Stability Analysis of Wilson-θ Method with Modified Acceleration

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Abstract: There remain two kinds of the Wilson-θ methods, namely, the Wilson-θ □ and □ methods. In the Wilson-θ □ method, the accelerations are not modified by the dynamic equilibrium equations; in the Wilson-θ □ method, the accelerations are modified. The amplification matrixes of the Wilson-θ □ and □ methods for single-degree-of-freedom system are derived. The stabilities of the Wilson-θ □ and □ methods are examined by the spectral radii of the amplification matrixes. The stability of the Wilson-θ □ method is unconditional. The calculation results indicate: the stability of the Wilson-θ □ method is not unconditional. The stability ranges of the Wilson-θ □ method are also put forward.

Key-Words: Wilson-θ method, stability, amplification matrix, spectral radius

1 Introduction

Wilson et al. extended the linear-acceleration method in a manner that makes it numerically stable [1]. The basic assumption of the Wilson-θ method is that the acceleration varies linearly over an extended time step $\theta\Delta t$. It has been shown that $\theta \geq 1.37$ will assure the unconditional stability regardless of the magnitude selected for the time step. For this reason, the Wilson-θ method is widely used. Several improvements have also been carried out [2~4]. But there remain two kinds of the Wilson-θ methods, namely, the Wilson-θ □ and □ methods. In the Wilson-θ □ method, the accelerations $\ddot{x}(t+\Delta t)$ at the time $t+\Delta t$ are not modified by the dynamic equilibrium equations [5]; in the Wilson-θ □ method, the accelerations are modified [6,7]. The stability of the Wilson-θ □ method is unconditional [1,4], but it is wrongly pointed out that the stability of the Wilson-θ □ method is also unconditional in some references [6, 7]. In this paper, the stability of the Wilson-θ □ method is proved to be conditional. The modal superposition method can reduce the response of a multi-degree-of-freedom (MDOF) system to the superposition of the single degree of freedom (SDOF) system responses for each mode, thus the stability for a SDOF system is equivalent to the stability for a MDOF system. In order to simplify the equation expression, only SDOF system is considered here. Naturally, the conclusions are applied to the MDOF system

2 The amplification matrixes of the Wilson-θ □ and □ methods

The following operator form can be summarized among various direct numerical integration methods including the Wilson-θ □ and □ methods:

$$\begin{Bmatrix} x(t+\Delta t) \\ \dot{x}(t+\Delta t) \\ \ddot{x}(t+\Delta t) \end{Bmatrix} = [A]\begin{Bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} F(t+\Delta t) \quad (1)$$

where $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ and $x(t+\Delta t)$, $\dot{x}(t+\Delta t)$, $\ddot{x}(t+\Delta t)$ are the displacements, velocities and accelerations at the time t and $t+\Delta t$ respectively; f_1 , f_2 , f_3 are the load operators, $F(t+\Delta t)$ is the dynamic load; $[A]$ is the amplification matrix. The stability criterion for a direct numerical integration method is $\rho(A) \leq 1$ [5], $\rho(A)$ is the spectral radius, $\rho(A) = \max|\lambda_i|, i = 1,2,3$, where λ_i are the eigenvalues of $[A]$. If $\rho(A) \leq 1$ is found to be true, then the method is stable at the value of the time step Δt .In Equation (1), $x(t+\Delta t)$, $\dot{x}(t+\Delta t)$, $\ddot{x}(t+\Delta t)$ are expressed as the function of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, so the algorithm of the Wilson-θ method would be rewritten as follows:

(1) Select the time step Δt and θ values, calculate $\tau = \theta\Delta t$ and the constants $\beta_0 = 6/\tau^2$, $\beta_1 = 3/\tau$, $\beta_2 = 6/\tau$, $\beta_3 = \tau/2$, $\beta_4 = \beta_0/\theta$,

$\beta_5 = -\beta_2/\theta$, $\beta_6 = 1 - 3/\theta$, $\beta_7 = \Delta t/2$, $\beta_8 = \Delta t^2/6$.

(2) Calculate the effective stiffness $\widetilde{K}$ and the effective force $\widetilde{F}(t+\tau)$ according to $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, stiffness *K*, damping coefficient *C*, mass *M*, and the dynamic force $F(t+\tau)$ by the following Equations:

$$\widetilde{K} = K + \beta_0 M + \beta_1 C \quad (2)$$

$$\widetilde{F}(t+\tau) = F(t+\tau) + B_1 x(t) + B_2 \dot{x}(t) + B_3 \ddot{x}(t) \quad (3)$$

where $B_1 = \beta_0 M + \beta_1 C,\ B_2 = \beta_2 M + 2C,$ $B_3 = 2M + \beta_3 C$

(3) Solve the pseudostatic equation $\widetilde{K}x(t+\tau) = \widetilde{F}(t+\tau)$ for $x(t+\tau)$:

$$x(t+\tau) = f_k + D_1 x(t) + D_2 \dot{x}(t) + D_3 \ddot{x}(t) \quad (4)$$

where $f_k = F(t+\tau)/\widetilde{K}$, $D_1 = B_1/\widetilde{K}$, $D_2 = B_2/\widetilde{K}$, $D_3 = B_3/\widetilde{K}$.

(4) Find $\ddot{x}(t+\Delta t)$, $\dot{x}(t+\Delta t)$, $x(t+\Delta t)$, by the following Equations:

$$\ddot{x}(t+\Delta t) = \beta_4 f_k + E_1 x(t) + E_2 \dot{x}(t) + E_3 \ddot{x}(t) \quad (5)$$

where $E_1 = \beta_4(D_1 - 1)$, $E_2 = \beta_4 D_2 + \beta_5$, $E_3 = \beta_4 D_3 + \beta_6$

$$\dot{x}(t+\Delta t) = \beta_7 \beta_4 f_k + G_1 x(t) + G_2 \dot{x}(t) + G_3 \ddot{x}(t) \quad (6)$$

where $G_1 = \beta_7 E_1$, $G_2 = \beta_7 E_2 + 1$, $G_3 = \beta_7(E_3 + 1)$.

$$x(t+\Delta t) = \beta_8 \beta_4 f_k + H_1 x(t) + H_2 \dot{x}(t) + H_3 \ddot{x}(t) \quad (7)$$

where $H_1 = \beta_8 E_1 + 1, H_2 = \beta_8 E_2 + \Delta t, H_3 = \beta_8(E_3 + 2)$

(5) Combining $x(t+\Delta t)$, $\dot{x}(t+\Delta t)$ obtained by Equation (6) and (7), recalculate $\ddot{x}(t+\Delta t)$ directly from the dynamic equilibrium equation at the time $t+\Delta t$ to reduce the calculation error, namely:

$$M\ddot{x}(t+\Delta t) + C\dot{x}(t+\Delta t) + Kx(t+\Delta t) = F(t+\Delta t) \quad (8)$$

$$\ddot{x}(t+\Delta t) = (F(t+\Delta t) - \beta_7 \beta_4 f_k C - \beta_8 \beta_4 f_k K)/M + J_1 x(t) + J_2 \dot{x}(t) + J_3 \ddot{x}(t) \quad (9)$$

where $J_1 = -(CG_1 + KH_1)/M$, $J_2 = -(CG_2 + KH_2)/M$, $J_3 = -(CG_3 + KH_3)/M$.

(6) Calculate the amplification matrixes: $\ddot{x}(t+\Delta t)$ obtained by Equation (5), is not modified by Equation (8), that is, using the Wilson-θ □ method, the amplification matrix is $[A_1]$; $\ddot{x}(t+\Delta t)$ obtained by Equation (9), meets the requirement of Equation (8), that is, using the Wilson-θ □ method, the amplification matrix is $[A_2]$. The expressions of $[A_1]$ and $[A_2]$ are as follows:

$$[A_1] = \begin{bmatrix} H_1 & H_2 & H_3 \\ G_1 & G_2 & G_3 \\ E_1 & E_2 & E_3 \end{bmatrix}, \quad [A_2] = \begin{bmatrix} H_1 & H_2 & H_3 \\ G_1 & G_2 & G_3 \\ J_1 & J_2 & J_3 \end{bmatrix}$$

3 Spectral radius ρ_2

ρ_2 is the spectral radius of $[A_2]$. For a SDOF system, the spectral radius ρ is the function of the period T, damping ratio ξ, θ value, and the time step Δt . If $\Delta t/T$ is small enough, the direct numerical integration method will be stable. The spectral radii ρ_2 for various values of θ and $\Delta t/T$ are listed in Fig. 1 and Table 1. Fig. 1 indicates the stability range shrinks when increasing θ value. It is indicated in Table 1 that with an increase in the value of θ, the value of ρ_2 increases also. According to the stability criterion $\rho_2 \le 1$, the Wilson-θ □ method is stable if $\Delta t/T \le 0.1$, with $\theta = 1 \sim 3$; or if $\Delta t/T \le \sqrt{3}/\pi$, with $\theta = 1$ (namely linear acceleration method). In a word, the Wilson-θ □ method is no longer unconditionally stable.

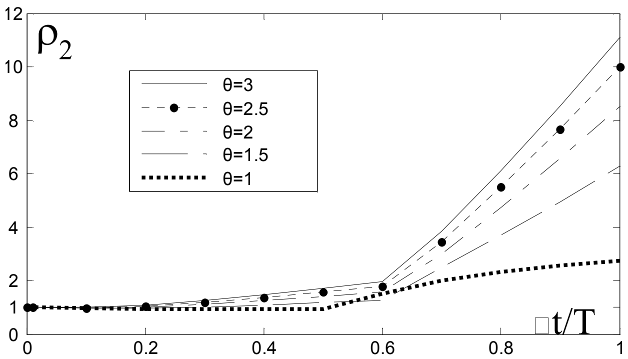

Fig. 1 Spectral radii ρ_2 for various values of θ and $\Delta t/T$

Table 1 Spectral radii ρ_2 for various values of θ and $\Delta t/T$

θ \ Δt/T	0.1	0.2	0.5	0.6
1.0	0.971	0.951	0.942	1.509
1.2	0.972	0.960	1.012	1.180
1.4	0.974	0.972	1.116	1.193
1.6	0.976	0.986	1.222	1.340
1.8	0.978	1.002	1.320	1.471
2.0	0.980	1.017	1.408	1.584
2.2	0.983	1.032	1.484	1.682
2.4	0.985	1.046	1.551	1.766
2.6	0.988	1.060	1.609	1.840

2.8	0.991	1.072	1.659	1.904
3.0	0.993	1.084	1.705	1.960

4 Conclusion

The above investigation has led to the following conclusions:

(1) The Wilson-θ method, in which the acceleration at the time $t+\Delta t$ meets the dynamic equilibrium equation, may reduces the calculation error, but it is conditionally stable only.

(2) The stability ranges of the Wilson-θ method are: $\Delta t/T \le 0.1$, with $\theta = 1 \sim 3$; or $\Delta t/T \le \sqrt{3}/\pi$, with $\theta = 1$.

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