a three-dimensional tetrahedral mesh for a cubic dam foundation is shown in Figure 3.

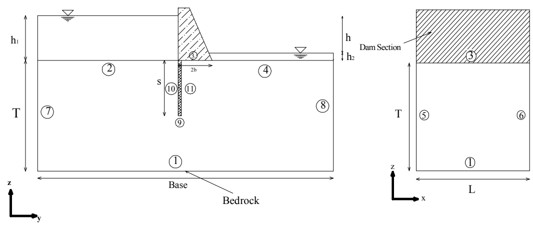

Figure2: Problem description of saltwater intrusion in a coastal confined aquifer [3]

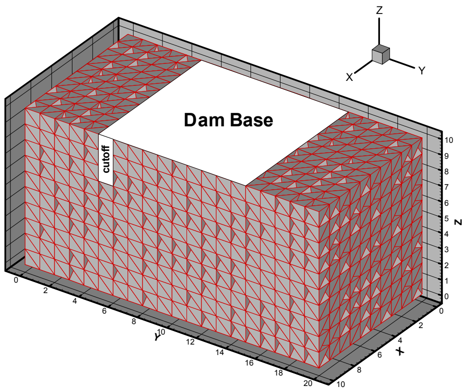

Figure 3: A three-dimensional tetrahedral mesh for dam foundation

Figure 4 shows a typical computed color coded surfaces of head in the homogeneous and isotropic sand foundation of dam with up stream cut off.

Figures 5 and 6, respectively, present typical computed color coded velocity vectors and flow net in a homogeneous and isotropic sand foundation of dam with up stream cut off.

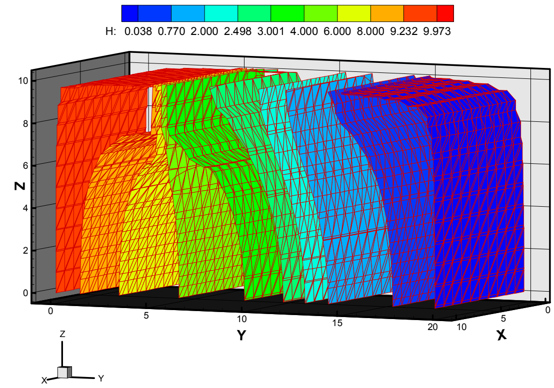

Figure 4: Typical computed color coded surfaces of head in the homogeneous and isotropic sand foundation of dam

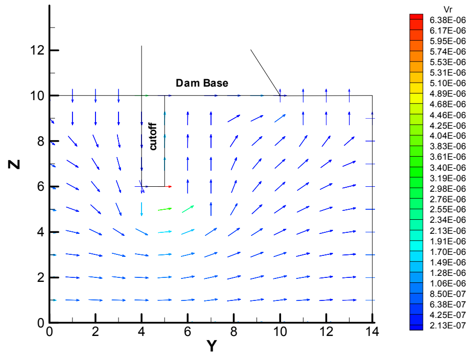

Figure 5: Typical computed velocity vectors in the homogeneous and isotropic sand foundation of dam

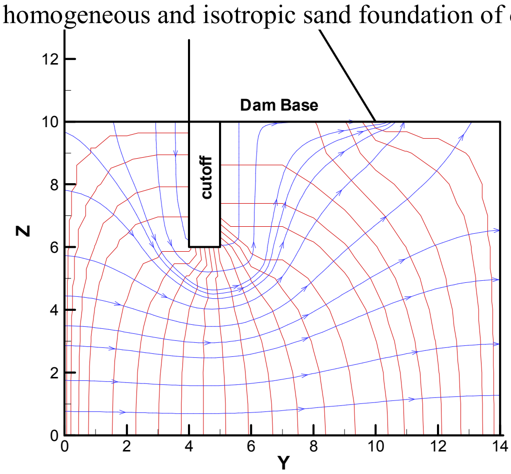

Figure 6: Typical computed flow net in the homogeneous and isotropic sand foundation of dam

Figure 7 presents plots of uplift pressure distribution underneath of dam with up stream cut off for various ratios of cut off wall over the dam base length (s/b) for a constant unit ratio of foundation depth over dam base ($T/b = 1$).

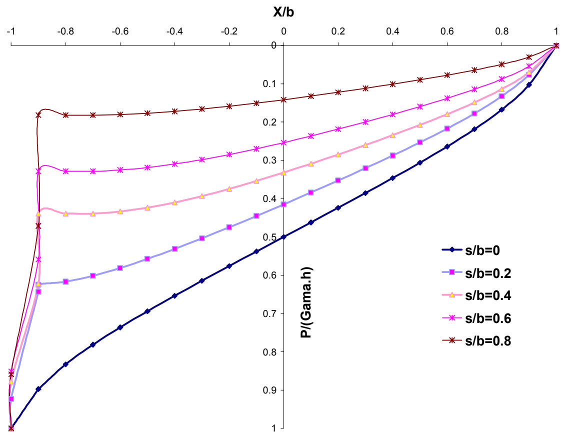

Figure 7: Uplift pressure distribution underneath of dam with up stream cut off for s/b ($T/b = 1$)

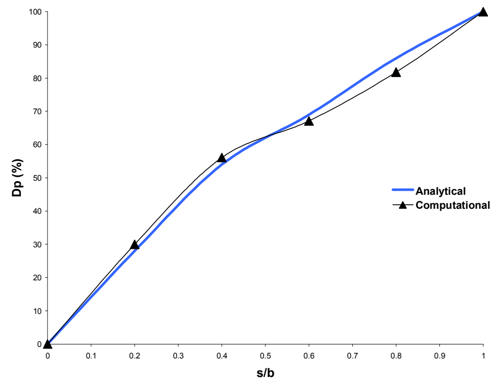

Figure 8: The comparison of the computed results for various (S/b) with the analytical solution of Pavlovsky,1956 [3]

Figure 8 presents plots of uplift pressure drop $D_p = (h_L - h_R)/h \times 100$ underneath of dam with up stream cut off for various ratios of cut off wall for a range of s/b for $T/b = 1$. In this relation h_L and h_R are pressure heads upstream and down stream of the cut wall and h is the difference of water heads at upstream and down stream of dam. The average error between numerical results and analytical solution is 0.56%, while the maximum error is computed as 7%.

As can be seen the accuracy of the results computed by present version of NASIR unstructured finite volume model for solution of seepage flow and computation of uplift pressure are quite acceptable.

Conclusion

In present paper, a 3D numerical model based on finite volume unstructured mesh (FVUM) method is developed for computing the seepage flow and uplift pressure under gravity dams with cut off wall. The model explicitly solves the equation of ground water flow on the three dimensional unstructured mesh using Galerkin Finite Volume Method developed for linear tetrahedral elements. The model can predict pressure head distribution in geometrical complex porous media. In order to verify the accuracy of model results, the seepage flow through a homogeneous and isotropic sand dam foundation is solved for various ratios of upstream cut off wall over the dam base length (s/b) for a constant unit ratio of foundation depth over dam base ($T/b = 1$). The computed results of uplift pressure distribution are compared with the analytical solutions obtained by application of conformal mapping technique by Pavlovsky,1956. Acceptable agreement between the results of the present simulation and analytical solutions encourages application of the model for modeling seepage flow in heterogeneous and anisotropic porous media.

References:

[1] Bear, J., "Hydraulics of Groundwater", McGraw-Hill, New Your, 1979.

[2] US Army Corps of Engineers, “Engineering and Design; Seepage analysis and control for dams”, EM 1110-2-1902, chapter2 (Determination of permeability of soil and chemical composition of water), April 1993.

[3] Lakshmi N. Reddi, 2003, “Seepage in Soils”, John Wiley & Sons Inc.

[4] Hoffmann K.A and Chiang S.T, "Computational Fluid Dynamic for Engineers", Engineering Education System, 1993. 165.

[5] Segol. G., Pinder, G. F. and Gray, W. G. (1975) “A galerkin finite element technique for calculating the transient position of the saltwater front.” Water Res., 11(2), 1975, 343-347.

[6] Sabbagh-Yazdi S.R. and Mastorakis E.N. (2007), "Efficient Symmetric Boundary Condition for Galerkin Finite Volume Solution of 3D Temperature Field on Tetrahedral Meshes ", The 5th IASME / WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment Vouliagmeni Beach, Greece

[7] Thompson J.F. , Soni B.K. and Weatherill N.P., Hand book of grid generation, CRC Press, New York, 1999.

[8] Reddy J.N. and Gartling D.K. , “The Finite Element Method in Heat Transfer and Fluid Dynamics”, CRC Press, 2000.